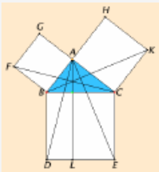
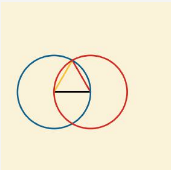
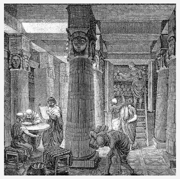


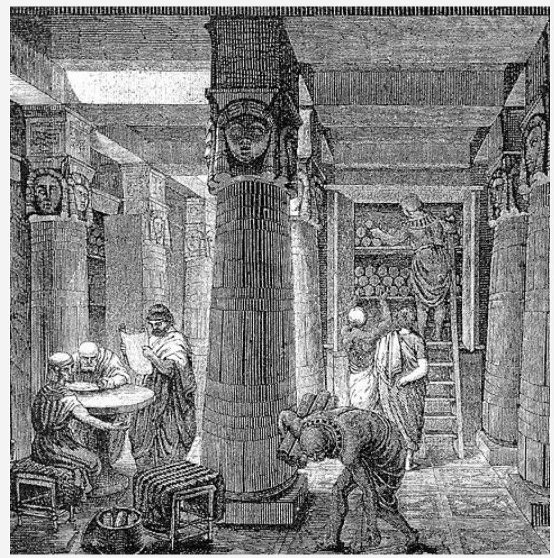
Euclid's Elements

Alexandria as context

Map of Ancient Hellenic Mathematics Encyclopedia Britannica

It is likely that Euclid worked and taught there

About 500,000 volumes



Nineteenth-century artistic rendering of the Library of Alexandria by the German artist O. Von Corven, based partially on the archaeological evidence available at that time

Euclid's Elements: What are they

Put these four steps of the axiomatic method in order from start to finish. (If some overlap or seem simultaneous, choose any order)

A.Use logical reasoning
B. Derive a new proposition
C. Establish definitions
D. Justify each step using axioms or previously proved propositions
E. Set up agreed principles (axioms)

Euclid's Elements

- Greek mathematical treatise
- Compiled around 300 BCE
- Attributed to Euclid
- Draws on centuries of accumulated mathematical work.
- Earliest extant work in **axiomatic method (**Propositions are arranged in a **logical progression**, where each step is justified by earlier results and axioms.)
- Its logical structure still underlies modern mathematics.

Covers
- **plane geometry** (shapes in the plane)
- **solid geometry** (shapes in space)
- **numbers and ratios** (divisibility, primes, proportions, irrationals)

Mathematics Is Built in The Elements: The Axiomatic Method

Axiomatic method:
- Starts with **definitions and agreed principles** (called **axioms** now**, postulates and common notions** in Euclid),
- New **propositions** are derived by **logical proof**;
- **Each step is justified** by
  - those principles or
  - by previously proved propositions.

Note: Nowadays, there are also **undefined terms.**

Euclid's Elements: Quick Overview

- **Definitions:** Terms like point, line, circle, angle, etc.
- **Postulates:** basic assumptions about geometry. (e.g., "*a straight line can be drawn between any two points*")
- **Common Notions**: intuitively obvious facts about magnitudes (e.g., "*things equal to the same thing are equal to each other*")
- **Propositions**:

  Theorems: In right-angled triangles, the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

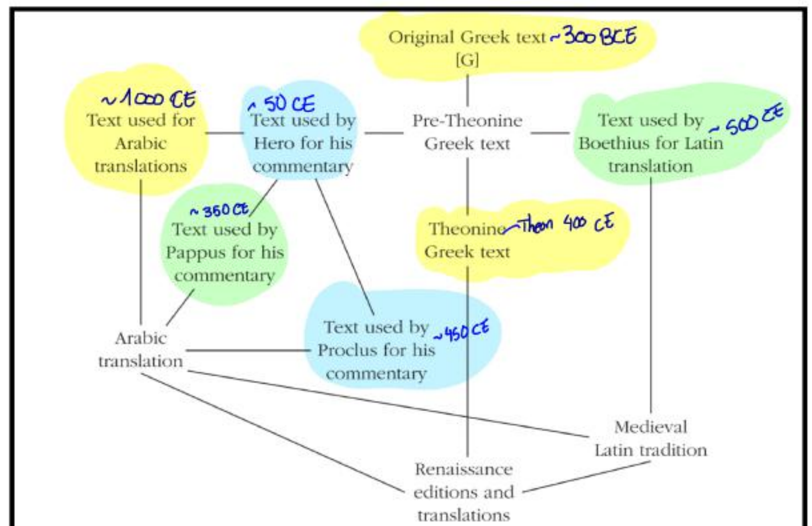
  Constructions: Construct an equilateral triangle, given a slide.

Axiomatic method: Starting from agreed principles (called **axioms**), new propositions are derived by logical proof; each step is justified by those principles or by previously proved propositions.

Euclid's elements	Today's math
Definitions	**Undefined** terms Definitions
Common notions Postulates **(self evident truths)**	Axioms **(assumptions wisely chosen)**
Propositions Constructions	Propositions Constructions

Axiomatic method: Starting from agreed principles (called **axioms**), new propositions are derived by logical proof; each step is justified by those principles or by previously proved propositions.

How the Elements reached us.
Impact and Variations



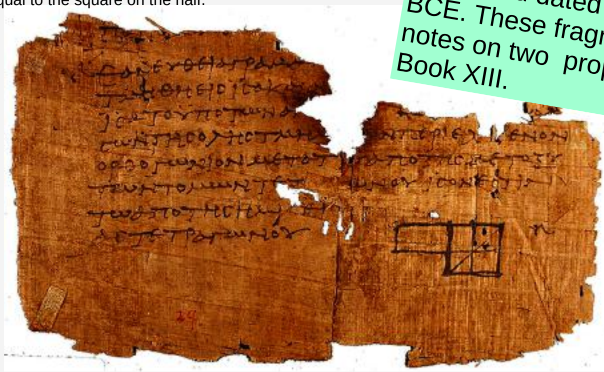
The medieval and Renaissance tradition of Euclid's *Elements* (adapted from Murdoch (1971))

Diagram from the book "Ancient Mathematics" by Serafina Cuomo

One of the Oldest Fragment of Euclid's Elements, dated from 1st century CE,

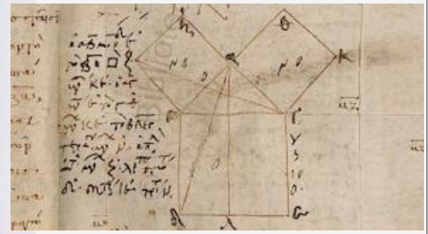
Proposition II.5: If a straight line is cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole, together with the square on the straight line between the points of the section, is equal to the square on the half.

Also there are fragments found in potsherds discovered in Egypt and dated from 225 BCE. These fragments contain notes on two propositions from Book XIII.



http://www.math.ubc.ca/~cass/Euclid/papyrus/

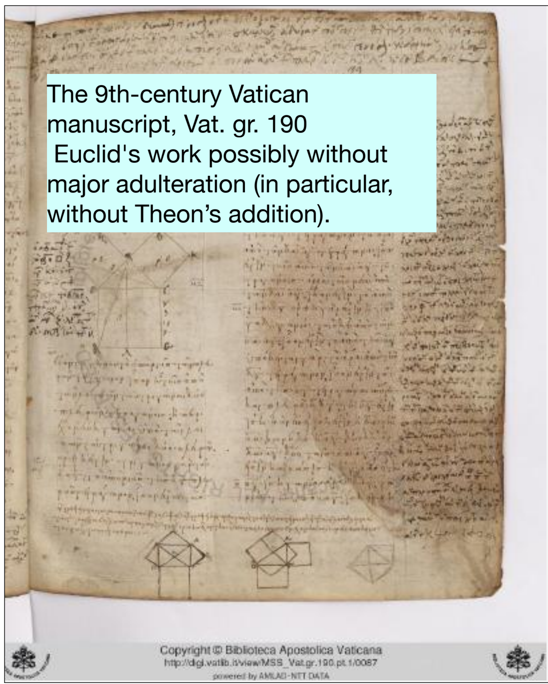
The 9th-century Vatican manuscript, Vat. gr. 190 Euclid's work possibly without major adulteration (in particular, without Theon's note).



Book I proposition 47

The Pythagorean Theorem

https://digi.vatlib.it/view/MSS_Vat.gr.190.pt.1/0087



The 9th-century Vatican manuscript, Vat. gr. 190 Euclid's work possibly without major adulteration (in particular, without Theon's addition).

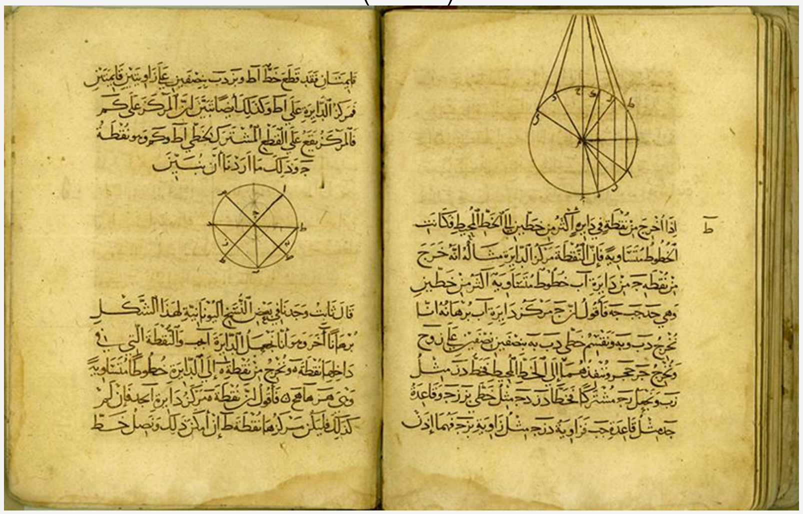
Heath (A History of Greek Mathematics (2 Vols.) (Oxford, 1921).) writes ".. *Thus Theon himself says that he edited the Elements and also that the second part of VI. 33, found in nearly all the MSS., is his addition....*"

This illustrates the many modifications the Elements have endured through the centuries.

https://digi.vatlib.it/view/MSS_Vat.gr.190.pt.1/0087

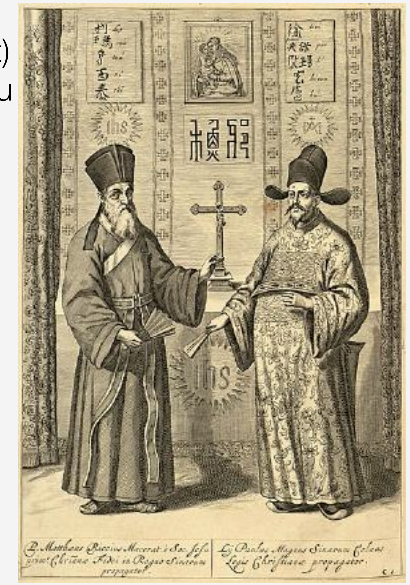
An illumination from a manuscript based on Adelard of Bath's translation of the *Elements*, *circa* 1309–1316; Adelard's is the oldest surviving translation of the *Elements* into Latin, done in the 12th-century work and translated from Arabic. (Wikipedia)

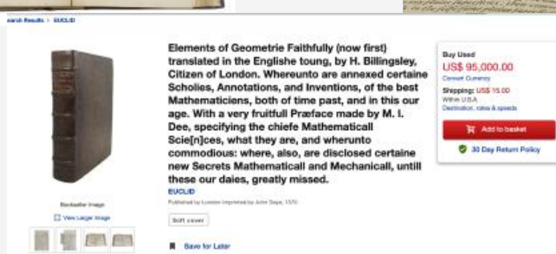
Early Translation of Euclid's Elements into Arabic (1466)

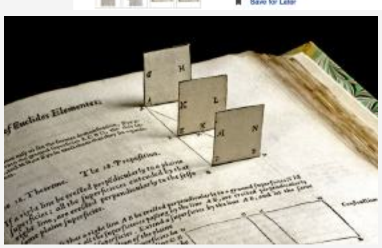


https://www.maa.org/press/periodicals/convergence/mathematical-treasure-early-translation-of-euclids-elements-into-arabic

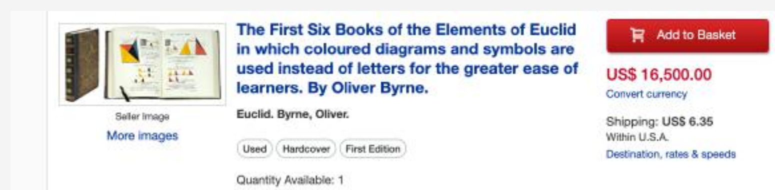
The Italian Jesuit Matteo Ricci (left) and the Chinese mathematician Xu Guangqi (right) published the Chinese edition of Euclid's Elements (幾何原本) in 1607. (Wikipedia)



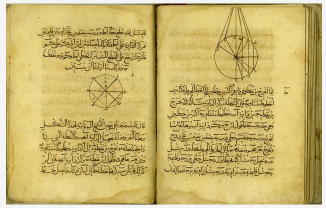


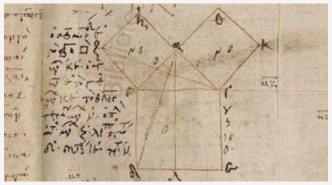


https://www.c82.net/euclid/



Euclid's element's





The 9th-century Vatican manuscript, Vat. gr. 190 Euclid's work possibly without major adulteration (in particular, without Theon's note).

were not created overnight

were not created by Euclid alone.

Definitions in the Elements

Group activity: Order these four Euclidean definitions from easiest to hardest to visualize and understand.
Then, explain the reasoning behind your order.
Hint: If you didn't already know the terms being defined, could you understand what the object is from the definition alone? Could you draw it?

Definition 1: A **point** is something that has no parts.
Definition 2: A **line** is length with no width.

Definition 10: When one straight line meets another and makes the two adjacent angles equal, each angle is called a **right** angle, and the two lines are said to be **perpendicular** to each other.

Definition 15: A **circle** is a plane figure whose boundary is a single [*curved*] line, such that all straight lines drawn from one interior point to the boundary are equal.

Definition 1.
A ***point*** is that which has no part.

Definition 2.
A ***line*** is breathless length.

Definition 3.
The ***ends*** of a line are points.

Definition 4.
A ***straight line*** is a line which lies evenly with the points on itself.

First four definitions of Book 1 of Euclid's Elements.

Axioms, postulates and common notions in the Elements

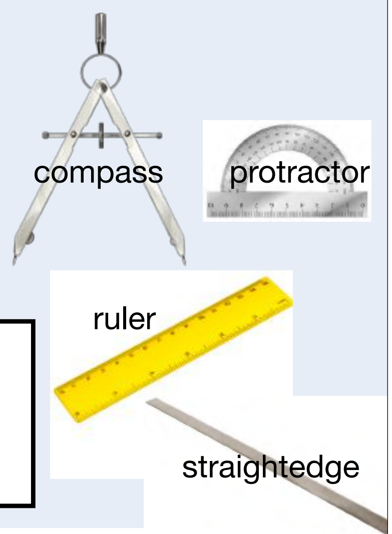
Common notions

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part

The Greek text of J.L. Heiberg translated by Richard Fitzpatrick https://farside.ph.utexas.edu/Books/Euclid/Elements.pdf

The first three postulates in the Elements

- Postulate I asserts that it is possible to draw a segment through any two given points.
- Postulate II says that any segment can be extended indefinitely in the same straight line.
- Postulate III states that it is possible to construct a circle with any given center and radius.



These first three postulates are associated with the tools that are used to implement them on a piece of paper.

Which tool is each of these postulates associated with? (Select the option with only the essential features)

Postulate 4 states
"*All right angles are equal one another.*"
This seems self-evident. Why does Euclid need to write it down as a postulate?

Definition 10: When one straight line meets another and makes the two adjacent angles equal, each of those angles is called a **right** angle.

Euclid’s five postulates

1.A straight line can be drawn from any point to any other point.

2. A finite straight line can be extended indefinitely in the same straight line.

3. A circle can be drawn with any center and any radius.

4. All right angles are equal to one another.

5. If a straight line crosses two straight lines and the interior angles on the same side add up to less than two right angles, then the two lines, if extended far enough, will meet on that side.

The first four postulates were straightforward; they were considered "truths". (Recall Aristotle's definition: "An axiom is a statement worthy of acceptance.")

The fifth postulate isn't simple or straightforward; it *feels* more like a proposition or theorem than an axiom.

Postulates

1.A straight line can be drawn from any point to any other point.

2. A finite straight line can be extended indefinitely in the same straight line.

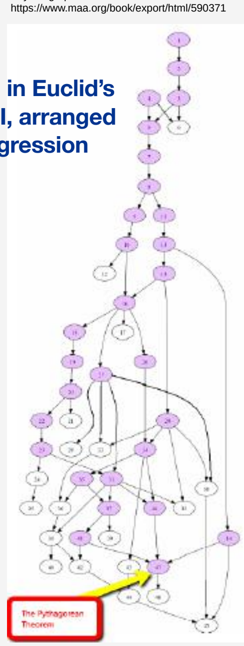
3. A circle can be drawn with any center and any radius.

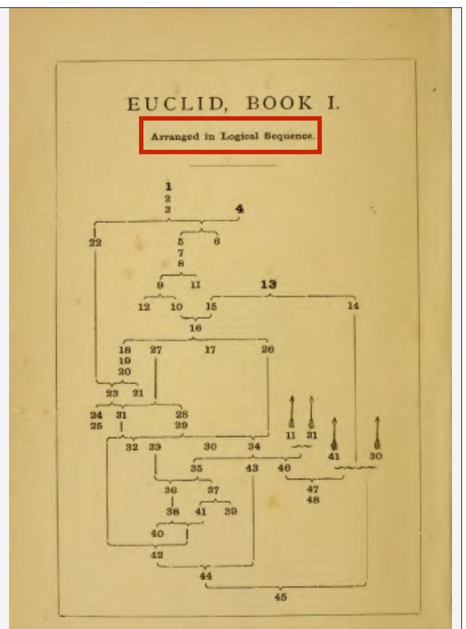
4. All right angles are equal to one another.

5. If a straight line crosses two straight lines and the interior angles on the same side add up to less than two right angles, then the two lines, if extended far enough, will meet on that side.

. Eyer's graph of Euclid's Book I
https://www.maa.org/book/export/html/590371

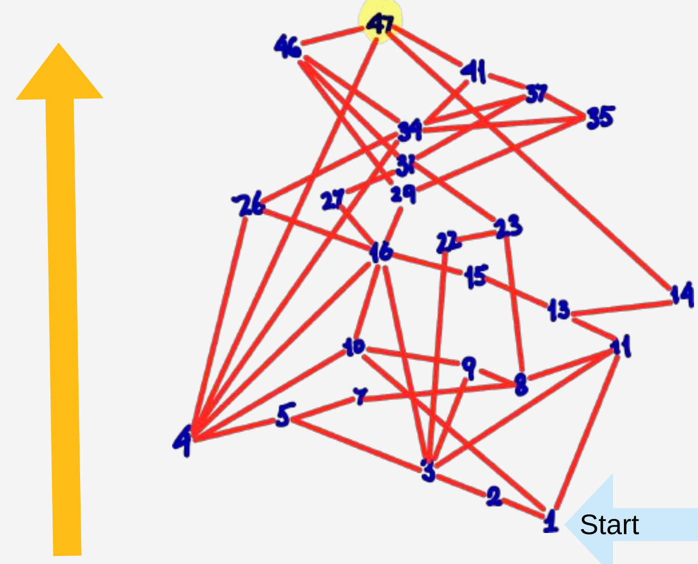
The proposition in Euclid’s Elements Book I, arranged in logical progression

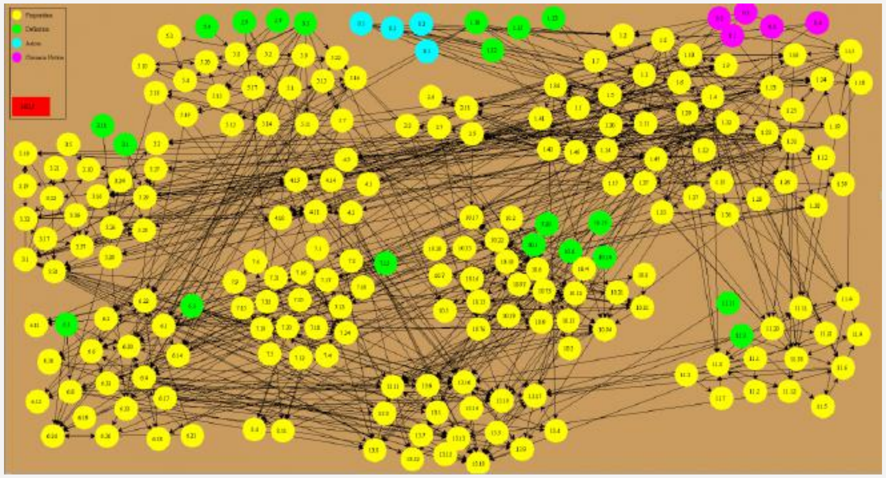




From Euclid and His Modern Rivals By Lewis Carroll

The structure of the proportions in Euclid’s Elements leading to the Pythagorean Theorem (Proposition 47)



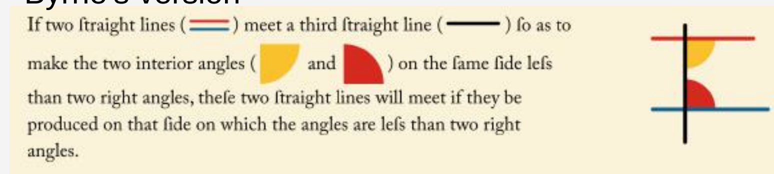


A graph of all 13 books of Euclid's *Elements*

https://www.maa.org/book/export/html/590371

Postulate 5'. (Playfair)
For any given point not on a given line, there is exactly one line through the point that does not meet the given line.

Byrne's version



Postulate 5.
If a straight line crosses two straight lines and the interior angles on the same side add up to less than two right angles, then the two lines, if extended far enough, will meet on that side.

Proposition I.1 from Euclid's Elemen ts: Construction of an equilateral triangle

An equilateral triangle is a triangle that… (complete the sentence, you have one minute)

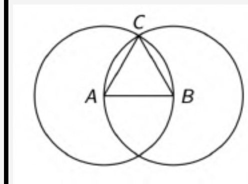
An equilateral triangle is a triangle that has three equal sides.

Proposition 1: Given a segment, it is possible to construct an equilateral triangle that has this segment as one of its sides.

Construction (Part 1 of the proof):

- Draw a circle with center A and distance AB.(Post.3)
- Draw a circle with center B and distance AB.(Post.3)
- Let C be a point where the circles intersect.
- Draw segments AC and BC. (Post.1)

- ABC is the required triangle.



1. Post 1. A straight line can be drawn from any point to any other point.

3. Post. 3: A circle can be drawn with any center and any radius.

Proposition 1 of Euclid's Elements. Construction of equilateral triangle

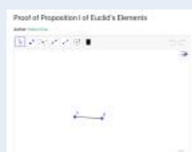
Join the lesson at www.geogebra.org/classroom with the code:

K8SG CX32

Or you can also share the following link with your students:

www.geogebra.org/classroom/k8sgcx32

https://www.geogebra.org/m/nessxbt3



1. **Construction**: In the Geogebra window, construct an equilateral triangle.
2. **Now explain in Wooclap: Why does this construction guarantee an equilateral triangle? (Hint: recall the definition of equilateral triangle)** (Use complete sentences)
3. **Optional**: Justify each step of your explanation using the Postulates and Common Notions below.
4. **Super Optional:** Compare the instructions given in 1. with this one , which is a translation of the original Euclid's Element. What are the differences?)

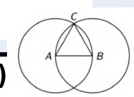
Why does this construction guarantee an equilateral triangle? Justify using the appropriate definitions, postulates and common notions. (Hint: recall the definition of equilateral triangle)

Proposition 1: Given a segment, it is possible to construct an equilateral triangle that has this segment as one of its sides.

Construction (Part 1 of the proof)

- Draw a circle with center A and distance AB.(Post.3)
- Draw a circle with center B and distance AB.(Post.3)
- Let C be a point where the circles intersect.
- Draw segments AC and BC. (Post.1)

- ABC is the required triangle.



Post 1. A straight line can be drawn from any point to any other point.

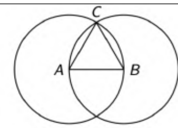
Post.3: A circle can be drawn with any center and any radius.

Definition 15. A circle is a plane figure whose boundary is a single [curved] line, such that all straight lines drawn from one interior point to the boundary are equal.

Definition 20: If all three sides are of the same length, then a triangle is **equilateral.**

Common notion 1: Things that are equal to the same thing are also equal to one another.

Proposition 1: Given a segment, it is possible to construct an equilateral triangle that has this segment as one of its sides.



a. AC = AB, since both are radii of the circle centered at A (Definition 15).
b. BC = AB, since both are radii of the circle centered at B (Definition 15).
c. AC = BC, since both equal AB (steps a, b, and Common notion 1).
d. By a, b and c and Definition 20 ABC is an equilateral triangle with AB as one of its sides.

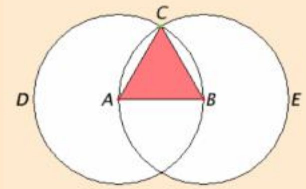
Definition 15: A circle is a plane figure whose boundary is a single [curved] line, such that all straight lines drawn from one interior point to the boundary are equal.

Definition 20: If all three sides are of the same length, then a triangle is **equilateral.**

Common notion 1: Things that are equal to the same thing are also equal to one another.

Proposition 1: Given a segment, it is possible to construct an equilateral triangle that has this segment as one of its sides.

Let *AB* be the given finite straight line.



It is required to construct an equilateral triangle on the straight line *AB*.

Describe the circle *BCD* with center *A* and radius *AB*. Again describe the circle *ACE* with center *B* and radius *BA*. Join the straight lines *CA* and *CB* from the point *C* at which the circles cut one another to the points *A* and *B*. I.Post.3 I.Post.1

Now, since the point *A* is the center of the circle *CDB*, therefore *AC* equals *AB*. Again, since the point *B* is the center of the circle *CAE*, therefore *BC* equals *BA*. I.Def.15

But *AC* was proved equal to *AB*, therefore each of the straight lines *AC* and *BC* equals *AB*.

And things which equal the same thing also equal one another, therefore *AC* also equals *BC*. C.N.1

Therefore the three straight lines *AC*, *AB*, and *BC* equal one another.

Therefore the triangle *ABC* is equilateral, and it has been constructed on the given finite straight line *AB*. I.Def.20

Q.E.F

Text and figure by David Joyce https://mathcs.clarku.edu/~djoyce/elements/bookI/propI1.html

Postulates

1. Let it have been postulated† to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).‡

Common Notions

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.†
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part.

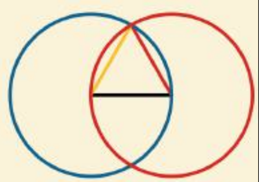
15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.

20. And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having three unequal sides.

Proposition 1: Given any finite segment, it is possible to construct an equilateral triangle having that segment as a side.

PROPOSITION I. PROBLEM.

O*N a given finite ſtraight line* ( ) *to deſcribe an equilateral triangle.*



Deſcribe and (poſtulate 3.); draw and (poſt. 1.). then will be equilateral.

For = (def. 15.);
and = (def. 15.),
∴ = (axiom. 1.);

and therefore is the equilateral triangle required.

Q. E. D. Byrne's Euclid

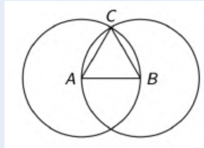
Is every step appropriately justified in Proposition I.1

Proposition 1: Given a segment, it is possible to construct an equilateral triangle that has this segment as one of its sides.

- Denote the segment by AB.
- Draw a circle with center A and distance AB.(Post.3)
- Draw a circle with center B and distance AB.(Post.3)
- Let C be a point where the circles intersect.
- Draw segments AC and BC. (Post 1)

- Claim: ABC is the required triangle.

a. AC = AB, since both are radii of the circle centered at A (Definition 15).
b. BC = AB, since both are radii of the circle centered at B (Definition 15).
c. AC = BC, since both equal AB (steps a, b, and Common notion 1).
d. By a, b and c and Definition 20 ABC is an equilateral triangle with AB as one of its sides.



Is every step appropriately justified?

Post 1.A straight line can be drawn from any point to any other point.

Post.3: A circle can be drawn with any center and any radius.

Common notion 1: Things that are equal to the same thing are also equal to one another.

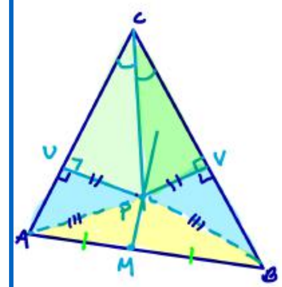
Definition 15. A circle is a plane figure whose boundary is a single [curved] line, such that all straight lines drawn from one interior point to the boundary are equal.

Definition 20: If all three sides are of the same length, then a triangle is **equilateral.**

- Let **M** be the midpoint of **AB**
- Let **P** be the intersection of the perpendicular bisector of **AB** and the angle bisector at **C**
- Let **U** be the foot of the perpendicular from **P** to line **AC**
- Let **V** be the foot of the perpendicular from **P** to line **BC**

Claim: The following triangles are congruent.

1. △**AMP** ≅ △**BMP**
   - AM=BM (M is midpoint of AB), MP=MP (common side), angles AMP and BMP are right angles. (SAS)
2. △**PCU** ≅ △**PCV**
   - PC=PC (common side), angle PCU = angle PCV (P lies on the angle bisector at C), angles PUC and PVC are right angles. (AAS)
3. △**APU** ≅ △**BPV**
   - AP=BP (from 1.), PU=PV (from 2.), angles PUA and PVB are right angles. (HL)



Find a statement proven by this argument

Are all triangles isosceles?

Lect 3

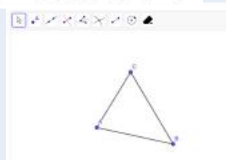
Join the lesson at www.geogebra.org/classroom with the code:

N3T4 FF7S

Or you can also share the following link with your students:

www.geogebra.org/classroom/n3t4ff7s

https://www.geogebra.org/m/uggnvm6x



Follow the instructions in the app. Answer in Wooclap: Based on your GeoGebra construction, what is the flaw in this 'proof' that all triangles are isosceles?

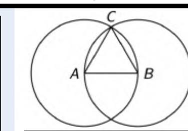
What can you conclude from the "all triangles are isosceles" fallacy?

Proposition 1: Given a segment, it is possible to construct an equilateral triangle that has this segment as one of its sides.

- Denote the segment by AB.
- Draw a circle with center A and distance AB.(Post.3)
- Draw a circle with center B and distance AB.(Post.3)
- Let C be a point where the circles intersect.
- Draw segments AC and BC. (Post 1)

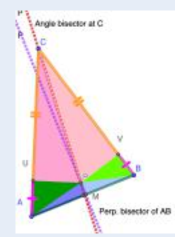
- Claim: ABC is the required triangle.

a. AC = AB, since both are radii of the circle centered at A (Definition 15).
b. BC = AB, since both are radii of the circle centered at B (Definition 15).
c. AC = BC, since both equal AB (steps a, b, and Common notion 1).
d. By a, b and c and Definition 20 ABC is an equilateral triangle with AB as one of its sides.



Post 1.A straight line can be drawn from any point to any other point.

Post.3: A circle can be drawn with any center and any radius.



Definition 15. A circle is a plane figure whose boundary is a single [curved] line, such that all straight lines drawn from one interior point to the boundary are equal.

Definition 20: If all three sides are of the same length, then a triangle is **equilateral.**

Common notion 1: Things that are equal to the same thing are also equal to one another.

Proposition 1 of Euclid's Elements. Construction of equilateral triangle

Bad news: There is a gap in the proof of Proposition I.1:
Euclid did not proved that the circles must intersect (remember that one can not "*prove by drawing*")

How do we deal with this issue nowadays:
Following **Hilbert**, modern geometry adds a **Completeness Axiom**, similar to an axiom you may know from Analysis.

Axiom of Completeness (for a line) which states.
If all points of a straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division.

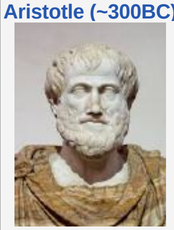
1. The first widely used Latin translation of Euclid's Elements reached medieval Europe through Arabic: Greek → Arabic (9th century) → Latin (12th century).
What does this suggest about how mathematical knowledge was transmitted across cultures?
2. Euclid's Elements remained one of the most widely used mathematics textbooks for nearly 2000 years. In your opinion, what features of the Elements might explain its extraordinary longevity? what features might explain its influence?
3. Why might Euclid — or later mathematicians — have found Postulate 5 suspicious? What propelled them to try to prove it?

Number theory in Euclid

Recall

- There are two distinct types of "quantities":
  - the continuous (magnitude)
    - *"A **magnitude** is that what which is divisible into divisible that are infinitely divisible."*
    - Example: lines, surfaces, bodies and time.
  - the discrete (number)
    - *A "**number**" quantities that is composed of distinct, separate units or parts that can be counted. (Examples 1, 2, 3, ...)*

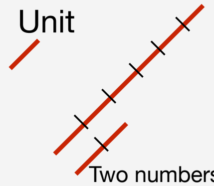
Aristotle (~300BC)



Book VII. Some Definitions

- A ***unit*** is (that) according to which each existing (thing) is said (to be) one.
- A ***number*** is a multitude composed of units.
- A number is ***a part of a number***, the less of the greater, when **it measures the greater;**
- A ***prime number*** is that which is measured by a unit alone.
- Numbers ***relatively prime*** are those which are measured by a unit alone as a common measure.
- A ***composite number*** is that which is measured by some number.

Unit



Two numbers, 2 and 6.

2 measures 6, then 2 is a part of 6

Proposition IX.20: The prime numbers are more than any collection of prime numbers.

Which statement best captures Euclid's meaning? Choose the statement that matches Euclid's exact reasoning, not just the modern summary. Explain your choice.

a) There are infinitely many primes
b) For any finite list of primes, there exists a prime not on that list
c) No finite collection contains all primes
d) There are more primes than composites.

Euclid's Elements

Proposition IX.20: The prime numbers are more than any collection of prime numbers.

What "collection" meant to Euclid
Collection here means a finite group of numbers — Greek mathematicians did not conceive of completed infinite collections.

Euclid never uses the word "infinite"

Two ways of thinking infinity

*The infinite has a **potential** existence but not an **actual** one. It exists in the sense that one thing after another can be taken without end.* Aristotle

Potential infinity: The property that there is always something beyond any given quantity.

Actual infinity: The idea that an infinite collection exists as a completed whole.

Euclid never uses the word "infinite"

Consider the primes A=7, B=11 and C=13.

Define E =A.B.C+1.

G be the smallest prime that divides E.

What is G?

Proposition IX.20: The prime numbers are more than any collection of prime numbers.

(Modern) Proof: Assume that p_1, p_2, and p_3 are prime.

Suppose that the primes A, B, C are 2, 3, and 5. Let N be as above and let G be the largest prime that divides N. What is G?

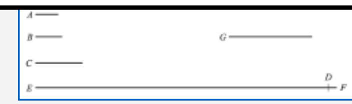
Euclid's Elements

This proposition is not used in the rest of the Elements.

Table 3

P1	P2	P3	P1.P2.P3+1	FACTORS	FACTORS	FACTORS	FACTORS	FACTORS
2	3	5	31	31				
3	5	7	106	2	53			
5	7	11	386	2	193			
7	11	13	1002	2	3	167		
11	13	17	2432	2	19			
13	17	19	4200	2	3	5	7	
17	19	23	7430	2	5	743		
19	23	29	12674	2	6337			
23	29	31	20678	2	7	211		
29	31	37	33264	2	3	7	11	
31	37	41	47028	2	3	3919		
37	41	43	65232	2	3	151		
41	43	47	82862	2	13	3187		
43	47	53	107114	2	7	1093		
47	53	59	146970	2	3	5	23	71
53	59	61	190748	2	43	1109		

Proposition IX.20: The prime numbers are more than any collection of prime numbers.



- Let *A, B,* and *C* be the assigned prime numbers.
- I say that there are more prime numbers than *A, B,* and *C.*
- Take the least number *DE* measured by *A, B,* and *C.* Add the unit *DF* to *DE.*
- Then *EF* is either prime or not.
- Let it, first of all, be prime. Thus, the (set of) prime numbers A, B, C, EF, (which is) more numerous than A, B, C, has been found.
- And so let EF not be prime.
- Thus, it is measured by some prime number [Prop. VII.31]. Let it be measured by the prime (number) G.
- I say that G is not the same as any of A, B, C. For, if possible, let it be (the same).
- And A, B, C (all) measure DE.
- Thus, G will also measure DE.
- And it also measures EF.
- (So) G will also measure the remainder, unit DF, (despite) being a number [Prop. 7.28]. The very thing (is) absurd. Thus, G is not the same as one of A, B, C.
- And it was assumed (to be) prime. Thus, the (set of) prime numbers A, B, C, G, (which is) more numerous than the assigned multitude (of prime numbers), A, B, C, has been found. (Which is) the very thing it was required to show.

Euclid's Elements

1. The first widely used Latin translation of Euclid's Elements reached medieval Europe through Arabic: Greek → Arabic (9th century) → Latin (12th century).
What does this suggest about how mathematical knowledge was transmitted across cultures?
2. Euclid's Elements remained one of the most widely used mathematics textbooks for nearly 2000 years. In your opinion, what features of the Elements might explain its extraordinary longevity? what features might explain its influence?
3. Why might Euclid — or later mathematicians — have found Postulate 5 suspicious? What propelled them to try to prove it?

1. Choose one (or more):
a) What surprised you most about how mathematics is organized in Euclid's Elements?
b) What is still confusing?
c) Name one idea from today that changes how you think about proofs or the Elements.

2. What is an axiomatic system?
Hint: chosen starting points (axioms), definitions, and proofs, logic.

3. Euclid never says "there are infinitely many primes." Instead, he writes that "the primes are more than any collection of primes." What does this wording tell you about how Euclid (and Greek mathematicians) thought about infinity?

4 Choose a or b (or both):
a) The "all triangles are isosceles" fallacy used a misleading diagram. What does that teach us about using figures in proofs?
b) Is 'all right angles are equal' a definition, postulate, or proposition? Why?

5 (Optional) Euclid vs. today:
Euclid treated postulates as self-evident truths; modern math treats axioms as chosen assumptions. What difference does this make for how we do mathematics?

Proof of Pythagorean Theorem

Persi Diaconis

- Ran away at 14 to be a professional magician.
- *"I thought I could do anything. ... So I bought Feller and I thought, "Well, I'll just read this book." And I couldn't read it. I didn't know calculus, or at least not enough."*
- BS at CUNY → PhD Harvard → Professor of Mathematics & Statistics, Stanford
- Studies randomness.
- Famous result: ~7 riffle shuffles randomize a deck.
- Tested and debunked psychics.
- Style: simple questions → deep mathematics.

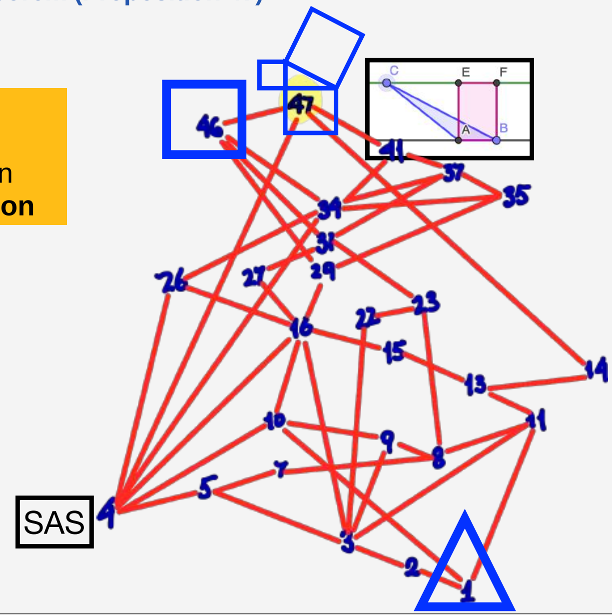
Today at 4:30 in SCGP-Della Pietra Auditorium - Understanding Coincidences

Tomorrow 10am: The Search for Randomness

15	Mar 24	Gauss on hyperbolic geometry Adam	History of the parallel postulate (see also here) Rahman
16	Mar 26	The Pentagram and the Discovery of Incommensurability Jibran	Gauss's First Argument for Least Squares Ryan

The structure of the proportions in Euclid's Elements leading to the Pythagorean Theorem (Proposition 47)

The proposition in Euclid's Elements Book I, arranged in **logical progression**

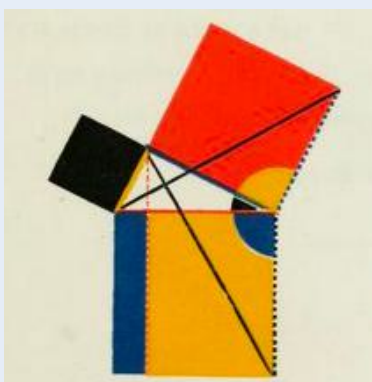


As you follow the next proof, pay attention to two key ideas:

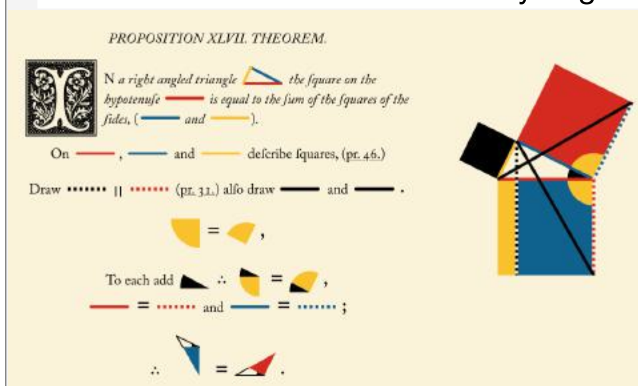
- How does the area of a triangle relate to the area of a rectangle?
- How are congruent triangles used to explain why certain areas are equal?

Try to connect what we saw in the GeoGebra activity to the steps in the proof. You'll be asked to explain this connection afterward.

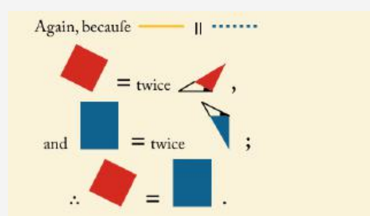
Later

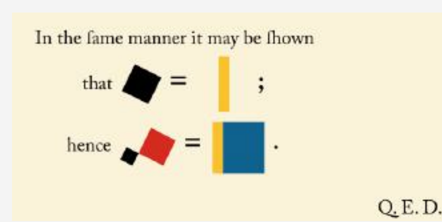


Euclid's Elements Proof of the Pythagorean Theorem

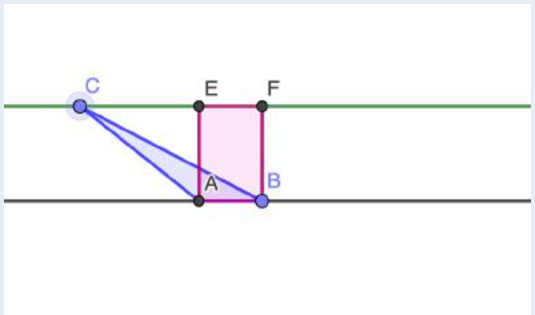


Byrne's Euclid





What is the ratio of the areas of triangle ABC and rectangle EFBA?
Does this ratio change when you drag point C along the green line? Explain why or why not.



One Step in the Proof of Pythagoras' Theorem

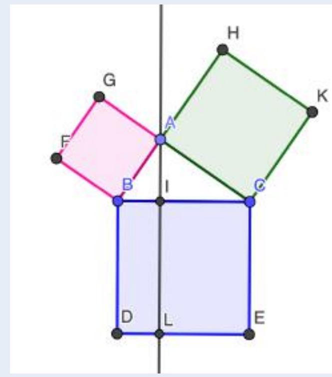
https://www.geogebra.org/m/xyfcawtm

Join the lesson at www.geogebra.org/classroom with the code:

MYYY ZAGA

Or you can also share the following link with your students:

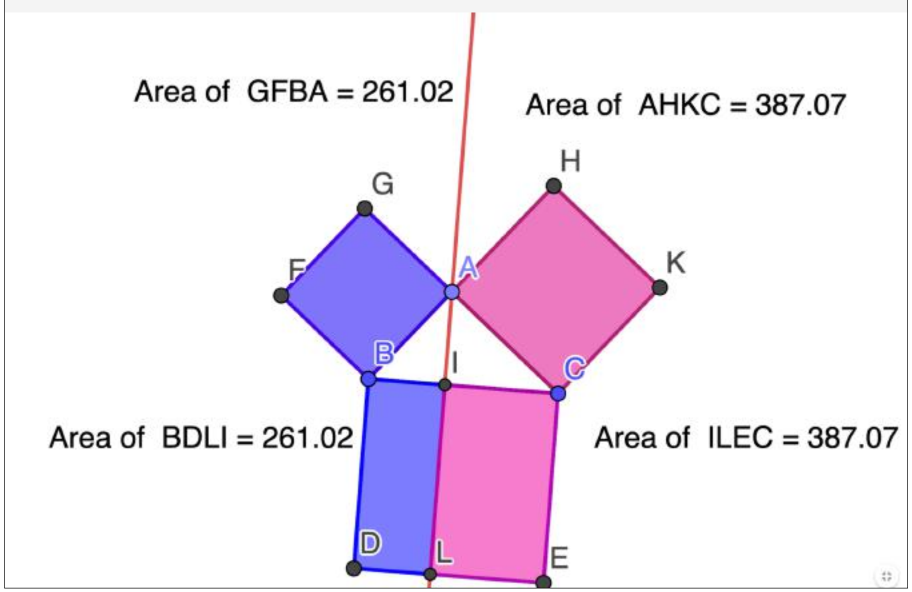
www.geogebra.org/classroom/myyyzaga



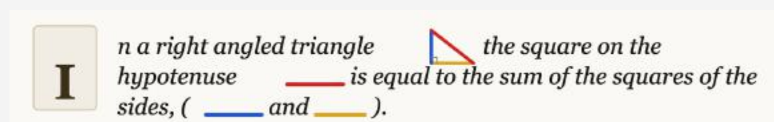
https://www.geogebra.org/classroom/fsty4v36

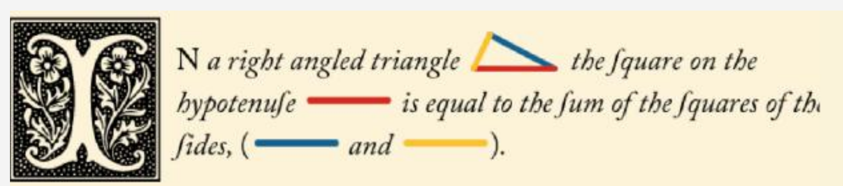
Geogebra (QR Code) Wooclap What relation holds between the areas of these figures?

Example

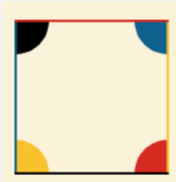


Euclid's Elements Proof of the Pythagorean Theorem:
The statement

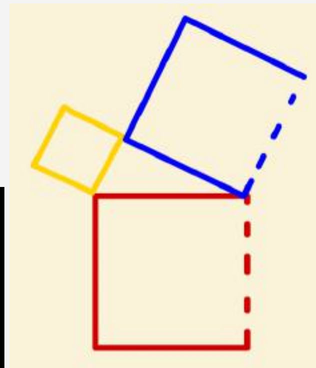




Euclid’s Elements Proof of the Pythagorean Theorem:

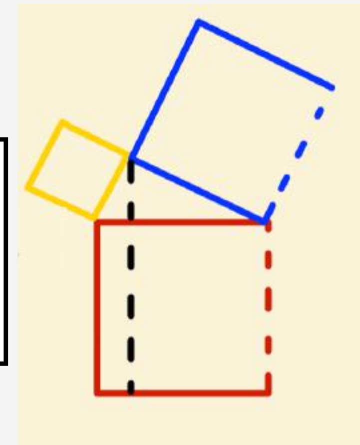


By Prop 46: Given a segment, it is possible to construct a square having that segment as one of its sides.

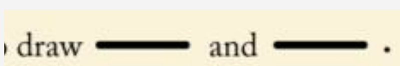


Euclid’s Elements Proof of the Pythagorean Theorem:

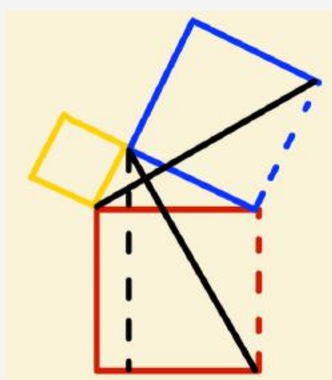
Proposition 31: Given a point and a line not passing through it, it is possible to construct a line through the point that is parallel to the given line.



Euclid’s Elements Proof of the Pythagorean Theorem:

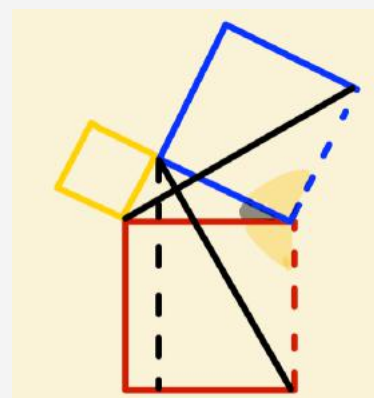


Post. 1: It is possible todraw a straight line from any point to any point

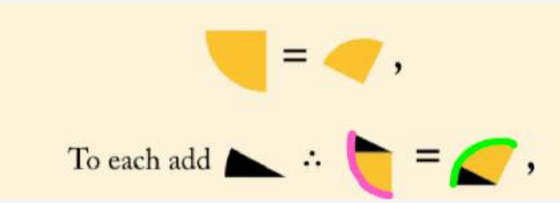


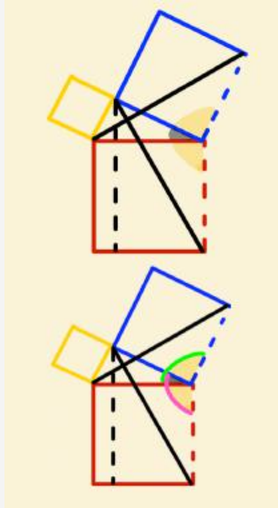
Euclid’s Elements Proof of the Pythagorean Theorem:

Postulate 4: All all right angles equal one another.



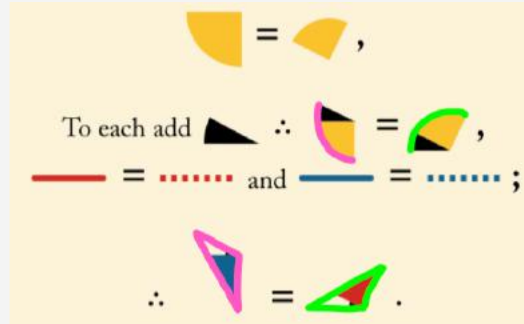
Euclid’s Elements Proof of the Pythagorean Theorem:

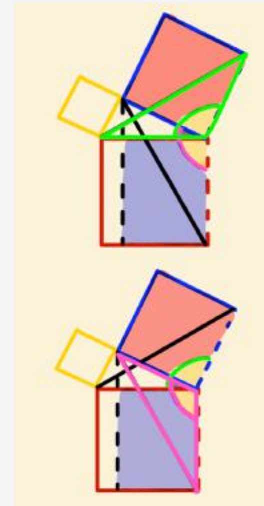




Common notion 2: If equals are added to equals, the results are equal.

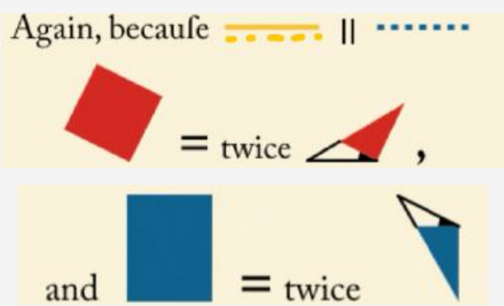
Euclid’s Elements Proof of the Pythagorean Theorem:

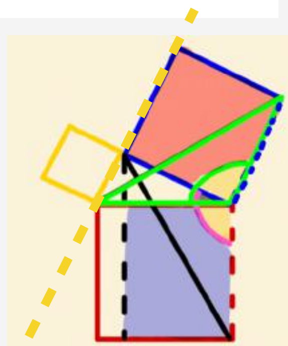




Prop. 4. SAS

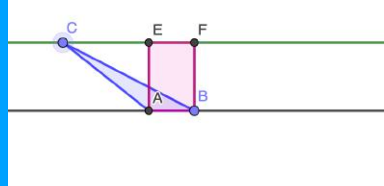
Euclid’s Elements Proof of the Pythagorean Theorem



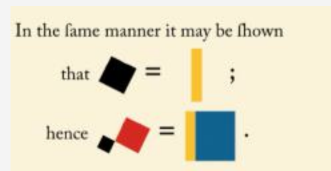


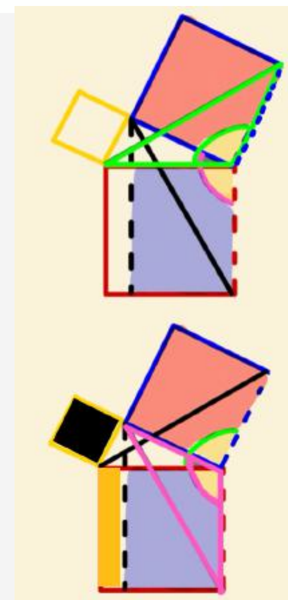
Proposition I.41
If a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle.

Recall



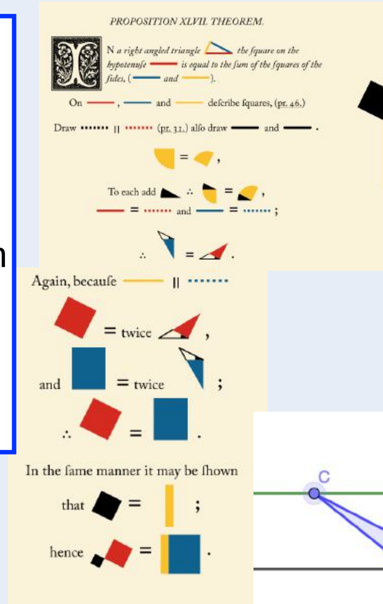
Euclid’s Elements Proof of the Pythagorean Theorem

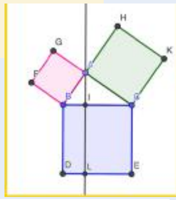


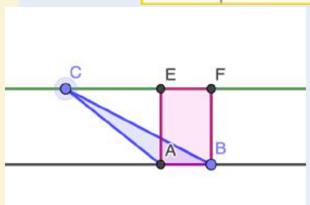


Euclid's Elements Proof of the Pythagorean Theorem

Describe how the Pythagorean Theorem is proved in Euclid's Elements using the relationship between triangles and rectangles, and the use of congruent triangles.







Byrne's Euclid

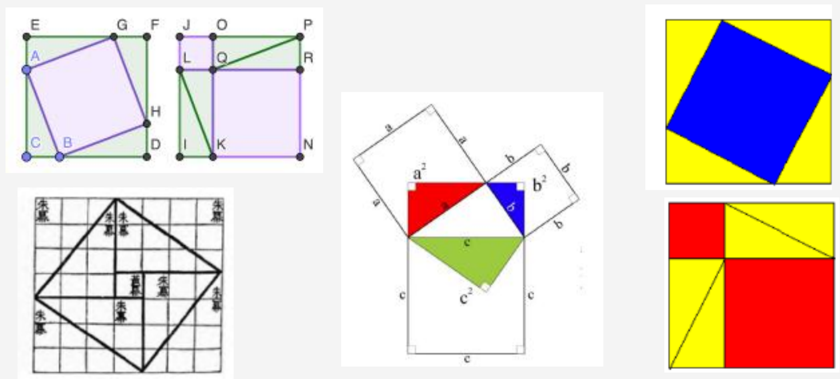
The Pythagorean Theorem Around the World

The Pythagorean Statement Across Time and Space

- **Mesopotamia** (c. 1900–1600 BCE):
- **India** (800–200 BCE):
- **China** (c. 500 BCE – 200 CE)
- **Greece** (c. 300 BCE)
- **Islamic World** (8th–15th c.)
- **Europe** (12th–16th c.)
- **Ancient Egypt and the Americas**:

The Pythagorean Statement Across Time and Space

- **Mesopotamia** (c. 1900–1600 BCE): Problem texts show step-by-step procedures to compute sides of right triangles; Plimpton 322 lists related numerical data, but their purpose is debated.
- **India** (800–200 BCE): Verbal general rule for rectangles Śulba Sūtras; used in altar geometry.
- **China** (c. 500 BCE – 200 CE) Gou-gu theorem, a general statement for right triangles in texts, explained later by commentators using geometric dissection.
- **Greece** (c. 300 BCE) first extant proof within an axiomatic system in Euclid's elements.
- **Islamic World** (8th–15th c.) Many geometric and algebraic proofs; extensions toward the law of cosines.
- **Europe** (12th–16th c.) Theorem transmitted via Arabic → Latin → printed Elements; used in surveying, art, and education.
- **Ancient Egypt and the Americas**: No clear textual evidence of a general statement or of algorithms for computing sides of right triangles (limited sources).



Euclid’s proof is deduced from axioms (postulates and common notions).

Chinese proofs are visual (dissection) arguments, not deductions from an explicit set of axioms.

In https://www.cut-the-knot.org/pythagoras/ you can find 122 proofs of the Pythagorean theorem.

The Gou-gu/Pythagorean theorem in Ancient China

Zhao Shuang (~3rd c. CE)

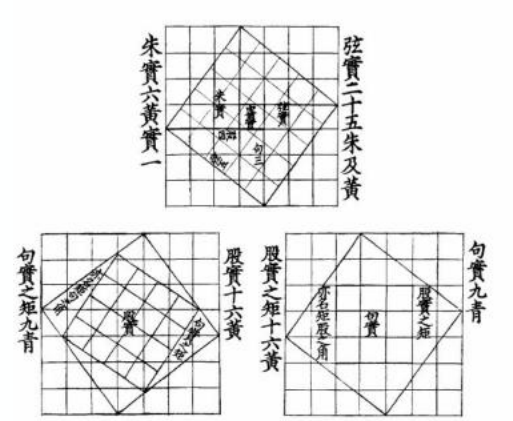
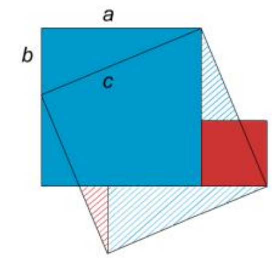


Figure 16. Hypotenuse, left and right diagrams from Song edition.

https://www.cambridge.org/core/books/abs/astronomy-and-mathematics-in-ancient-china/zhao-shuang-and-pythagoras-theorem/B2113BA11AFCEE471E42F2439709E740

Liu Hui (3rd c. CE)



https://mathvoices.ams.org/mathmedia/tonys-take-september-2023/

https://mathvoices.ams.org/mathmedia/tonys-take-september-2023/#two

https://www.tandfonline.com/doi/epdf/10.1080/00029890.2024.2370240?needAccess=true


The Guardian

News Opinion Sport Culture Lifestyle

New Orleans

This article is more than 2 years old

US teens say they have new proof for 2,000-year-old mathematical theorem

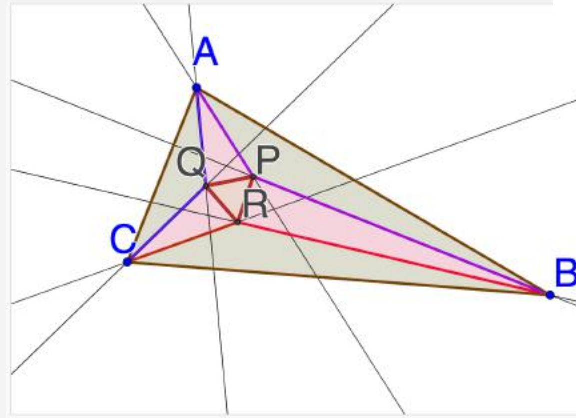
New Orleans students Calcea Johnson and Ne'Kiya Jackson recently presented their findings on the Pythagorean theorem

More About Euclid’s Elements

Are there statements that follow from Euclid's postulates and common notions that are not proved in the Elements? Explain.

Morley's trisector theorem, 1899

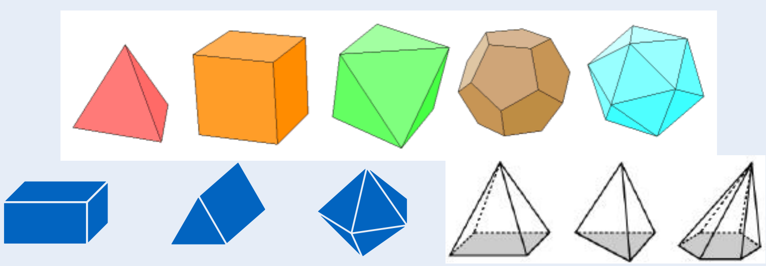
https://ggbm.at/FRZ9Nfec



Compute V, E, F and then

V – E + F

for two or more of the polyhedra below.
What value do you get?
Do you think this will always give the same value for *any* polyhedron?
(V is the number of vertices, E is the number of edges, and F is the number of faces.)



Euler's Polyhedral Formula
In a (convex) polyhedron, the following equation holds: **V – E + F = 2**, where V is the number of vertices, E is the number of edges, and F is the number of faces. Descartes (~1600), Euler (~1700)

If some propositions already proved in Euclid's Elements are added as axioms, what happens?

a) A contradiction can be deduced from the new (larger) set of axioms.
b) All the propositions in Euclid's Elements can still be proven with the new (larger) set of axioms.
c) Some, but not all, of the propositions in Euclid's Elements can be proven with the new (larger) set of axioms.

Choose one option and explain your reasoning.

Hint: Think of axioms as tools in a toolbox. Propositions are things you build with those tools. If you put something you already built back into the toolbox, can you still build everything you built before? Could you accidentally "build something wrong" with the new set of tools?

Full length portraits of Euclid and Ptolemy, each holding a compass; a sphere on a staff is between them.
- Cardano, Girolamo, 1501-1576,

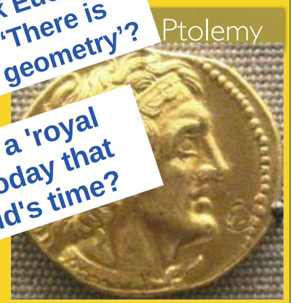
According to Proclus, Ptolemy (the pharaoh) once asked Euclid if there was not a shorter road to the knowledge of geometry than by the study of the Elements, and Euclid replied:

There is no royal road to geometry.

What do you think Euclid meant by saying 'There is no royal road to geometry'?

Do you think there's a 'royal road' to geometry today that didn't exist in Euclid's time?

The Philosophers (Ptolemy and Euclid with Their Pupils) Pietro della Vecchia - 1600



Tetradrachm of Ptolemy I, British Museum, London