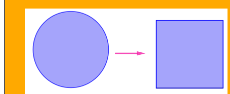
MAT 336 Hellenic Mathematics before Euclid's Elements.



??

- Thales
- Hippocrates
- Quadrature Hippias

Hellenic Mathematics: Proof, Persuasion, and the Power of Mathematical Reasoning

Ancient Greece 101 I National Geographic, https://youtu.be/6bDrYTXQLu8

Greek mathematics was highly influential. Why might it be absent from this overview?

Greek or Hellenic Mathematics (Between ~600 BCE and 300 AD)

- In mathematics, the question
  "**why?**"
- was now asked together with the older question
  "**how?**"

Greek or Hellenic Mathematics
Two important aspects

1. Emphasis on proofs
2. Involvement with specific, challenging problems.
   - need to decide which new arguments counted as genuine proof
   - in this way, the problems led to
     - new discoveries
     - increasingly sophisticated ideas about proofs

Greek or Hellenic Mathematics (Between ~600 BCE and 300 AD) Primary Sources

- No tablets or papyri or …
- Only copies of copies of copies of…
  - Restored versions of texts of Euclid, Archimedes, Apollonius and other.
- To study how this mathematics was developed we rely on these copies and remarks of writers of the time.

Early Greek mathematics

Early Greek mathematics was not one but many; there were ***various levels of practice,*** *from*

- *calculations on the abacus to*
- *indirect proofs concerning incommensurable lines, and varying attitudes,*
- *from laughing off attempts to square the circle to*
- *using attempts to square the circle as examples in a second-order discussion about the nature of demonstration.*

Cuomo, Serafina. Ancient mathematics. Routledge, 2005.

Early Greek mathematics

In sum, ***different forms of mathematics were used for different purposes by different groups of people.*** *Perhaps one common feature is clearly distinguishable:* ***mathematics was a public activity,***

- *it was played out i**n front of an audience**, and*
- *it fulfilled functions that were **significant at a communal level**,*
  - *be they counting revenues,*
  - *measuring out land or*
  - *exploring the limits of persuasive speech.*

Cuomo, Serafina. Ancient mathematics. Routledge, 2005.

Sound arguments Public life

The Greek city-states fostered an environment of intellectual competition and debate. In the agora (public square), scholars engaged in **discussions and debates**, which encouraged the refinement of ideas and the need for rigorous argumentation.

"*Recent studies of Babylonian sources have shown that we must revise former estimates of the extent to which the Greeks were indebted for the details of astronomy to the Babylonians. This debt proves to have been much greater than they have been imagined and further research may prove to be greater still"* Sir Thomas Heath,1932.

"It is gradually beginning to be realized that many of the achievements of Greek culture in the fields of astronomy and mathematics did not spring, fully armed, from the Hellenic brain, but had their more remote origins in the civilizations of the ancient east." Professor Filon.

"*Whatever the Greeks receive from the barbarians, they bring to a finer perfection.*" Plato (Epinomis 987d–e; probably by Philip of Opus)

What do these three quotes together suggest about the relationship between Greek and Babylonian mathematics?

Adapted from GREEK ASTRONOMY AND ITS DEBT TO THE BABYLONIANS, LEONARD W. CLARKE The British Journal for the History of Science, Jun., 1962, Vol. 1, No. 1 (Jun.,1962), pp. 65-77

Mesopotamian Influence on Greek Mathematics - What many historians accept today

- Greek mathematics did not develop in isolation; it emerged within a broader eastern Mediterranean intellectual world.
- Greek mathematical astronomy shows clear dependence on Babylonian numerical methods.
- Near Eastern computational techniques likely circulated widely before the classical Greek period.
- Certain problem traditions (e.g., algebraic-type problems) likely moved across cultures.

The jurors were called upon to be not just witnesses, but also calculators – in principle, a *logistes* could tell with absolute certainty whether the account presented to him worked out or not, whether everything added up. It was this kind of persuasiveness, mathematical persuasiveness, that both speakers wished to claim for their arguments. (…) For now, let me observe that accounts – collective, public counting – were a pervasive practice in classical Athens. We find them not only in inscriptions, but also in various genres of literature. Along with its practical functions, public counting was associated with political accountability, and in fourth-century legal speeches it seems increasingly to symbolize the role itself of the Athenian citizen.

Write one short answer to each question and save them.

1. **In what sense were jurors acting like mathematicians?**
2. **How is "counting" connected to political accountability?**
3. **How is this different from how we usually think about mathematics?**

Cuomo, Serafina. Ancient mathematics. Routledge, 2005.

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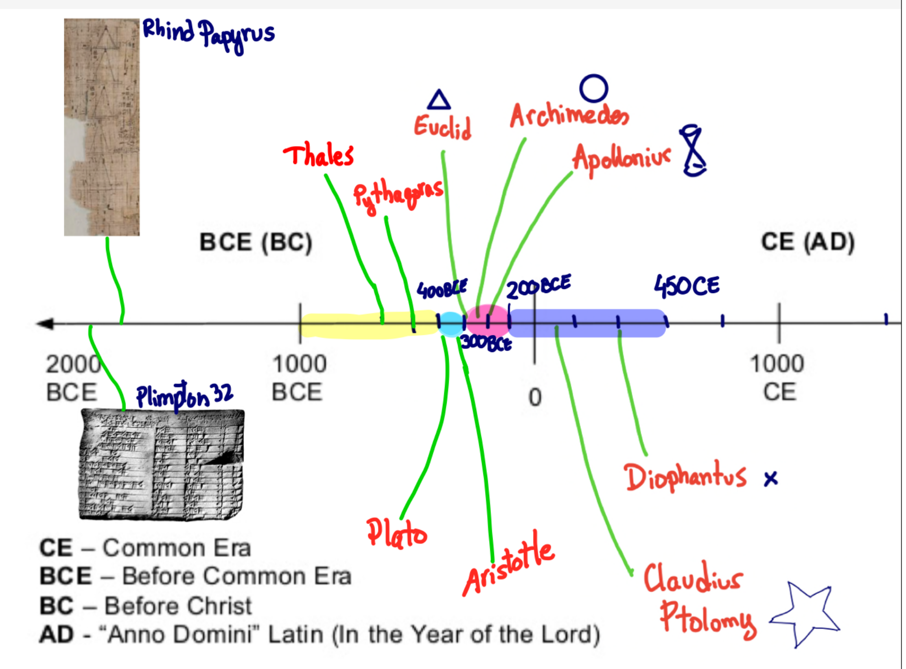
Groups of 3–4. Produce one written answer per question.

1. **In what sense were jurors acting like mathematicians?**
2. **How is "counting" connected to political accountability?**
3. **How is this different from how we usually think about mathematics?**

Cuomo, Serafina. Ancient mathematics. Routledge, 2005.

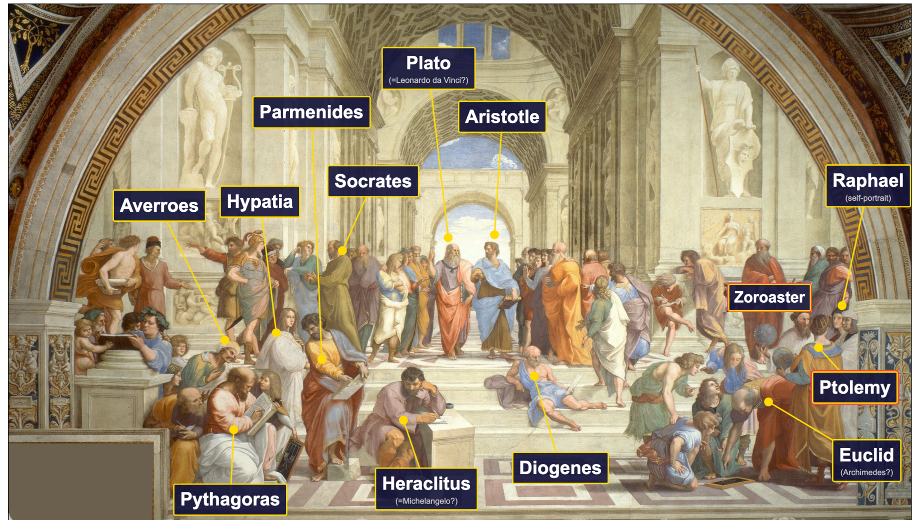


Map of Ancient Hellenic Mathematics Encyclopedia Britannica

Mathigon timeline - Hellenic

mathigon.org

The school of Athens, by Raffaele



The school of Athens, by Raffaele

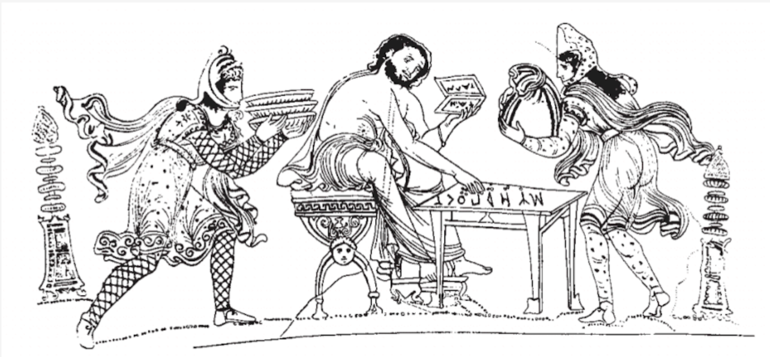
You can place the labels yourself in this link

Remarks

- Raphael painted The School of Athens (1509–1511) in the Vatican, during the Renaissance.
- The painting gathers thinkers from different centuries into one imagined scene.
- The setting is imagined classical architecture inspired by Roman forms, not an actual Greek building.
- The composition is symmetrical, but the left and right sides differ in mood and activity.
- The vanishing point is between Plato and Aristotle.
- The mathematicians are placed closest to the viewer: Pythagoras (left) and Euclid (right) are prominently placed at the front. Raphael may be making a statement about the status of mathematics.
- A tablet near Pythagoras includes numerical ratios and a harmonic diagram.
- A geometric figure (often identified as Euclid or Archimedes) appears in the right foreground using a compass to demonstrate a construction.
- Ptolemy (astronomer) holds a terrestrial globe.
- Students are physically gathered around the mathematicians, emphasizing teaching.
- Mathematics is depicted as active, demonstrative, and teachable.

Numbers as a Gift from the Gods

300BCE - Scene depicted on a vase, likely with a counting board.



300BCE - Scene depicted on a vase, likely with a counting board.

Cuomo, Serafina. Ancient mathematics. Routledge, 2005.

Prometheus at Rockefeller Center, sculpture by Paul Manship, 1933

Paul Manship, Public domain, via Wikimedia Commons

Prometheus Bound By Aeschylus

PROMETHEUS: Yes, and from it [the fire] they'll learn many arts […] hear what wretched lives people used to lead, how babyish they were – until I gave them intelligence, **I made them masters of their own thought**. […] they knew nothing of making brick-knitted houses the sun warms, nor how to work in wood. They swarmed like bitty ants in dugouts in sunless caves. They hadn't any sure signs of winter, nor spring flowering, nor late summer when the crops come in. All their work was work without thought, until I taught them to see what had been hard to see: where and when the stars rise and set. **What's more, I gave them the numbers, chief of all the stratagems.** And the painstaking, putting together of letters: to be their memory of everything, to be their Muses' mother, their handmaid! […]

Codex-Style Vessel CE 700–750. Two teaching scenes with Itzam—an elderly creator/atlas-like deity—instructing four students

https://kimbellart.org/collection/ap-200404

Itzam ID cues: Aged face; netted headdress with a brush tucked in.

Scene 1: Itzam uses a pointer and a folded codex; a speech thread with **bar-and-dot numerals suggests arithmetical/calendrical calculation**.

Scene 2: Itzam taps the ground while addressing two students; spoken glyphs on a speech thread.

Takeaway: Elite scribal education in mathematics/ calendar lore framed as instruction from a creation deity.

Birds, by Aristophanes performed in 414 BC

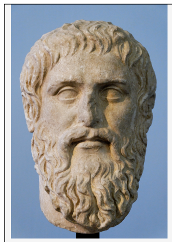
A new city has to be founded from scratch. The main character, Peisthetaerus, is visited by various people who offer their services.

METON : With the straight rod I measure out, that **so the circle may be squared**; and in the centre a market-place; and streets be leading to it straight to the very centre; just as from a star, though circular, straight rays flash out in all directions.
PEISTHETAERUS : **Why, the man's a Thales!**

Translation Cuomo, Serafina. Ancient mathematics. Routledge, 2005.

Now, we will discuss some remakable people and ideas, in a not necessarily chronological way.

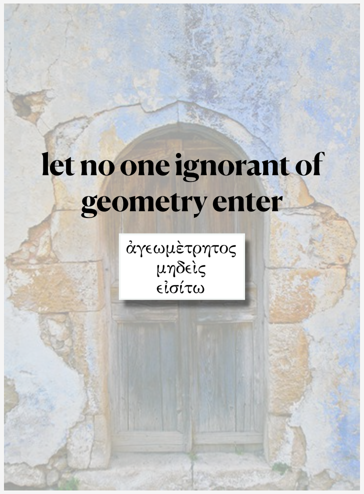
Plato ~ 400 BC

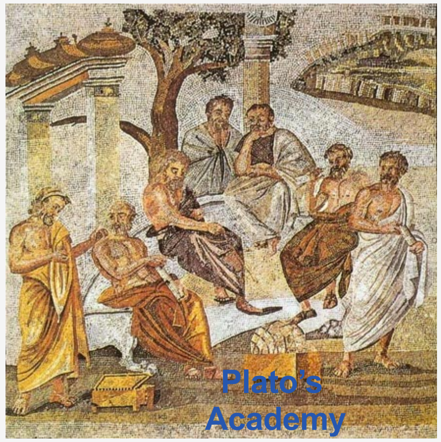


Plato (~400 BC)

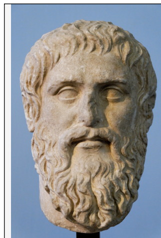
- **'greatly advanced mathematics in general, and geometry in particular, because of his zeal for these studies'** (Proclus)
- was an **important influence** upon the mathematicians of his time, by inspiration and direction.
- the works of Plato are some of our fullest and **best source of information about the mathematical developments at that time.**
- many scholars credit Plato with laying the conceptual foundations of Western thought.

Plato (~400 BC)





- Emphasized distinction between pure math and “merely useful”
- Math education is essential for the mind (of an elite)



Plato (~400 BC)

In the philosophical school of thought of Plato’s (429–348) academy, **mathematics was shaped as ideal case of purely deductive science**, which has influenced the development of this science enormously up to the present day. Plato argued that **mathematics had an intermediate position between the realm of mere ideas and the world of empirical objects.**

5000 Years of Geometry Mathematics in History and Culture
By Christoph J. Scriba, Peter Schreiber · 2015

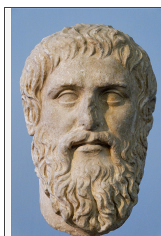
Prompt (at least one per group):

1.Why would Plato post ‘Let no one ignorant of geometry enter’?

2. What qualities was he actually seeking in students?

3. Elitist gatekeeping or reasonable prerequisite—defend your position.

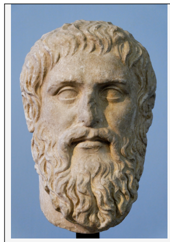
There is no direct historical or archaeological evidence of such inscription but it reflects Plato's authentic emphasis on the importance of geometric reasoning.



Plato (~400 BC)

Plato greatly advanced mathematics in general and geometry in particular because of his zeal for these studies. It is well known that his writings are thickly sprinkled with mathematical terms and that he everywhere tries to arouse admiration for mathematics among students of philosophy.

Proclus (~400 CE, Platonist philosopher)



Plato (~400 BC)

In sum, what information we can get from Plato is fascinating and rich, and fraught with problems. Beyond issues of detailed reconstructions and precise attributions, which may never be solved with any certainty, **he definitely is testimony to the vitality of mathematics at the time, and to the interest that mathematical issues aroused in the educated public at large.**

Cuomo, Serafina. Ancient mathematics. Routledge, 2005.

Aristotle ~ 350 BCE

Aristotle (~350BCE) Wrote about the structure of mathematical knowledge.

- Through deductive reasoning, **necessary conclusions** follow from accepted **starting points.** (Note: starting points=axioms)
- Demonstration produces knowledge of **why** something must be so.

https://plato.stanford.edu/entries/aristotle-mathematics/

Aristotle (~350BCE) Wrote about the structure of mathematical knowledge.

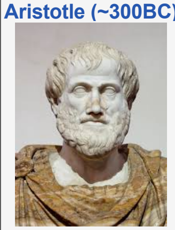
- An **axiom** is a statement worthy of acceptance*.
- Example: when equals taken from equals the remainders are equal.

*worthy of acceptance: in modern terms, self-evident.

https://plato.stanford.edu/entries/aristotle-mathematics/

- There are two distinct types of "quantities":
  - the continuous (magnitude)
    - *"A **magnitude** is that what which is divisible into divisible that are infinitely divisible."*
    - Example: lines, surfaces, bodies and time.
  - the discrete (number)
    - *A "**number**" quantities that is composed of distinct, separate units or parts that can be counted. (Examples 1, 2, 3, ...)*

Aristotle (~300BC)



An old question in modern terms: Is it possible to find integers p and q such that $\sqrt{2} = p/q$?n In other words, is $\sqrt{2}$ rational?

Here is Aristotle's take

…when something impossible results from the assumption of its contradictory; **e.g.** **that the diagonal of the square is incommensurate with the side, because odd numbers are equal to evens if it is supposed to be commensurate. One infers syllogistically that odd numbers come out equal to evens, and one proves hypothetically the incommensurability of the diagonal, since a falsehood results through contradicting this.**

Aristotle - The Organon ANALYTICA PRIORIA Book 1 Part 23- ~350

Proof of the irrationality of $\sqrt{2}$ discussed by Aristotle

It is clear then that the ostensive syllogisms are effected by means of the aforesaid figures; these considerations will show that reductiones ad also are effected in the same way. For all who effect an argument per impossibile infer syllogistically what is false, and prove the original conclusion hypothetically when something impossible results from the assumption of its contradictory; **e.g.** **that the diagonal of the square is incommensurate with the side, because odd numbers are equal to evens if it is supposed to be commensurate. One infers syllogistically that odd numbers come out equal to evens, and one proves hypothetically the incommensurability of the diagonal, since a falsehood results through contradicting this**. For this we found to be reasoning per impossibile, viz. proving something impossible by means of an hypothesis conceded at the beginning. Consequently, since the falsehood is established in reductions ad impossibile by an ostensive syllogism, and the original conclusion is proved hypothetically, and we have already stated that ostensive syllogisms are effected by means of these figures, it is evident that syllogisms per impossibile also will be made through these figures. Likewise all the other hypothetical syllogisms: for in every case the syllogism leads up to the proposition that is substituted for the original thesis; but the original thesis is reached by means of a concession or some other hypothesis. But if this is true, every demonstration and every syllogism must be formed by means of the three figures mentioned above. But when this has been shown it is clear that every syllogism is perfected by means of the first figure and is reducible to the universal syllogisms in this figure.

Aristotle - The Organon ANALYTICA PRIORIA Book 1 Part 23- ~350 BCE

Reasoning pattern: Proof by contradiction:

assume opposite →

derive impossibility →

conclude what we wanted to prove.

Some reasons why √2 irrationality matters

Proof by contradiction: A reasoning pattern

(Uses: diagnosis, debugging, risk, legal).

Frameworks have limits: find something not fitting → recognize limits → build better systems

(Uses: floating-point, measurement, model failure).

Paradigm shock: a core belief shatters → the field rebuilds.

(Uses: Progress of science)

Why might Plato have seen mathematics as an 'ideal case of purely deductive science'? Choose one option and explain your reasoning.

A) Because mathematical truths, once proved from fixed axioms, don't rely on sense experience..

B) Because mathematics starts from basic definitions and axioms, then derives new truths through logical reasoning alone

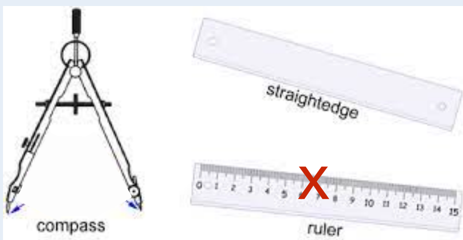
C) Because mathematics only deals with abstract numbers, not physical objects

About last class

- Was Plato a mathematician? Or "just" a philosopher?
  - Not exactly a mathematician. Plato was a philosopher. He discussed the mathematics of his time mainly to support philosophical arguments about knowledge and reasoning.
  - The idea of *mathematician* did not exist a that time.
  - The School of Athens depicts philosophers, so of whom we will call now mathematicians.
- Greeks took a lot of inspiration from the Babylonians
- The Greeks placed great emphasis on public counting and mathematical persuasiveness.
- About the role of mathematics in ancient Greek society: jurors were expected to act like "calculators," checking whether financial accounts added up. This shows that mathematics was not only an abstract subject but also an important practical tool in public life.

Impossible problems of the Antiquity ~5th century BCE

Greek geometers limited constructions to a straightedge and a compass.
Why only these two tools? What's special about them? Hint: Think about geometry.

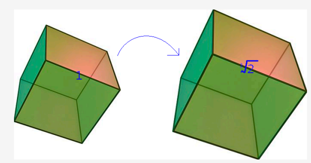


Doubling of the cube

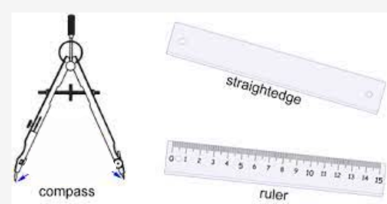
If cube C has side length 1 inch, what is the side length of a cube with twice the volume of C?

Three "impossible" problems

Doubling a cube: Given a cube, **construct** another cube with twice its volume.



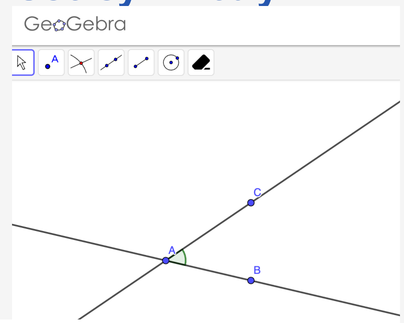
"Construct" means "construct using only straightedge and compass



These problems originated around 400 BCE

Trisection of the angle

Bisect the angle using only straightedge and compass. Hint: Use symmetry.



Join the lesson at www.geogebra.org/classroom with the code:

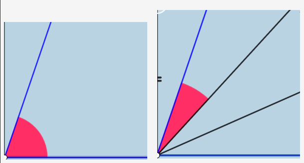
CKCZ PUQU

Or you can also share the following link with your students:

www.geogebra.org/classroom/ckczpuqu

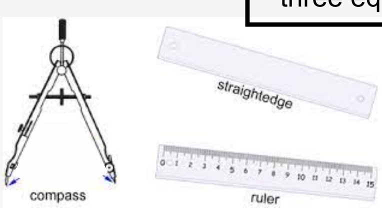
https://www.geogebra.org/m/av94xc8u

Three "impossible" problems



Trisecting an angle: Given an angle between two straight lines, **construct** two lines that divide the angle into three equal parts.

"Construct" means "construct using only straightedge and compass

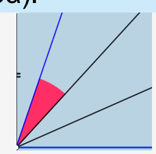


These problems originated around 400 BCE

Trisecting an angle

Constructing an angle α from the angle 3α — equivalent — Constructing a segment of length cos(α) from a segment of length cos(3α).

Also, $\cos(3\alpha) = 4\cos^3(\alpha) - 3\cos(\alpha)$.
If we set $\cos(\alpha)=x$ and $\cos(3\alpha)=a$ then
$4x^3 - 3x - a = 0$



Thus, trisecting an angle is equivalent to solving certain cubic equation

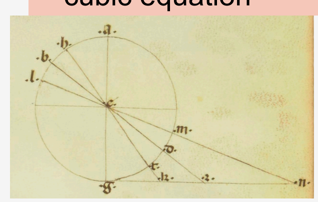


Diagram from 14th-century manuscript copy of Ptolemy's Almagest, folio 22 recto.
Manuscript owned and digitized by gallica.bnf.fr / Bibliothèque nationale de France.

It is likely that the question of trisection of angles arose when trying to construct a table of chords for astronomical purposes. (A chord for the angle 3° can be constructed so it would have been natural to try to get a chord for the angle 1° from the chord for the angle 3°).

Quadrature of the circle

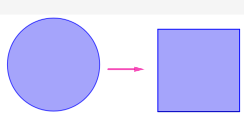
If a circle has radius 1 inch, what is the side length of a square that has the same area as the circle?

Three "impossible" problems

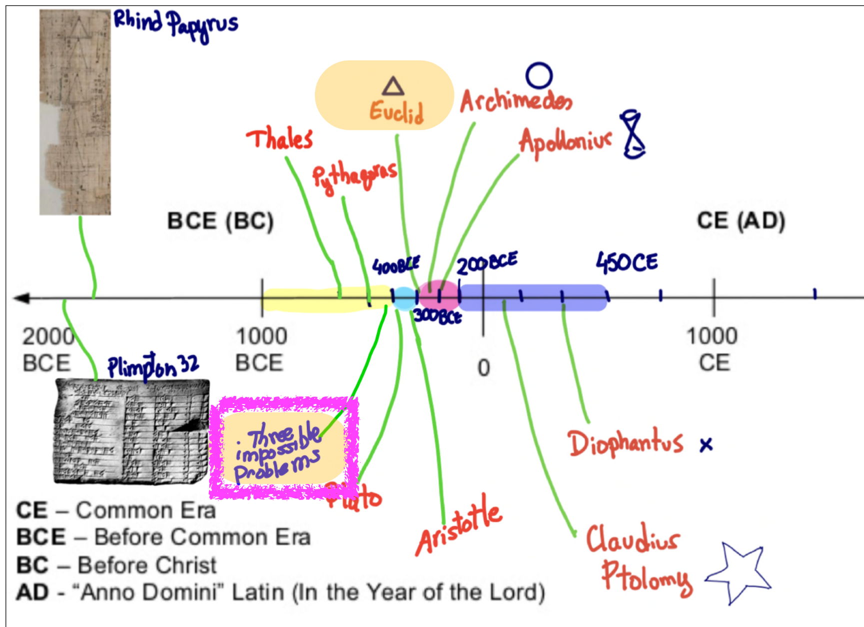
Squaring a circle is closely related to finding its area

"Construct" means "construct using only straightedge and compass

Squaring a circle: Given a circle, **construct** a square with the same area.

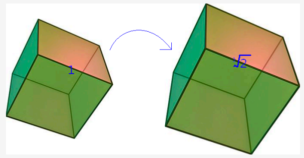


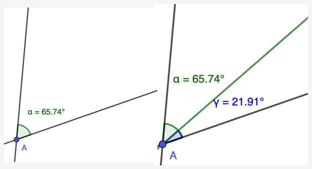
These problems originated around 400 BCE



Three “impossible” problems

Doubling a cube: Given a cube, **construct** another cube with twice its volume.

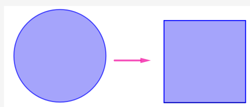




Trisecting an angle: Given an angle between two straight lines, **construct** two lines that divide the angle into three equal parts.

“Construct” means “construct using **only straightedge and compass**

Squaring a circle: Given a circle, **construct** a square with the same area.



These problems originated around 400 BCE

Greek or Hellenic Mathematics
Two important aspects

1. Emphasis on proofs
2. Involvement with specific, challenging problems.
   - need to decide which new arguments counted as genuine proof
   - in this way, the problems led to
     - new discoveries
     - increasingly sophisticated ideas about proofs

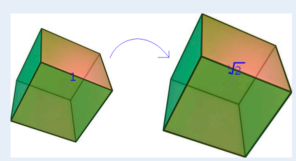
The key mathematical idea behind impossibility

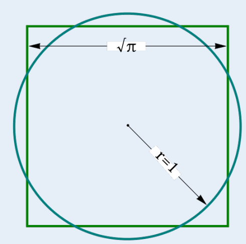
- ***Constructible numbers*** are those obtained by the combining the following operations
  - addition
  - subtraction
  - multiplication
  - division
  - square roots
- **Straightedge–compass constructions → only constructible numbers**
- The numbers π, $\sqrt[3]{2}$, and cos(α) (for most angles α) are not constructible

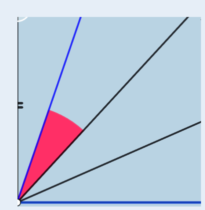
In your opinion, which of the three classical problems was the hardest to prove impossible — and why?

The three problems are :
- Squaring the circle
- Trisecting an angle
- Doubling the cube

Using only straightedge and compass.





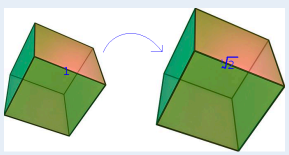


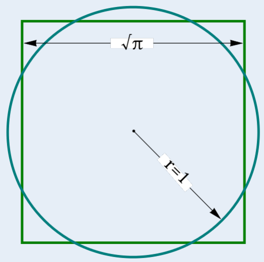
All three problems were eventually proven impossible. Who proved each one impossible, and in what year?. You may use Google and AI to answer this question.

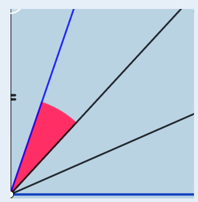
The three problems are :

- Squaring the circle
- Trisecting an angle
- Doubling the cube

Using only straightedge and compass.







The mathematical problems that the Greeks tackled, and indeed the whole geometrical bias of so much Greek mathematics, both **stimulated and was stimulated by construction methods**, and the question naturally arose as to which geometrical problems can be solved by the basic **line-and-circle** constructions, and which ones cannot.

The History of Mathematics: A Source-Based Approach, Vol. 1, (2014) June Barrow-Green, Jeremy Gray, Robin Wilson.

Takeaway: Problems shaped the tools, and the **tools shaped the problems.**

Can you think of a practical technology that may trace back to line-and-circle constructions?

Hint: Think parabola, ellipse, triangles/triangulation.

Examples of practical applications

Constructions → conics (parabola / ellipse / hyperbola)

- Parabola → headlights, satellite dishes, solar ovens.
- Ellipse → planetary/satellite orbits; whispering galleries.
- **AI:** camera calibration and conic/ellipse detection in computer vision.

Proof culture → surveying / architecture / engineering

- Triangulation → land surveys, mapmaking, GPS logic.
- Geometry of structure → arches, trusses; constraints in CAD.
- **AI:** formal verification/specs for algorithms; constraint reasoning in planning and robotics.

Constructibility → algebra & number theory → real-world tech

- Public-key crypto (RSA, ECC) → HTTPS, banking, messaging (protects AI models/ data).
- Error-correcting codes → QR codes, Wi-Fi, storage (reliable datasets/training).
- **AI:** hashing/modular arithmetic for indexing, sharding, and fast retrieval.

Obsession with Proofs and Abstract constraints (line & circle) → powerful ideas → tools embedded in today's communications, navigation, engineering, AI, …

Pythagoras(~600BC?)

The Pythagoreans

Pythagoras (~600BC?) - The Pythagoreans

- **No primary sources**, only accounts written centuries after Pythagoras would have lived.
- We know almost nothing about Pythagoras himself. Some scholars even doubt he existed. Most believed he as a Holy Man.
- The Pythagoreans form a very secretive sect.
  - The mathematical and the mystical were merged.
  - They believed that numbers reveal hidden structure in nature (music, geometry, astronomy).

No primary sources ☹

All things are number

Doll by UneekDollDesigns

Pythagoras (~600BC?) - The Pythagoreans

> They [the Pythagoreans] were the first to advance this study [of mathematics], and having been brought up in it they thought its principles were the principles of all things. Since of these principles numbers are by nature the first, and in numbers they seemed to see many resemblances to the things that exist and come into being—…—and similarly almost all other things being numerically expressible; since, again, they saw that the attributes and the ratios of the musical scales were expressible in numbers; since, then, all other things seemed in their whole nature to be modeled after numbers, and **numbers seemed to be the first things in the whole of nature**, they supposed the elements of numbers to be the elements of all things, and t**he whole heaven to be a musical scale and a number.**
>
> Aristotle - Metaphysics

Do Pythagorean ideas still shape today's science? Why or why not?
Hint: think measurement, models, data, music/acoustics.

The Pythagoreans

Number rules the universe. ⟷ Great influence on scientists from then on

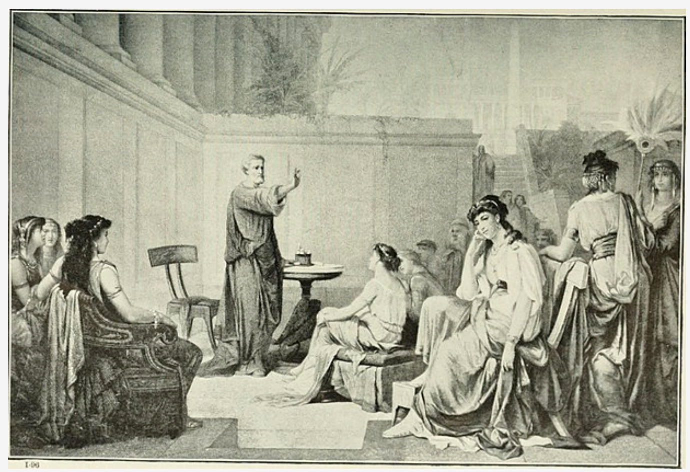
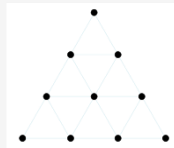


Illustration from 1913 showing Pythagoras teaching a class of women.

Pythagorean "mystical" views of number

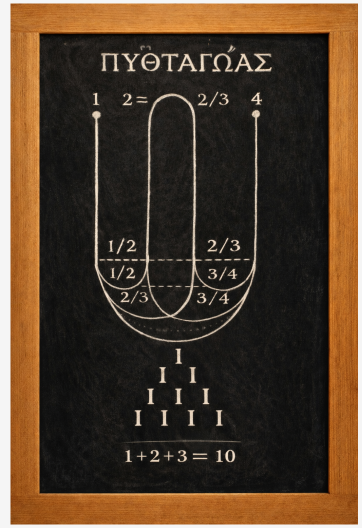
- Number rules the universe.
- **1** = source of all things; neither even nor odd.
- Odd = masculine;
- even = feminine;
- **5** = marriage (**2 + 3**).
- **7** considered sacred: "seven planets," seven strings on a lyre; Apollo's day = the 7th.
- **10** called "perfect" because 1 + 2 + 3 + 4 = 10. Never meet in groups larger than 10!



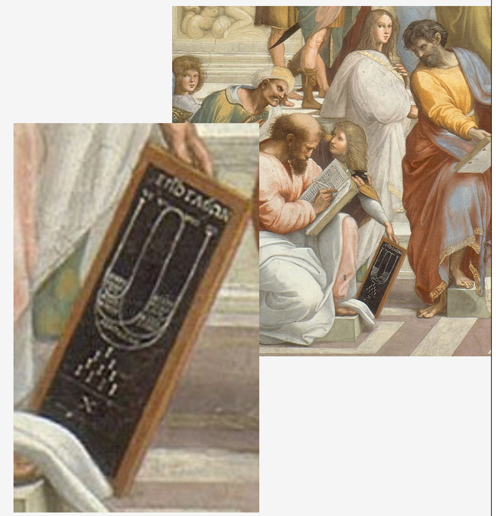
Pythagorean Mathematical Music

Whole-number ratios → musical consonance

ChatGPT interpretation



Pythagoras by Raphael in The School of Athens



Pythagorean Mathematical Music

According to a legend (reported much later than Pythagoras, around 200 CE.), Pythagoras discovered the numerical ratios underlying musical consonances after hearing blacksmiths' hammers and investigating their pitches.

- The connection between music and numerical ratios (e.g.,
  - octave = 2:1,
  - fifth = 3:2,
  - fourth = 4:3)

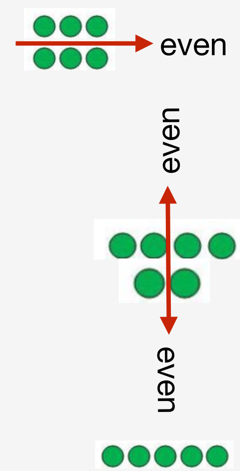
  was associated with the Pythagorean tradition.

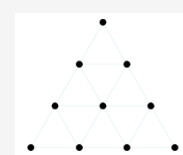
- Related tuning practices predate or extend beyond the Greek and the Pythagoreans.

Pythagoras by Raphael in The School of Athens

The Pythagoreans

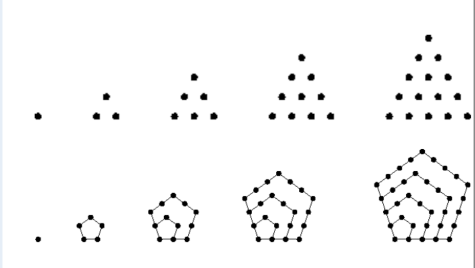
- Recall Dichotomy between odd and even.
- Pythagoreans probably represented numbers with pebbles.
  - **Example**: A number is **even** if it can be represented by a configuration of pebbles that than be divided into two equal parts. Otherwise is odd. (Well, 1 was not considered odd, nor even)
    - Proof: 4+2 is even.
    - Proof: An even sum of odd numbers is even. (Similar to the proof of 4+2 is even.)





The Pythagoreans - Figurate numbers

The *gnomon* is the piece of the figure that needs to be added to a given figurate numbers in order to get the next greater figurate number.



What is the 6th triangular number
What is the 6th pentagonal number
If you finish early: Can you find a formula for the n-th triangular number? What about the n-th pentagonal number?

Western history of the Pythagorean theorem and how it was attributed to Pythagoras

- **c. 1800–1600 BCE (Mesopotamia):** Precise approximation of $\sqrt{2}$. Plimpton 322? New research discards Pythagorean links.
- **6th–5th c. BCE (Pythagoreans):** All things are number.
- **4th c. BCE — Plato:** *Meno* (diagonal doubles a square); *Republic/Timaeus* elevate math, discusses Pythagorean links.
- **4th c. BCE — Aristotle:** *Metaphysics* A5: Pythagoreans first to advance mathematics.
- **c. 300 BCE — Euclid's Elements** First extant general proof.
- **5th c. CE — Proclus:** Attributes to Pythagoras; legend spreads.
- **Medieval → modern West:** Greek historiography + Euclid's prestige fix the "Pythagorean" name despite earlier/independent proof.

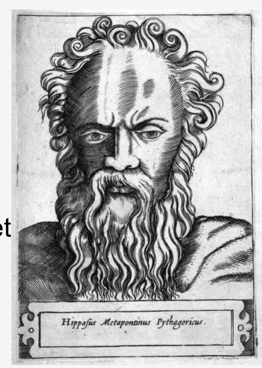
1) What does this change about how we view mathematical discoveries, names, and credit? 2) (optional) Does it matter who first discovered an idea? Why or why not?

Most scholars will agree that there was a Pythagorean school of philosophy from the sixth until probably the fourth century BC, that they were involved in politics and that they had certain beliefs about life and the universe, including perhaps the tenet that 'everything is number', or that number holds the key to understanding reality. But most scholars today also think, for instance, that Pythagoras never discovered the theorem that bears his name.
Cuomo, S. 2001. Ancient Mathematics, Science of Antiquity, Routledge.

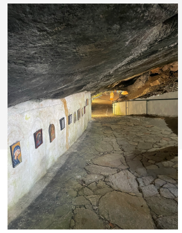
The Discovery of √2: Drama Among the Pythagoreans?

- **Discovery**: The incommensurability of the square's diagonal with its side was likely discovered in the 5th century BCE.
- **Hippasus of Metapontum** (possibly 5th century BCE) is described by Aristotle as an early Pythagorean, though not in connection with incommensurability.
- **The legend**: Later writers (3rd–4th century CE) claim that Hippasus revealed a Pythagorean secret—sometimes linked to incommensurability—and was punished by drowning.
- **Historically unverified**: the sources for the legend are late, inconsistent, and not considered reliable evidence.

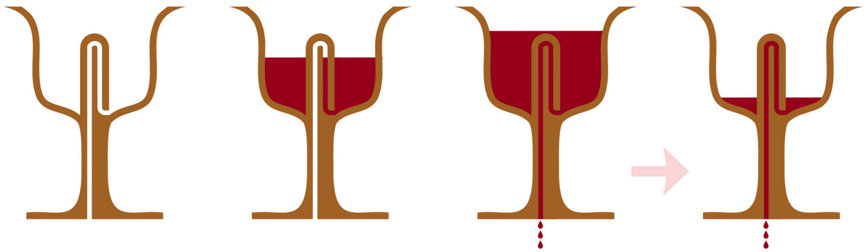
Hippasus of Metapontum



Hippasus, engraving by Girolamo Olgiati, 1580



Cross section of a Pythagorean cup being filled: at B, it is possible to drink all the liquid in the cup; but at C, the siphon effect causes the cup to drain



https://commons.wikimedia.org/wiki/File:Physagorian_Pythagoras_Greedy_Tantalus_cup_05.svg

No beans for the Pythagoreans!

French manuscript from 1512/1514, showing Pythagoras turning his face away from fava beans in revulsion

French early 16th century. http://www.nga.gov/content/ngaweb/Collection/art-object-page.86044.html

Authors discuss many reasons

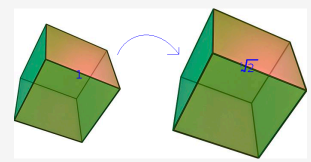
- Beans were associated with souls of the dead.
- Beans were linked with generation and decay (and connected with life and reincarnation)
- beans were used in voting (white and black beans). Avoiding beans could symbolize avoiding politics or public life.

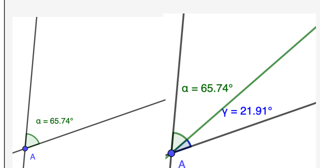
Ancient sources agree on one fact:
The Pythagorean community had a rule not to eat beans.

Incommensurable magnitudes

Three “impossible” problems

Doubling a cube: Given a cube, **construct** another cube with twice its volume.

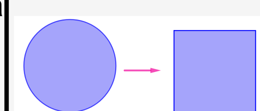




Trisecting an angle: Given an angle between two straight lines, **construct** two lines that divide the angle into three equal parts.

“Construct” means “construct using only straightedge and compass

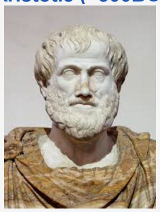
Squaring a circle: Given a circle, **construct** a square with the same area.



These problems originated around 400 BCE

- There are two distinct types of “quantities”:
  - the continuous (magnitude)

Aristotle (~300BC)



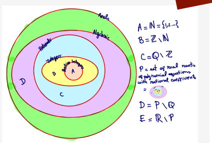
    - *“A **magnitude** is that what which is divisible into divisible that are infinitely divisible.”*
    - Example: lines, surfaces, bodies and time.
  - the discrete (number)
    - *A “**number**” quantities that is composed of distinct, separate units or parts that can be counted. (Examples 1, 2, 3, ...)*

$\sqrt{2}$ is not rational

No pattern!

Does it have a “nice” expression in terms of whole numbers?

Commensurability



π is not rational
π is not even algebraic.

https://cosmosmagazine.com/mathematics/the-square-root-of-2/

Holy beard of Zeus! It's expanding irrationally and infinitely! Credit: Jeffrey Phillips

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset$ **Algebraic** $\subset \mathbb{R}$

Positive integers (1,2,..)

$\cap$

Integers Z - (-2,-1,0, 1,2..)

$\cap$

Rationals Q (3/2, 1/2, 1, -1, 3/100,…)

$\subset$

Algebraic - roots of integer-coef polynomials - (1, -1/2, √3,³√2…)

$\cup$

Real numbers (1, -1/2, √3, ³√2, π, e, …)

Positive integers

Integers

Rationals Q (3/2, 1/2, 1, -1, 3/100,…)

Algebraic - roots of polynomials with integer coefficients - (1, -1/2, √3,³√2…)

Real numbers (1, -1/2, √3, ³√2, π, e, …)

Inner: $\mathbb{N}$ Positive integers

Next: **B** Integers \ A

Next: **C** Rationals \ $\mathbb{Z}$

Next: **D** Algebraic irrationals (e.g., $\sqrt{2}$, $\sqrt[3]{2}$)

Outer: **E** Transcendentals (π, e)

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset$ **Algebraic** $\subset \mathbb{R}$

In which of the following sets is:

i. 11/5

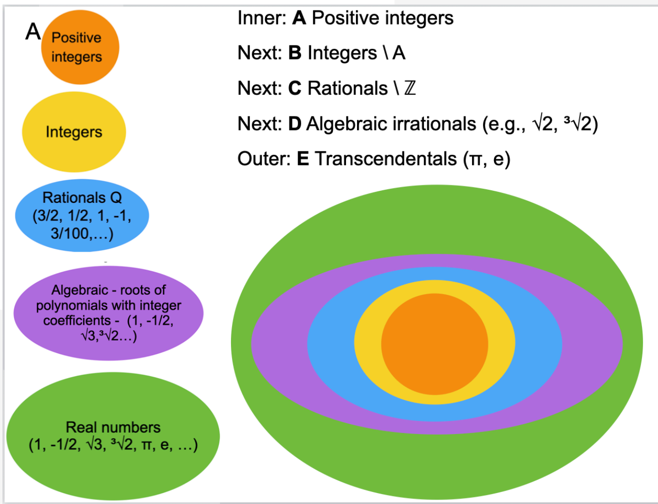
ii. $\sqrt{2}$

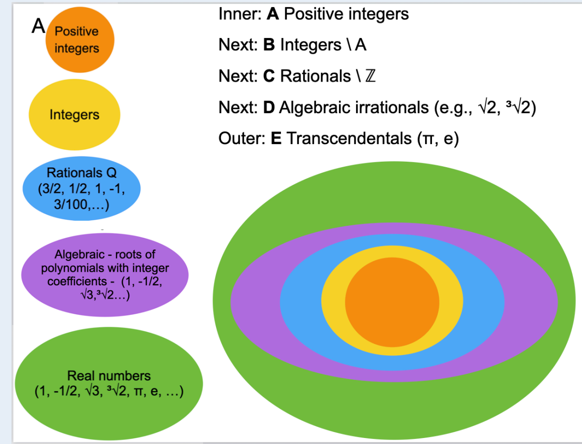
iii. E

iv. -1000

v. $2^{1/3}$

vi. -11/5



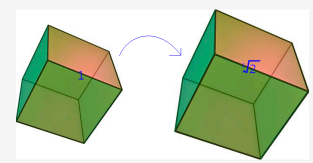


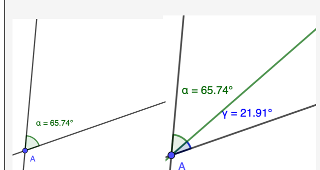
Write one number for each set ℕ, B, C, D, E. Write one number for each set N, B, C, D, E (no repeats). Add a one-line justification showing the evidence/test you used. Example: Example: 7 — "integer, ≥1."

Commensurability

Three "impossible" problems

Doubling a cube: Given a cube, **construct** another cube with twice its volume.





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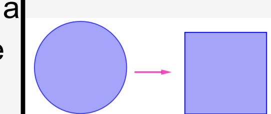


These problems originated around 400 BCE

Zeno (~450BCE) and Paradoxes

The android is the one in grey, wearing pajamas-like clothes.
Why do you think the android self-destructs at the end of the clip?
(Optional) What does this suggest about how the android's reasoning differs from human reasoning?

A clip from Star Trek

https://youtu.be/EzVxsYzXI_Y?si=gfZ16Y6C5nQYfjcL

Paradox

Greek
- **para**: distinct from
- **doxa**: opinion, belief.

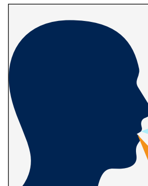
Colloquial

A statement contrary to common belief or expectation

Logical/Philosophical

A statement that starts from acceptable premises and follows sound reasoning, yet reaches a self-contradictory or illogical conclusion.

Which type of paradox destroyed the android?



I am a barber.
I shave anyone in my town who does not shave himself, and no one else.

Do I shave myself?

Barber paradox

Analogous to Russell's paradox

Analogous the paradoxes "*This sentence is false*" and the one in the Star Trek clip

Liar paradox

I am a liar

Is my previous sentence true of false?

Zeno’s Dichotomy paradox

> The first (argument) is the one which declares movement to be impossible because, however near the mobile is to any given point, it will always have to cover half, and then the half of that, and so on without limit before it gets there.
>
> Aristotle - Physics

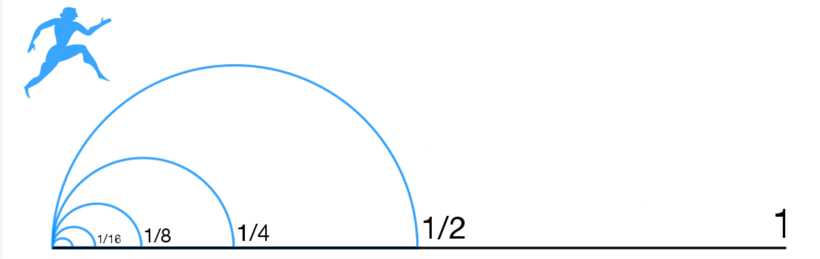
Read Aristotle’s explanation. Then discuss:
What has to happen every time you try to reach a point?
Why does this create a logical problem?

Zeno’s Dichotomy paradox

To go from A to B, you have to

- go midway to $A_{1/2}$,
- midway from A to $A_{1/2}$, $A_{1/4}$,
- and midway from A to $A_{1/4}$, $A_{1/8}$,
- and so on.

So you have to pass through infinite points in finite time, which is impossible.



How many Zenos does it take to change a light bulb?

Infinite. One to screw it in halfway, one to screw it in half of what's left, another to screw it in half of what remains, and so forth...

Zeno showing the Doors to Truth and False

Youths the Doors to Truth and False (Veritas et Falsitas) by Pellegrino Tibaldi

Zeno’s Achilles and the Tortoise paradox

Zeno's logic seems sound, but we know Achilles actually catches the tortoise.

Video from Open University https://www.youtube.com/watch?v=skM37PcZmWE

Zeno's logic seems sound, but we know Achilles actually catches the tortoise.

1 What assumption about space and time makes Zeno's argument work? (Hint: What does he keep dividing?)

2 How is the Achilles paradox similar to the Dichotomy paradox we discussed earlier?

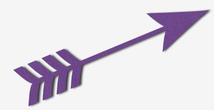
3: Zeno assumes infinite steps = infinite time. Why would ancient Greeks find this assumption convincing? Do you agree with Zeno? Why or why not?

Zeno’s Achilles and the Tortoise paradox

“In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead.”

Aristotle, Physics, VI:9

Zeno’s Arrow Paradox



- At any instant in time, an arrow in flight occupies a precise position in space.
- At that instant, it's not moving (motion requires time to pass).
- If it's motionless at every instant, when does it move?
- The paradox: If time is composed of instants, and the arrow is frozen at each instant, motion is impossible.

Is movement impossible?

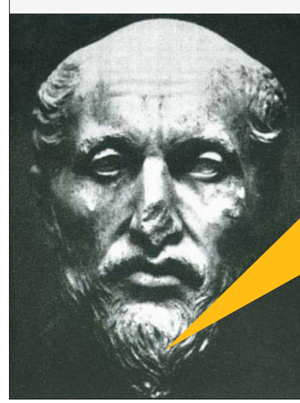
- *Nude descending a staircase*, by Marcel Duchamp 1912
- Provoked an scandal when first shown.
- The object at each moment is at a fixed position (according to Zeno at rest) how can it then move?

Why are paradoxes important in mathematics? (Hint: What happens when mathematicians try to resolve a paradox? What might they discover or create in the process?)

Thales ~ 600 BC

About Thales de Miletus ~ 600 BC

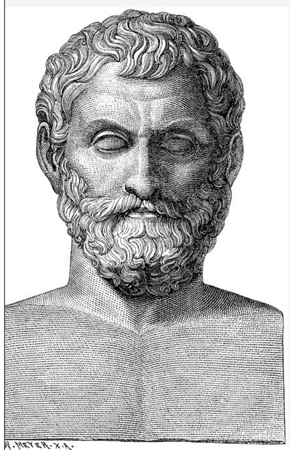
No primary sources



Hey, I am Proclus and I wrote:
"Thales was the first to go to Egypt and this study [geometry] bring back to Greece
He himself discovered many propositions and disclosed the underlying principles of many others to his successors, in some case his method being more general, in others more empirical..."

- *Proclus's Summary*
- ***written around 450 CE***
- *of Eudemus' History of Geometry*
- ***written around 320 BC***

Thales de Miletus ~ 600 BC



- We know very little.
- Earliest Greek mathematical investigations (and first proof!) that we know of
- According to Proclus
  - “*Thales travelled to Egypt and was the first to introduce mathematics into Greece”.*
  - was the first to demonstrate that
    - *“the circle is bisected its diameter”.*
    - *“A triangle inscribed on a circle, with a diameter as one of its sides is a right triangle.”*

No primary sources ☹️

ANSWER TWO OR MORE .

1. **Why the name stuck: The first extant proof is Euclid (~300 BCE), and the personal attribution to Pythagoras appears only with Proclus (5th c. CE). Explain why the label “Pythagorean theorem” nevertheless persisted.**
2. **Plato: Give one reason the Academy valued geometry and one trait Plato sought in students.**
3. **Aristotle: Define an axiom in plain language and state what it is used for**