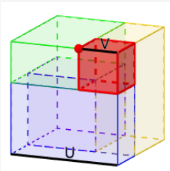
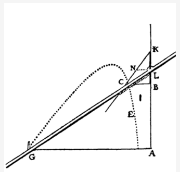
Some topics in Renaissance Mathematics



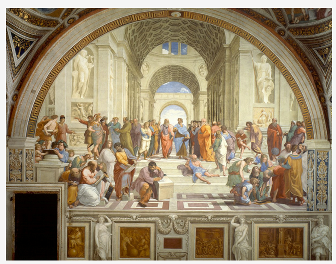


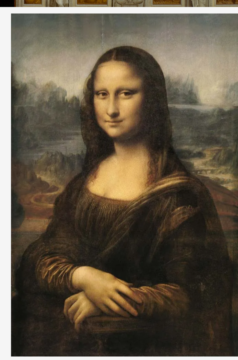
- Mathematics and the Renaissance
- The emergence of modern algebraic notation
- The solution cubic equations
- Algebra and the solution of equations during the Renaissance and after.
- Bombelli and the development of Complex Numbers
- History of negative and complex numbers
- Fermat, Cavalieri, Pascal
- Descartes on curves
- Napier and logaritms.

Mathematics and the Renaissance

Renaissance ~ 1300 to 1600

- Revival of Greek and Roman learning
- Human-centered worldview (humanism)
- Emphasis on observation, critical inquiry, and individual reasoning
- Broad cultural, artistic, and intellectual renewal in Europe





Reconstruction of Gutenberg's printing press

https://www.printmuseum.org/gutenberg-press

List some possible ways in which the printing press could have impacted the way people thought about mathematics and knowledge in Europe

How did the printing press in Europe change the way people thought about mathematics or knowledge?

- **Before**: manuscripts, errors introduced, limited circulation, knowledge controlled by scribes/ monasteries
- **After**: same text available for all→ scholars could build on each other's work, challenge and verify claims

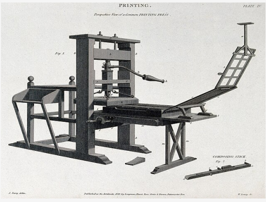
https://www.printmuseum.org/gutenberg-press

- Eventual **shift from authority to verification**: "I can check the source myself"
- Knowledge became cumulative, not repetitive
- Enabled rapid dissemination and standardization.
- Also
  - Freezes errors as well as insights
  - Creates priority disputes and authorship claims

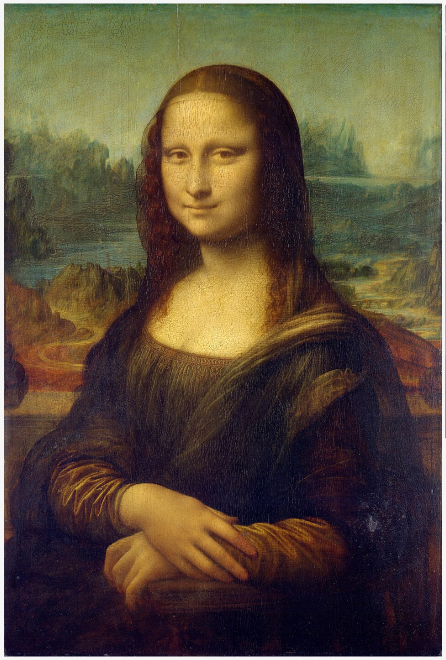
Renaissance ~ 1300 to 1600 - Causes

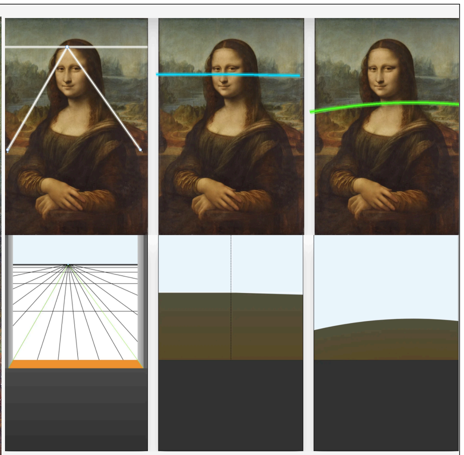
- **Printing press**
- **Wealth & patronage**: Italian city-states (Florence, Venice) funded scholars/ artists
- Political fragmentation in Europe, creating **competition among courts and universities.**
- **Trade expansion**: Contact with Islamic world brought preserved Greek works + new mathematics
- **The "new world discovery:** New wealth, navigation challenges, expanded worldview.
- Fall of Constantinople Greek scholars fled west with ancient texts

- Willingness to **question previously held truths and search for new answers** resulted in a period of major scientific advancements.



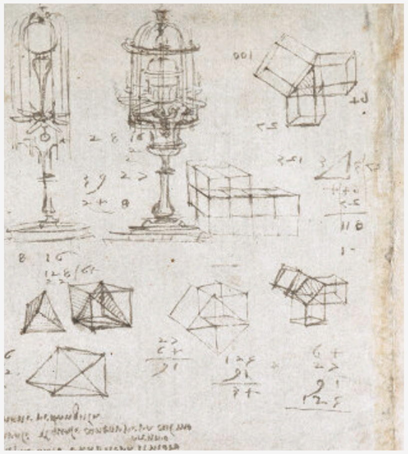
Printing press from 1819





https://nicofranz.art/en/leonardo-da-vinci/mona-lisa

Leonardo da Vinci trying to extend Pythagoras's theorem on right-angled triangles from squares to cubes.



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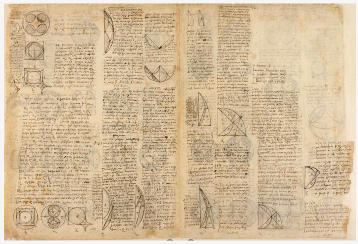
See Chousing, R. (2008). "Leonardo and Theoretical Mathematics." *Nexus Network Journal*, 10(1), pp. 39–52. https://link.springer.com/content/pdf/10.1007/s00004-007-0055-9.pdf

Leonardo thinking about the squaring of the circle



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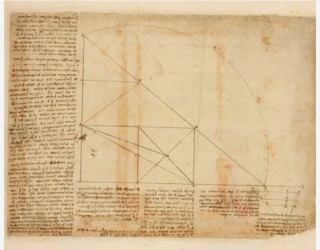
Leonardo thinking about the duplication of the cube



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Medieval

Maestà by Cimabue's (**1280**)

Ognissanti Madonna by Giotto's (c. **1310**)

Renaissance

Flagellation of Christ (Piero della Francesca)

The Last Supper - Leonardo da Vinci

Pick one question or answer both. 1. What's different between the medieval painting and the Renaissance paintings?
2. The Renaissance paintings look 3D even though they're flat. Have you seen this 'trick' anywhere in modern life?

1. Medieval vs. Renaissance Perspective

Medieval *(Cimabue's Maestà)*: Flat, gold background; figures don't recede in space; symbolic/hierarchical arrangement; no realistic depth.

Renaissance *(Flagellation, Ognissanti Madonna)*: Realistic 3D depth; perspective with vanishing point; architectural elements recede realistically; figures have volume and spatial relationships.

Note the difference in how size is used:

- **Medieval**: Size indicates importance (spiritual hierarchy) — Christ/Mary are largest because most important, not because they're closer.
- **Renaissance**: Size indicates distance (mathematical perspective) — larger figures are closer, smaller are farther, following geometric rules.

2. Modern Examples

- Video games (3D graphics on 2D screens)
- Movies/TV (2D screen shows 3D world)
- Photography
- Virtual reality interfaces
- Architectural drawings/blueprints
- Any screen-based 3D visualization

School of Athens - Raphael -

Christ Giving the Keys to St. Peter - Perugino

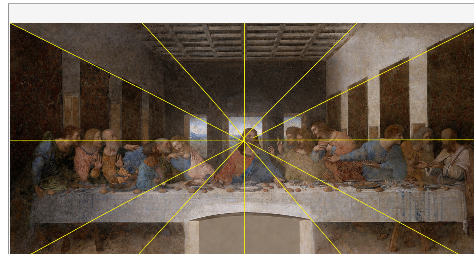
Choose one (or more) of the four paintings and for each do the following

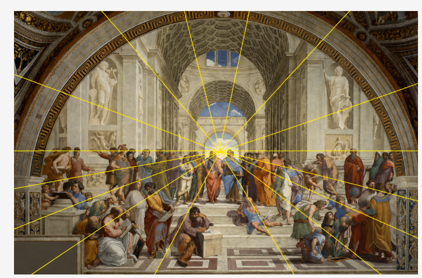
1. Look for lines that seem to go back into the distance — edges of ceilings, floor tiles, walls, beams.
2. Using a ruler (or a straight edge), draw these lines extended across the painting.
3. The point where these lines meet is called the vanishing point (this is an approximation — the lines may not meet exactly). Mark the vanishing point with an X.
4. Take a photo of your work and upload it to Wooclap.
5. List one or more of the mathematical ideas artists needed to create this illusion of depth.

Flagellation of Christ (Piero della Francesca)

The Last Supper - Leonardo da Vinci

Date ~1500

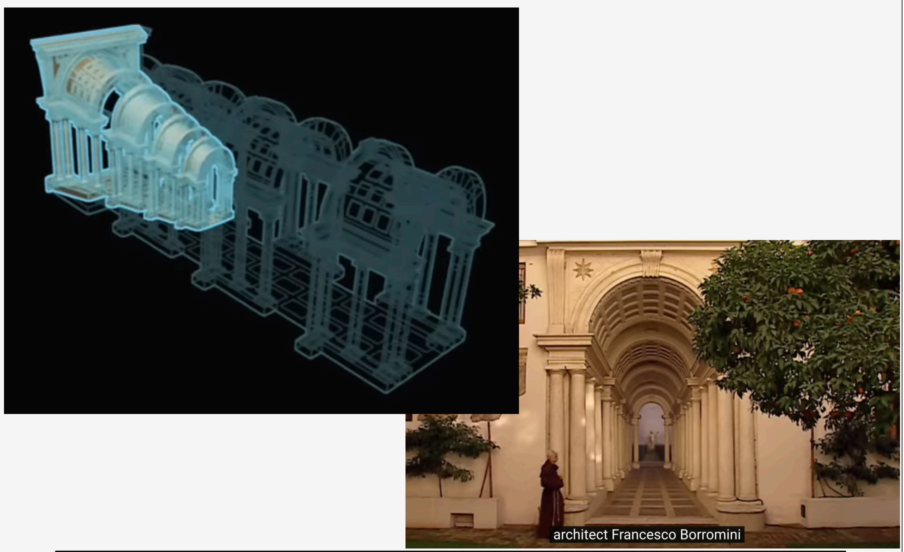




Estimate the length of this corridor (in feet). The monk is about 5'7''

Borromini Corridor, Palazzo Spada, Rome, by Francesco Borromini ~ mid 1600

What is the actual length of the corridor?



From one column to the next, the ratio

(column height) / (spacing to the next column) stays constant.

Renaissance ~1300–1600 - Science

Observational revolution:

- Copernicus: heliocentric model
- Galileo: telescope, experimental method. Nature written in "language of mathematics”
- Kepler: elliptical orbits, mathematical laws

Scientific Method emerges.

Shift from Aristotelian logic → empirical observation + mathematical description.

Machines → study of motion and theoretical mechanics

Optimism about reason

Early calculus: Work on motion, area, infinitesimals.

European Renaissance Mathematics ~ 1300 to 1600

Math shifted from

- Geometry (ideal forms)
- Practical tool

To

Describing motion and change

Culminated: calculus of Isaac Newton and Gottfried Leibniz

European Renaissance Mathematics ~ 1300 to 1600

Symbolic algebra

- Symbolic notation
- Cubic equations
- Operations with imaginary numbers.

Practical math:

- Double-entry bookkeeping.
- Decimal positional number system.
- Navigation, surveying, and accounting → trigonometry, new ways of computing, etc.
- Machines → study of motion and theoretical mechanics
- Artillery and fortification → early ballistics

European Renaissance Mathematics ~ 1300 to 1600

Art impact

- Science and art intertwined:
- Artists→ geometric methods.

Abstract math

- Work on Greek texts.
- Influence of Islamic mathematics.
- Expectation that nature could be described mathematically
- Idea of proof (as opposed to believe based on authority)

- **Antiquity → 1200s:** Roman numerals and the counting board were the standard system for calculation and record-keeping in Europe.
- **Late 1200s–1400s**: Transition period. After **Fibonacci**'s Liber Abaci (1202), Hindu–Arabic numerals began entering commercial education, but adoption was slow.
- **1400s–1600s**: Mixed use. Ledgers in some regions (especially Italy) began to adopt Hindu–Arabic numerals; other regions kept Roman numerals longer.
- **By 1600s**: Hindu–Arabic numerals dominated European bookkeeping and calculation; Roman numerals survived mainly in headings, dates, and decoration.

Why do you think the Hindu–Arabic numeral system and its algorithms faced so much initial resistance?

How did the Renaissance context (art, exploration, printing press, etc.) shape the mathematics that developed during this period?

How did the Renaissance context (art, exploration, printing press, etc.) shape the mathematics that developed during this period?

It pushed mathematics to grow in new directions by changing what people cared about and what problems they needed to solve:

Art demanded geometry for perspective → work on proportion, projection, similar triangles.

Exploration and navigation needed better trigonometry, tables, and computation.

The **printing press** made mathematical texts easier to copy, compare, criticize, and build on → faster spread and standardization.

Humanism encouraged revisiting Greek sources and valuing reasoning over authority → revival of Euclid, Archimedes, Ptolemy.

Scientific curiosity (astronomy, mechanics) required quantitative methods → math became a tool for describing nature.

In summary: the Renaissance created practical and intellectual pressures that pushed mathematics toward geometry, algebra, and quantitative science.

Exploration - Navigation In Renaissance Impact of mathematics

Created

- practical problems requiring mathematical solutions
- demand for mathematically trained professionals

- Navigation required better astronomical calculations, trigonometry, cartography
- Surveying new lands required geometric methods
- **Practical needs funded and directed mathematical research**

Contact with the Islamic World

- Brought preserved Greek texts + original Islamic innovations
- Renaissance Europeans accessed Euclid, Archimedes, Apollonius through Arabic translations
- Also gained algebra (al-Khwarizmi), algorithms, trigonometry, astronomical tables
- Trade routes brought mathematical knowledge along with goods
- **Built on two traditions simultaneously : Greek geometry + Islamic algebra**

Wealth and Patronage

- Mechanism: Italian city-states funded scholars and competitions
- Mathematicians competed for positions at courts and universities
- Public mathematical contests (like Tartaglia vs. Fiore) created pressure to solve problems
- Wealthy patrons supported mathematical work
- **Economic incentives impulsed mathematical innovation**

Fall of Constantinople (mid 1400)

- The Ottoman sultan Mehmed II conquered Constantinople, ending the Byzantine Empire and the last remnant of the Roman Empire.
- Greek scholars fled west with original Greek manuscripts and knowledge.
- Europeans could now read Greek mathematics in original language, in addition to Arabic translations
- Sparked effort to "recover" and surpass ancient knowledge

The emergence of modern algebraic notation

The emergence of modern algebraic notation

At the end 1500s, algebra began to look more like it does today.

Notations similar to those we use started to appear in the work of many mathematicians.

The emergence of modern algebraic notation

François Viète: 1540–1603

- **François Vietè** worked at the French court as a **cryptographer**. (The king of Spain accused Viète of having used magical powers!).
- Used
  - A, E, I, O, U for unknown quantities
  - consonants for known quantities.

A cubus+B plano in A, aequetur C solido

Confectarium: Itaque si A cubus + B plano 3 in A, aequetur Z solido 2,

Si A cubus + B plano in A, equetur, Z solido". | $A^3 + D A = Z$

The analytic art . . . claims for itself the greatest problem of all, which is *to solve every problem* François Viète, The Analytic Art

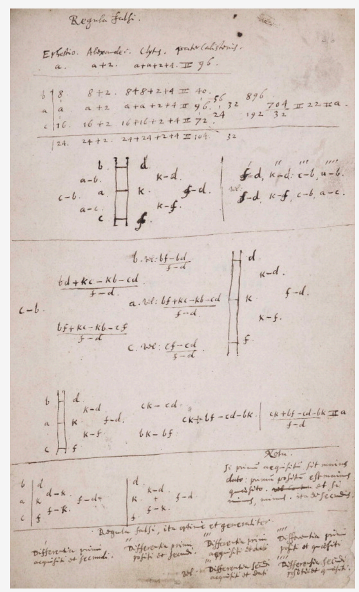
What was François Viète's main contribution to algebraic notation, and why did this matter?

Viète systematically used letters to represent both known quantities (parameters) and unknowns in equations. He used consonants (B, C, D...) for known values and vowels (A, E, I...) for unknowns. This allowed:

1. **Generality:** Writing one equation to represent entire families of problems
2. **Symbolic manipulation:** Operating on symbols according to rules, not just on specific numbers
3. **Pattern recognition:** Seeing relationships between problems more clearly

This shifted algebra from "calculating with specific numbers" to "reasoning with general relationships." It's the foundation of modern algebraic thinking—we now write $ax^2 + bx + c = 0$ to represent all quadratics, not just individual cases.

The emergence of modern algebraic notation



Thomas Harriot Manuscript.

Thomas Harriot (1560-1621)

- English mathematician
- His work on **solutions of equations** looks remarkably **modern**
- Pure symbolic notation:
  - Invented *ab* for multiplication, √ for roots
  - Worked with negative and complex roots symbolically
- Relationship with Viète unclear—mutual influence debated

The emergence of modern algebraic notation

- Descartes Géométrie (1637) build on Viéte by:
  - rejecting the need to adhere to homogeneity:
  - Using the letters at the beginning of the alphabet for known quantities and the letters at the end of the alphabet for unknowns.

It is not however the same thing when unity is determined, because unity can always be understood, even when there are too many or too few dimensions; thus, if we are required to extract the **cube root of aabb - b [a^2b^2 - b] we must consider the quantity aabb to be divided once by unity, and the quantity b multiplied twice by unity.**

The impact of modern algebraic notation

Symbolic Algebra: Breaking Free from Concrete Reality

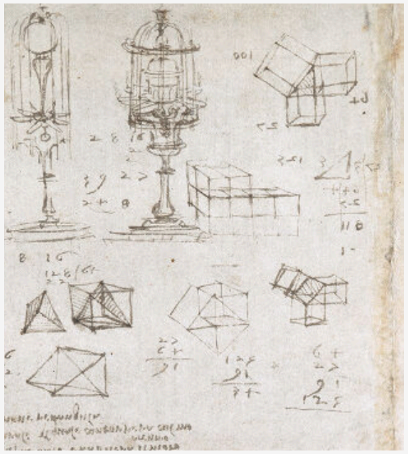
- **Before:** Numbers tied to **concrete** objects
- **After:** Letters represent **abstract** quantities

Revolutionary shift:

- Manipulate symbols without concrete meaning:
  - $x + 3 = 1 \rightarrow x = 1 - 3$ instead of “"a number which is increased by three to give one”"
- Equations previously "impossible" now solvable. ($x^2 = -3$)
- Mathematics defined by *internal consistency*, not external reality.
- Math becomes self-generating rather than descriptive.

New mathematical objects emerge: Negative numbers, imaginaries, quaternions,

Leonardo da Vinci trying to extend Pythagoras’s theorem on right-angled triangles from squares to cubes.



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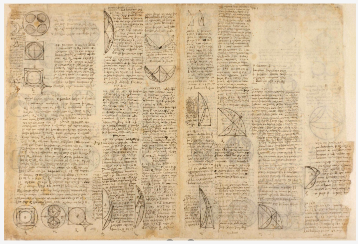
Leonardo thinking about the squaring of the circle



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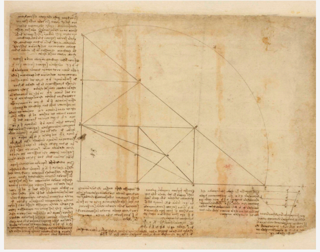
Leonardo thinking about the duplication of the cube



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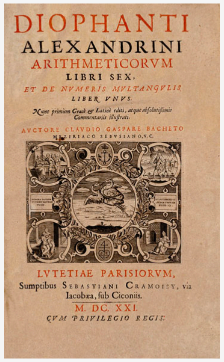
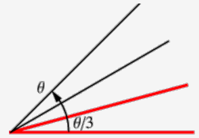
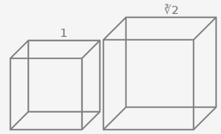
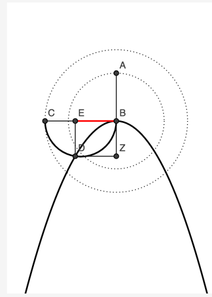
Solution of the Cubic Equation Circa 1500





$\cos\theta = 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}$





Cubic equations we've seen before

These ideas resurfaced in the Renaissance.

Consider a positive number A.

1. **Give a definition of *square root of A, √A***
2. **Find all solutions of the equation $x^2 = A$.**

1. Definition of √A:
   - **Algebraic definition:** √A is the unique positive number whose square is A. **Note:** √A specifically means the *positive* square root (by convention)
2. Solutions of $x^2 = A$:
   - x = √A or x = -√A

In general, equations can have multiple solutions, while the √ notation means the positive root (it is a particular choice of a solution)

In the beginning of the Renaissance somebody said: "*Solving a cubic equation is as hard as squaring the circle.*"

What do you think is the meaning of that statement?

Find the coefficient of x^2 of $p(x-A/3)$ where $p(x)$ is the cubic polynomial

$p(x) = x^3 + A x^2 + B x + C$

Find the coefficient of x^2 of $p(x-A/3)$ where $p(x)$ is the cubic polynomial

$p(x) = x^3 + A x^2 + B x + C$.

From $(x - A/3)^3$:

- x^2 appears in: $3x^2(-A/3)^1 = -Ax^2$

From $A(x - A/3)^2$:

- x^2 appears in: $A \cdot x^2 = Ax^2$

From $B(x - A/3)$:

- No x^2 term ✗

From C:

- No x^2 term ✗

The coefficient is $-A+A = 0$

Depressed 😞 Cubics

If $p(x) = x^3 + A x^2 + B x + C$ then

$p(x - A/3) = x^3 + (B - A^2/3) x + (2A^3/27 - AB/3 + C)$ has no quadratic term.

This "depressed cubic" substitution (eliminating x^2) was known before Luca Pacioli (~1500)

Renaissance mathematicians only worked with positive coefficients, they treated the following as different equations:

$x^3 + ax = b$ and $x^3 = b + ax$

My book ended up stating that it **seemed impossible** to find a formula that yields the roots of the equations $x^3 + a \cdot x = b$ and $x^3 = b + a \cdot x$

I am **Luca Pacioli**. In 1494, I wrote a book containing all that was known about arithmetic, algebra, geometry and trigonometry.

We used to call the unknown the "thing" (cosi, or co)

A portrait of mathematician Luca Pacioli and Unknown Young Man, attributed to the Italian Renaissance artist Jacopo de' Barbari, circa 1495-1500, and now housed in The Capodimonte Museum in Naples. Photo: Scala

When the squares and the things are equal to a number, first you must reduce all the equation to one square, that is if there is less than one square you must equally restore and make good. And if there is more than one square you must reduce to one square, and reducing is done by dividing the whole of the equation by the amount of the squares. And when you have done this, halve the things, and multiply one half by itself. The number is added to this product, and the root of this sum minus the half of the things is the value of the thing required

Recall Al-Khwarizmi

Luca Pacioli, Summa de arithmetica . . . (1494), in Fauvel and Gray, p. 251.

A portrait of mathematician Luca Pacioli and Unknown Young Man, attributed to the Italian Renaissance artist Jacopo de' Barbari, circa 1495-1500, and now housed in The Capodimonte Museum in

PACIOLI, Luca (Lucas de Burgo S. Sepulchri; c.144...

15th Century

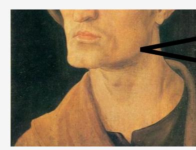
Estimate

USD 1,000,000 - 1,500,000

Price Realised

USD 1,215,000

https://www.christies.com/en/auction/Summa-de-Arithmetica-The-Birth-of-Modern-Business-28300?salenumber=17644&saleroomcode=nyr&lid=1&dt=9420199363&saletitle=



Some time after 1500, I, **Scipione del Ferro**, found a general solution to the equation **'a cube and things equal to numbers'** which you might know as

$x^3 + ax = b$

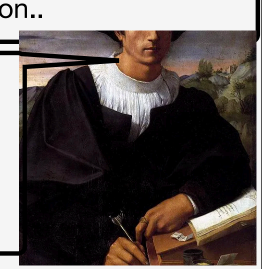
Here, a and b are positive numbers (I did not know that something called "negative numbers existed)

In my deathbed, I told my student, **Antonio Maria del Fiore** about the solutions.

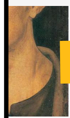
I died in 1525, without publishing my discovery, leaving notes containing the solution..

I am **Antonio Maria del Fiore**, del Ferro student.

I tried to use my teacher discovery to gain some fame.I challenged **Tartaglia** to thirty questions, all related to the solution of $x^3 + ax = b$,



solution to ‘a cube and things equal to numbers’ $x^3 + ax = b$



Ferro

Taught

Antonio Maria Fiore

Luca Pacioli

My lack of money put some obstacles to my education. I could never learn Latin. But in Math…

My name is **Niccolo Fontana.** I was born in Italy, in 1499.My life was not an easy one…

My father was murdered by robbers in 1506.

In 1512 the French troops invaded my town and a soldier wounded me and left me for death.

My mother nursed me back to health, but I was left with an speech impediment that gave me the nickname “**Tartaglia**”, which means stutterer.

Later in life, a beard covered the scars of my injuries

RECALL: I am **Antonio Maria del Fiore**, del Ferro student. I tried to use my teacher discovery to gain some fame. I challenged **Tartaglia** to thirty

del Ferro challenged me

Eight days before the the public debate ended, I found the solutions of

$x^3+ax = b$ (cube and thing equal to number)

and **$x^3+b=ax$.**

I was able to solve all the questions of del Ferro, and he couldn’t solve any of mine.

General Treatise on Number and Measure (~1550)

I, **Gerolamo Cardano**, was born in Italy in 1501. My parents were not married, and this was a big deal when I was alive. I had an enormous amount of health problems that started with a complicated birth…

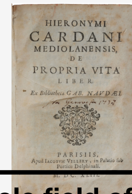
I studied Medicine at the University of Padua, and… guess how I occasionally supplemented my small income?

Gambling! I introduced the idea of probability and wrote a treatise about it

In fact, I wrote **412** books about medicine, popular science and astrology…

Also, I was jailed by heresy.

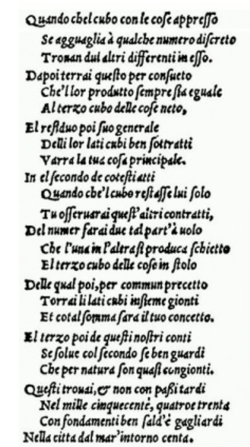
Girolamo Cardano, De Vita Propria (“The Book of My Life”)



- Written around 1573–1575, finished it shortly before his death in 1576.
- not published during his lifetime.
- First publication: 1643, in Amsterdam

In arithmetic I advanced almost the whole field of the science including the sections treating, as they call it, of algebra; my discoveries dealt with properties of numbers, especially of those having similar ratios among themselves. I also expounded the numerical functions already discovered, showing either a simplified treatment or some uncommon formula method, or both. In geometry I dealt with confused and reflex proportions, and the treatment of infinity with finite numbers and through finite, although it was first discovered by Archimedes.

After I won, and then, Cardano begged me for the solution and I finally sent it encoded in a poem:


Quando chel cubo con le coſe appreſſo
Se agguaglia à qualche numero diſcreto
Trouan dui altri differenti in eſſo.
Dapoi terrai queſto per conſueto
Che'l lor produtto ſempre ſia eguale
Al terzo cubo delle coſe neto,
El reſiduo poi ſuo generale
Delli lor lati cubi ben ſottratti
Varra la tua coſa principale.
In el ſecondo de coteſti atti
Quando che'l cubo reſtaſſe lui ſolo
Tu oſſeruarai queſt'altri contratti,
Del numer farai due tal part'à uolo
Che l'una in l'altra ſi produca ſchietto
El terzo cubo delle coſe in ſtolo
Delle qual poi, per commun precetto
Torrai li lati cubi inſieme gionti
Et cotal ſomma ſara il tuo concetto.
El terzo poi de queſti noſtri conti
Se ſolue col ſecondo ſe ben guardi
Che per natura ſon quaſi congionti.
Queſti trouai, & non con paſſi tardi
Nel mille cinquecentè, quatroe trenta
Con fondamenti ben ſald'è gagliardi
Nella citta dal mar'intorno centa.

When the cube and the things together
Are equal to some discrete number,
Find two other numbers differing in this one.

Then you will keep this as a habit
That their product should always be equal
Exactly to the cube of a third of the things.

The remainder then as a general rule
Of their cube roots subtracted
Will be equal to your principal thing.

In the second of these acts,
When the cube remains alone
You will observe these other agreements:

You will at once divide the number into two parts
So that the one times the other produces clearly
The cube of a third of the things exactly.

Then of these two parts, as a habitual rule,
You will take the cube roots added together,
And this sum will be your thought.

The third of these calculations of ours
Is solved with the second if you take good care,
As in their nature they are almost matched.

These things I found, and not with sluggish steps,
In the year one thousand five hundred, four and thirty
With foundations strong and sturdy

In the city girdled by the sea.

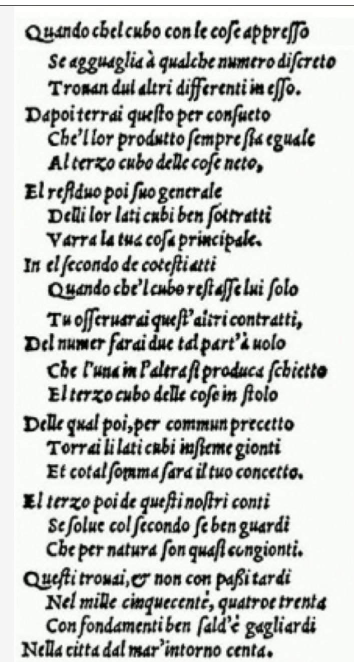
I am Ludovico Ferrara and arrived at Cardano's house when I was 14 years old, to work as a servant.

When Cardano learnt that I could read and write, he took me as his secretary. Soon we started collaborating in math work.

I discovered the solution of the quartic equation (with a quite beautiful argument I must say) which relied on the solution of cubic equations.

Because of this, it could not be published before the solution of the cubic had been published.

For more about this story, see, for instance, https://mathshistory.st-andrews.ac.uk/Biographies/Ferrari/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Tartaglia Solution to $x^3 + ax = b$ Cube and thing equal to number

First three verses of the poem

Denote the "two other numbers" (in poem) by u and v.
Using Tartaglia's hints, write as many equations as you can relating u, v, x, a, and b
It might be useful to set $U=u^{1/3}$ and $V=v^{1/3}$.

When the cube and the things together
Are equal to some discrete number,
Find two other numbers differing in this one.

Then you will keep this as a habit
That their product should always be equal
Exactly to the cube of a third of the things.

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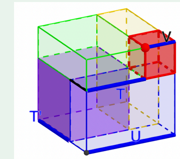
Tartaglia's Solution to $x^3 + ax = b$ Cube and thing equal to number



x^3+ax
When the cube and the things together
Are equal to some discrete number, b
Find two other numbers differing in this one.
u and v — u-v = b

Then you will keep this as a habit
That their product should always be equal
Exactly to the cube of a third of the things. $u.v = (a/3)^3$

$x^3+ax = b$
$u - v = b$

The remainder then as a general rule
Of their cube roots subtracted
Will be equal to your principal thing.

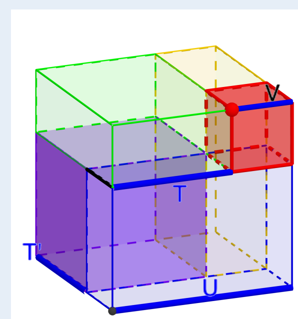
Set $U=u^{1/3}$, $V=v^{1/3}$,
Then $x=U-V$, and
- $UV = a/3$
- $U^3 - V^3 = b$

First three verses of the poem

How did Tartagilia find the solution?

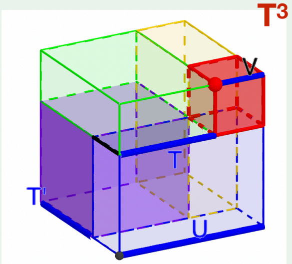
A conjecture.

The large purple cube has side length T = U - V. Express its volume of the purple cube (T^3) in in terms of the volumes of the colored pieces shown below.



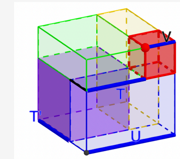
https://www.geogebra.org/m/byuvvryg

Write the volume of the purple cube (T^3) in terms of the volumes of the other solids in the figure below. (Hint: start with $T^3 = U^3 - \ldots$



$T^3 = (U-V)^3 = -3U \cdot V(U-V) + U^3 - V^3$

$x^3 + ax = b, a>0, b>0$



T=U-V

T^3 is the volume of the purple cube

When the cube and the things together (x^3+ax)
Are equal to some discrete number, (b)
Find two other numbers differing in this one. (u and v; $u-v = b$)

Then you will keep this as a habit
That their product should always be equal
Exactly to the cube of a third of the things. ($u \cdot v = (a/3)^3$)

The remainder then as a general rule
Of their cube roots subtracted
Will be equal to your principal thing.

$x^3 + ax = b$

$u - v = b$

Set $U=u^{1/3}$, $V=v^{1/3}$.

Then $x=U-V$, and

- $U \cdot V = a/3$
- $U^3 - V^3 = b$

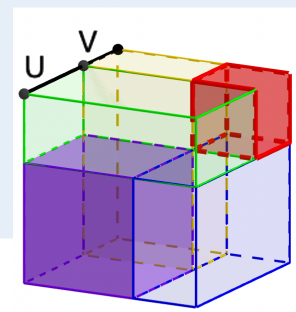
$T^3 = (U-V)^3 = -3UV(U-V) + U^3 - V^3$

$x^3 \quad = \quad -a. \ x \quad + \ b$

Verses 4, 5, 6

Write down the equations corresponding to the highlighted part of Tartaglia's poem.
Hint: See figure on the right and

$x^3 = ax + b, a>0, b>0$



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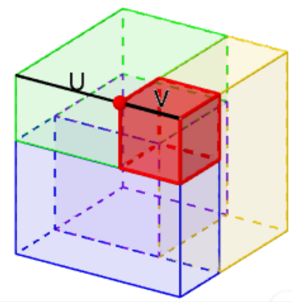
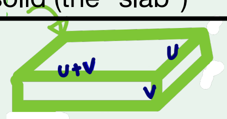


$(U + V)^3$ = volume of a cube
U^3 and V^3 = volumes of the two small cubes
$U \cdot V\,(U + V)$ = volume of the rectangular solid (the "slab")



Set $U = u^{1/3}$, $V = v^{1/3}$. Hence,

the root is U + V, so x = U + V

$U \cdot V = a/3$
$U^3 + V^3 = b$

$(U + V)^3 = U^3 + 3\,U^2 V + 3\,U V^2 + V^3$

$(U + V)^3 = 3\,U \cdot V\,(U + V) + (U^3 + V^3)$

$x^3 \quad = \quad a \cdot x \qquad\qquad + b.$

Beware of choices when computing roots

A solution of the depressed cubic

$$x^3 + p\,x + q = 0 \text{ is}$$

$$x = A + B,$$

where

$$A^3 = -\,q/2 + (\,(q/2)^2 + (p/3)^3\,)^{1/2}$$

$$B^3 = -\,q/2 - (\,(q/2)^2 + (p/3)^3\,)^{1/2}$$

and we choose the cube roots so that

$$A \cdot B = -\,p/3.$$

Find at least one solution of the equation

$x^3=15x+4$ cubus.aeq.15cos.p.4

(Hint: One of the solutions is a small positive integer)

Find at least one solution of the equation

$x^3=15x+4$ cubus.aeq.15cos.p.4

(Hint: One of the solutions is a small positive integer)

Solutions $x=4$, $-2+\sqrt{3}$, $-2-\sqrt{3}$

$x^3=15x+4$

Find a solution of the equation using Tartaglia's method

Tartaglia's solution is $x = U+V$ where

$$\begin{cases} U.V = a/3 \\ U^3+V^3 = b \end{cases}$$

$a=15$
$b=4$

$a=15$
$b=4$

$U.V = a/3$

$U^3+V^3=b$

Bombelli's solution $x^3=15x+4$

$$x = \sqrt[3]{\frac{b}{2}+\sqrt{\frac{b^2}{4}-\frac{a^3}{3^3}}} + \sqrt[3]{\frac{b}{2}-\sqrt{\frac{b^2}{4}-\frac{a^3}{3^3}}}$$

$$x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}.$$

$(2 + \sqrt{-121})^{1/3} = a + b\sqrt{-1}$

$(2 - \sqrt{-121})^{1/3} = a - b\sqrt{-1}$

Then

$2 + \sqrt{-121} = (a + b\sqrt{-1})^3$

$2 - \sqrt{-121} = (a - b\sqrt{-1})^3$

Solve $(a+bi)^3=2+11i$

real: $a^3 - 3ab^2 = 2$

imag: $3a^2b - b^3 = 11$.

Solve, get $a=2$, $b=1$

Plugin $a=2$, $b=1$

$x = 2+\sqrt{-1} + 2-\sqrt{-1} = 4$

Working with the cubic equations paved the way for the use of negative and complex numbers.

- When solutions were expressed in algebraic terms, **mathematicians noted that there was always the same *trick*** (the same algorithm).
- The **formulae** sometimes were so **complicated** that it was not clear it was giving the right answer($= \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$)

Contrast with Atiyah's statement "Algebra is like a Devil's gift.

From the solution of cubic equations to negative and complex numbers

Bombelli

- Italian engineer, wrote book about arithmetic that would be understandable for people who were not highly educated.
- In this book, he tries to explain how to solve the cubic.
- Arrives to square roots of negative numbers (even when he knows there is a solution).
- Bombelli says something like: **suppose these weird expressions are numbers. Even if the intermediate steps are strange, you manipulate these expressions you get to the answer at the end.**

Contrast with Atiyah's statement "Algebra is like a Devil's gift.

Why did Renaissance mathematicians eventually have to accept complex numbers, even though they initially called them "impossible" or "sophistic"?

From the solution of cubic equations to negative and complex numbers

> We distinguish two types of discoveries according to whether the person who makes the discovery knew what he was looking for and tried to find it, or whether he happened upon something - a physical object, a mathematical result, a medical procedure, for examples - and subsequently found a use for the discovery . The initial discovery of complex numbers falls into the latter category.
>
> - Jacques Hadamard

What does Hadamard's quote suggest about the role of 'unexpected discoveries' in mathematics?

Some remarks about the history of negative and complex numbers

Why do you think negative numbers didn't appear in ancient Greek mathematics?

Greece: Pythagoreans: **number** → "*a multitude of units*". One was not a number.

Greece: Aristotle and later Euclid's Elements discuss number and magnitude ("that which is divisible into divisibles that are infinitely divisible")..

500 — 400 — 300 — 100 — 0

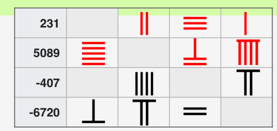
In Greece, magnitudes are associated with geometric objects, so they are always positive.

The Problem of Meaning

Negative numbers sneak in intermediate computations

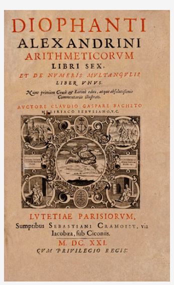
500 — 400 — 300 — 100 — 0

China: In The Nine Chapters on the Mathematical Art, negative numbers were used when solving systems of equations.



Source: Wikipedia

0 — 200



Diophantus' *Arithmetica(~200 CE)*:

- Stated Law of signs. Example
  - *"a number to be subtracted, multiplied by a number to be subtracted, gives a number to be added."*
- but did not accept negative solutions:
  - On a problem expressed now as **4 = 4x + 20** says, this is

    "***absurd****; for the 4 units must not be less than the 20 units*".
- Diophantus changes the initial hypotheses.

Negative numbers sneaked in intermediate computations

Middle East: al-Khwarizmi → negative numbers used in intermediate computations.

India: Bhaskara → negative roots for quadratic equations but…

The second value is in this case not to be taken, for it is ***inadequate****; people do not approve of negative roots*"

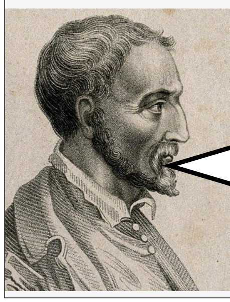
0 — 600 — 800 — 1100

600 C.E., India: Brahmagupta → rules for operations with negative numbers.

- A debt minus zero is a debt; an asset minus zero is an asset.
- Zero minus zero is zero.
- Zero minus a debt is an asset. Zero minus an asset is a debt.
- The product (quotient) of a debt and an asset is a debt, of two debts or of two assets is an asset.
- The product of zero with an asset, a debt or with zero is zero.

Negative numbers in intermediate computations

Cardano

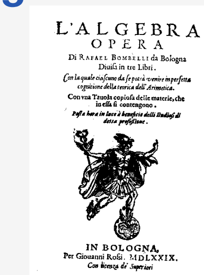


Note that √9 is either +3 or -3, for a plus [times a plus] or a minus times a minus yields a plus.

Therefore, √-9 is neither +3 or -3 but is some recondite third sort of thing.

Forced Acceptance Through Algebra

Bombelli (1572) Even if intermediate steps involve impossible numbers, manipulating these expressions gives the correct answer



Plus of minus times plus of minus makes minus $[+\sqrt{-n} \times +\sqrt{-n} = -n]$
Plus of minus times minus of minus makes plus $[+\sqrt{-n} \times -\sqrt{-n} = +n]$
Minus of minus times plus of minus makes plus $[-\sqrt{-n} \times +\sqrt{-n} = +n]$
Minus of minus times minus of minus makes minus $[-\sqrt{-n} \times -\sqrt{-n} = -n]$

Note: "plus of minus n" means +√-n

Revolutionary idea: Mathematics can work with entities that have no concrete meaning—IF the rules are consistent

Still Resistance

Chuquet → negative numbers referred to them as *"absurd numbers."*

1448

Cardano → recognized that some of his equations had negative solutions, which he called "*fictitious (for such we call that which is a debitum or negative)*". Described imaginary numbers as "*mental tortures*".

Bombelli → used **m.** to denote a negative number, and **p.** to denote a positive number. He gave rules to multiply complex numbers.

1545

Stifel refers to negative numbers as "absurd" or "fictitious below zero". However, he did accept them as coefficients, and exponents.

1550

Tycho Brahe, the astronomer, referred to negative numbers as "privative"

From Resistance to Understanding

1600-1700, even if not comfortable many mathematicians worked with negative and imaginary numbers in the theory of equations and in the development of the calculus.

- John Wallis **late 1600**s gave meaning to negative numbers by inventing the **number line.**
- **Late 1700s**: Gauss still doubtful about "true metaphysics of √-1
- **1799-1831**: Complex plane (Wessel, Argand, Gauss) → Imaginaries get geometric meaning (rotation) Gauss (1831): If we'd called them "direct, inverse, lateral" instead of "positive, negative, imaginary," there would be no mystery

Representation of the complex numbers → impact on acceptance.

Electricity and complex numbers

- **Early 1880s:** The first practical electrical systems used **direct current (DC)**, the type we still use in batteries.
- **Early–mid 1880s:** Edison promoted **DC** distribution in the United States.
- **Mid–late 1880s Alternating current (AC)** systems were developed; in AC the direction of current changes periodically. Because of this oscillation, **the mathematics of AC circuits is more involved than that of DC, but AC can be transmitted over long distances with far less loss.**
- **~1887–1893:** During the "War of Currents," Edison supported Harold P. Brown, an anti-**AC** activist seeking to portray **AC** as dangerous; Brown designed the first electric chair, which used **AC**.
- **From 1893** onward: Later, Charles Proteus **Steinmetz** (and independently A. E. **Kennelly**) showed that **AC** circuit analysis becomes much simpler when expressed using complex numbers.

Algebra and the solution of equations during the Renaissance and after.

A **solution in radicals** of a polynomial equation is an expression involving :

coefficients of the equation,

+,-, x, %

nth root extraction.

Joseph Luis Lagrange - Mid 1700's

Instead of looking directly for a method of solution of equations of the fifth degree,

made an **inventory of all the existing methods for solving equations of a degree 1, 2, 3, and 4,** trying to **understand** what the mathematical properties of these equations and what were the underlying **ideas in the methods that allowed their solution.**

Joseph Luis Lagrange - Early 1700's

$P(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$

coefficients a_0, a_1, a_2, a_3 ? roots x_1, x_2, x_3, x_4

Elementary symmetric functions

$\sigma_1 = x_1 + x_2 + x_3 + x_4 = -a_3$

$\sigma_2 = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_3x_4 = a_2$

$\sigma_3 = x_1x_2x_3 + x_1x_2x_4 + x_2x_3x_4 = -a_1$

$\sigma_4 = x_1x_2x_3x_4 = a_0$

4 unknowns (x_1, x_2, x_3, x_4)

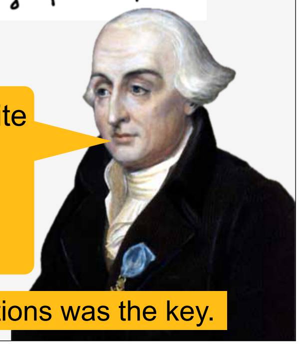
4 equations ....

but very special equations

All other symmetric functions can be written in terms of the elementary ones, (as polynomials with rational coefficients)

Example

$x_1^3 + x_2^3 + x_3^3 + x_4^3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$



What can I write in terms of the coefficients a_0, a_1, a_2, a_3?

Counting outcomes of symmetric functions was the key.

Algebra in the Renaissance

Two interconnected questions about equations

- what **algorithms** exist to find the solution(s), accurately and if possible automatically?
- how can these solutions can be **expressed**?

Solutions were found by

- **geometry**,
- **manipulation of signs and symbols** led to the development of new algorithms
- **solution of equations of the third and fourth degree using radicals** (that is, writing solutions in terms square roots, cubic roots, etc of functions of the coefficients.)

For a short period, it was thought that the equation of degree 5 would be solved soon… until

A **solution in radicals** of a polynomial equation is an expression involving :

coefficients of the equation,

+,-, x, %

nth root extraction.

Abel-Ruffini Theorem (1799-1824): There is no solution in radicals to the *general* polynomial equations of degree five or higher.

Galois theory (1830) gives a method of deciding, for any given equation, whether it is solvable in radicals.

Algebra in the Renaissance

It was proved that the **general equation of degree 5 cannot be solved it by radicals.**

This discovery resulted in a revolution in the conception of the nature of mathematical activity: **a change from mathematics being about numbers and quantities, to mathematics being about relations and structures.**

History

1.The method of false position.

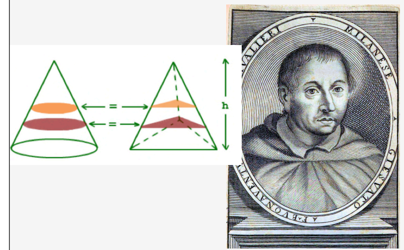
2.The birth of algebra

3.Methods using radicals

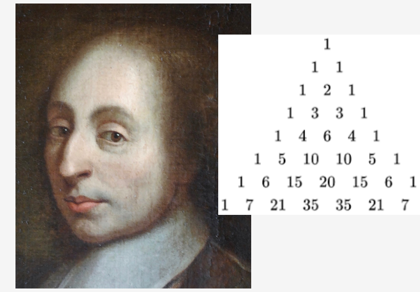
4.The emergence of the concept of a group.

Four remarkable Renaissance mathematicians

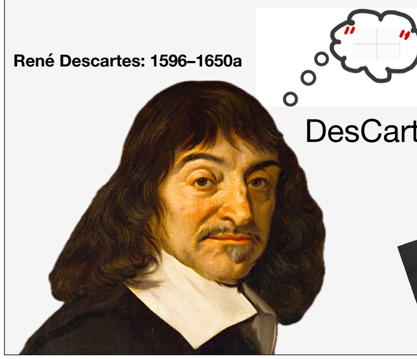
Bonaventura Cavalieri: 1598–1647



Blaise Pascal: 1623–1662

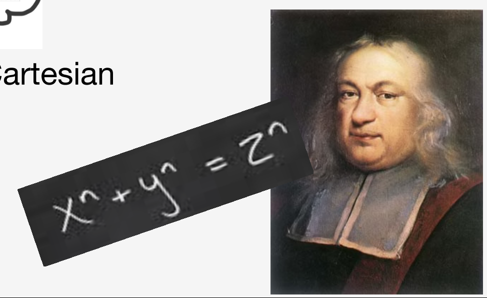


René Descartes: 1596–1650a



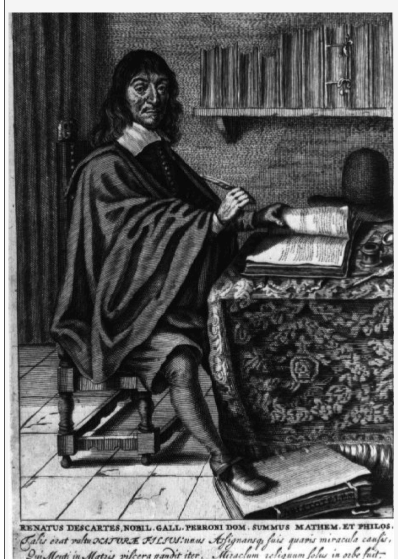
DesCartesian

Pierre de Fermat: 1607–1665



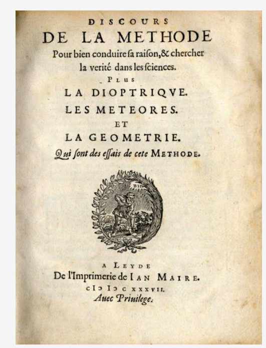
Descartes on Curves

Discourse on the Method of Rightly Conducting the Reason in the Search for Truth in the Sciences



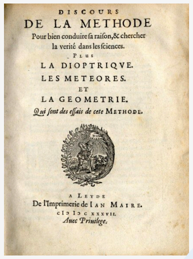
Appendix: The geometry

Discourse on Method



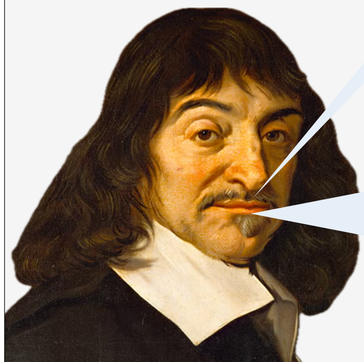
Rene Descartes ~1600

Discourse on Method


DISCOURS
DE LA METHODE
Pour bien conduire sa raison, & chercher
la verité dans les sciences.
Plus
LA DIOPTRIQVE.
LES METEORES.
ET
LA GEOMETRIE.
Qui sont des essais de cete METHODE.

A LEYDE
De l'Imprimerie de IAN MAIRE.
Avec Privilege.

Je pense, donc je suis
Cogito, ergo sum
I think, therefore I am



Rational inquiry could lead to knowledge in science. In other words, there is a **general method of thinking to facilitate inventions and find the truth in sciences.**

“The Geometry” is an example of how this should be done.

Descartes’ method method for seeking truth

The first rule was **never to accept anything as true unless I recognized it to be certainly and evidently such**: that is, carefully to avoid all precipitation and prejudgement, and to include nothing in my conclusions unless it presented itself so clearly and distinctly to my mind that there was no reason or occasion to doubt it.

The second was to **divide each of the difficulties** which I encountered into **as many parts as possible**, and as might be required for an easier solution.

The third was to **think in an orderly fashion** when concerned with the search for truth, b**eginning with the things which were simplest and easiest to understand**, and gradually and by degrees reaching toward more complex knowledge, even treating, as though ordered, materials which were not necessarily so.

The last was, both in the process of searching and in reviewing when in difficulty, always to **make enumerations so complete**, and reviews so general, that I would be certain that **nothing was omitted.**

Discuss one of Descartes’ rules for seeking knowledge. State whether you agree or disagree and explain why (you can choose more than one)

What we do NOT find in La Geometrie

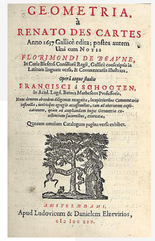
- Cartesian coordinates.
- Analytic geometry of the straight line, or of the circle, or of the conic sections.
- Curves plotted from its equation.

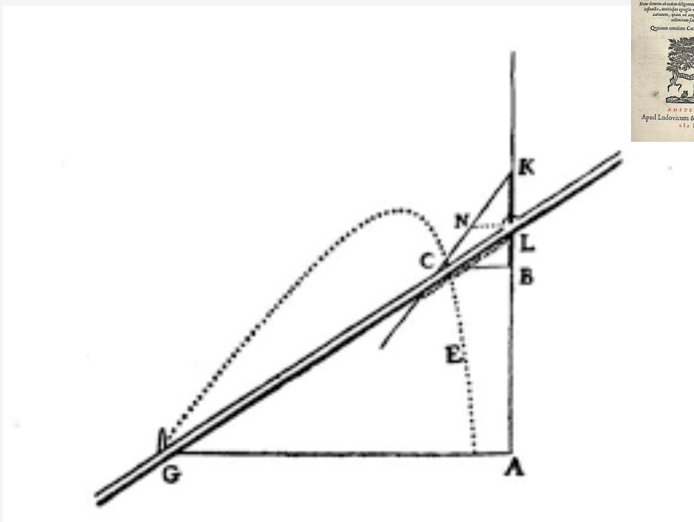
What we do find in La Geometrie

- Only **curves constructible by some mechanical device** that draws them according to specified rules.
- .The goal of solving problems.
- The specific purpose of the book is to answer questions like "What is the locus of a point such that a specified condition is satisfied?" And the answer to these questions must be **geometric**. Not "it is such-and-such a curve," or even "it has this equation," but "it is this curve, it has this equation, and it can be constructed in this way." Everything else in the Geometry-and that does include algebra, theory of equations, classifying curves by degree, etc.-are just means to this geometric end. **To solve a problem in geometry, one must be able to construct the curve that is its solution.**

From *Descartes and Problem-Solving* by Judith Grabiner

From La Géométrie -appendix to René Descartes' Discours de la méthode

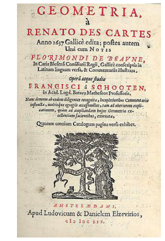




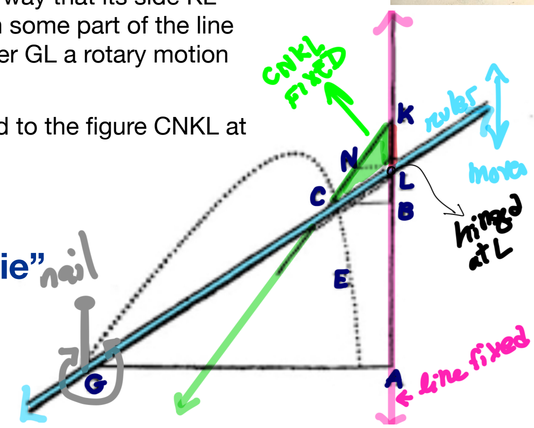
https://www.math.stonybrook.edu/~tony/archive/336s20/documents/from%20La%20Geometrie.pdf

"Suppose the curve EC to be described by the intersection of

- the ruler GL
- and the rectilinear plane figure CNKL whose side KN is extended indefinitely in the direction of C, and which, being moved in the same plane in such a way that its side KL always coincides with some part of the line BA, imparts to the ruler GL a rotary motion about G
- (the ruler being hinged to the figure CNKL at L).

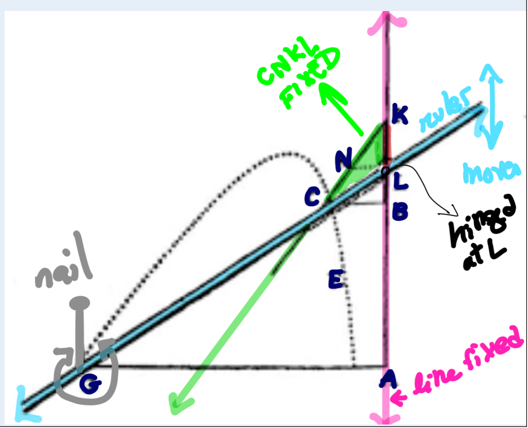


From "La Geometrie" by Descartes



Educated Guess: What is the curve described by Descartes? Give reason(s) for your answer.

https://www.geogebra.org/m/zwwbjdcx



- If I wish to find out to what class this curve belongs, I choose a straight line, as **AB**, to which to refer all its points, and in **AB** I choose a point like A at which to begin the investigation. [The class of the curve in independent of these choices].
- Then I take on the curve an arbitrary point, such as **C**, at which we will suppose the instrument applied to describe the curve.
- Then I draw through **C** the line **CB** parallel to **GA**.
- Since **CB** and **BA** are unknown and indeterminate quantities, I shall call one of them **y** and the other **x**.
- To the relation between these quantities I must consider also the known quantities which determine the description of the curve, as **GA**, which I shall call **a**; **KL**, which I shall call **b** and **NL** parallel to **GA**, which I shall call **C**.

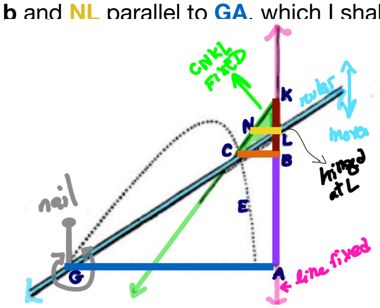
CB = y

BA = x

GA = a

KL = b

NL = c



Not exactly Cartesian

coordinates!

(even if Descartes is writing)

Then I say that as NL is to LK, or as c is to b, so CB, or y, is to BK, which is therefore equal to (b.y/c). Then BL is equal to (b.y/c)-b, and AL is equal to x+(b.y/c)-b.

c/b=y/BK. Then BK= b.y/c

BL = BK - KL = b.y/c - b

AL = x+(b.y/c)-b

CB = y

BA = x

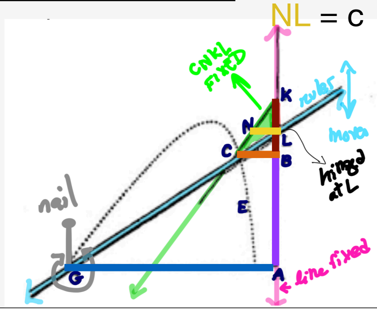
GA = a

KL = b

NL = c

Moreover, as CB is to LB, that is, as y is to (b.y/c)-b so AG or a is to LA or x+(b.y/c)-b

y/(b.y/c - b) = a/(x+(b.y/c)-b)



(Added parenthesis for clarity)

c/b=y/BK. Then BK= b.y/c

BL = BK - KL = b.y/c - b

AL = x+(b.y/c)-b

y/(b.y/c - b) = a/(x+(b.y/c)-b)

CB = y

BA = x

GA = a

KL = b

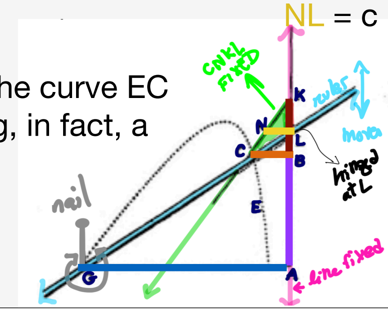
NL = c

Multiplying the second by the the third

we get (ab/c)y – ab equal to xy + (b/c) y^2 - by, which is obtained by multiplying the first by the last. Therefore the required equation is

y^2 = cy – (cx /b) y + ay – ac.

From this equation we see that the curve EC belongs to the first class, it being, in fact, a hyperbola."



(Added parenthesis for clarity)

$$y^2 + (a-c)y + \frac{c}{b}xy + ac = 0$$

A hyperbola is the set of point in the plane whose differences of distance to two given points (the foci) is constant.

$$A_{xx}x^2 + 2A_{xy}xy + A_{yy}y^2 + 2B_x x + 2B_y y + C = 0,$$

conic sections

$$D := \begin{vmatrix} A_{xx} & A_{xy} \\ A_{xy} & A_{yy} \end{vmatrix} < 0.$$

$$D = \det\begin{pmatrix} 0 & c/2b \\ c/2b & 1 \end{pmatrix} = -\left(\frac{c}{2b}\right)^2$$