

10101101

Number Systems

- The goal of these slides is to give an overview of ideas related number representation, so we can later understand better how different societies represented numbers.
- We will briefly discuss number systems in Egypt, Mesopotamia, Greece, China and the Mayans.

Summary of previous lectures

- All societies develop ideas of number.
- Counting as one-to-one correspondence
- Different ways of recording number (what do these ways depend on?)
- Formal definition of number is very hard (difficulty related to the barber paradox)
- **Primary sources**: A primary source is an original, firsthand, or direct piece of evidence or material that provides information about a particular topic or event.
- A **secondary source** is a document or material that is created based on information derived from primary sources. In academic research and historical analysis, secondary sources interpret, analyze, or comment on primary sources. They are **one or more steps** removed from the original events or materials and often involve synthesis, interpretation, or commentary by the author.
- The moment when the helper lowers their 10 fingers and the second helper lifts 1.

Image credit: https://creativekindergartenblog.com/one-to-one-correspondence-intervention-for-kindergarten/

It is crucial to use reliable sources of information.

From the Boston University Website https://www.bu.edu/africa/outreach/teachingresources/outreach-teachingresources-mathematics/

The Lebombo bone (top) is the oldest known mathematical artifact. It is a tally stick with 29 distinct notches that were deliberately cut into a baboon's fibula. It was discovered within the Border Cave in the Lebombo Mountains of Eswatini. The Lebombo bone (bottom) resembles a calendar stick still used in Namibia. See more about these artifacts under "Other Resources" below.

Mathematical Treasure: Ishango Bone MAA website https://www.maa.org/press/periodicals/convergence/mathematical-treasure-ishango-bone

Number systems

Number systems

A ***number system*** consist on a set of symbols, called ***numerals***, and a set of ***rules*** for writing this numerals to represent ***numbers***.

Examples

number system	*numerals*	*numbers*	*rules*
Hindu-Arabic ("ours")	1,2,3,4,5,67,8,9,0	71, 7,	
Roman	V, I, L...	VII	
Binary	0,1	1000, 1, 101,	

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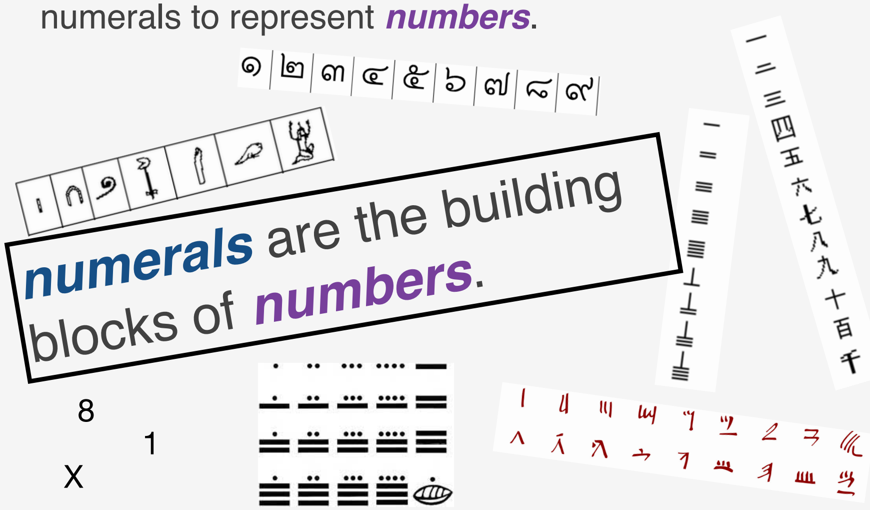
Examples

number system	*numerals*	*numbers*	*rules*
Hindu-Arabic ("ours")			
Roman			
Binary			

Give examples of numeral, numbers and rules.

Number systems

A ***number system*** consist on a set of symbols, called ***numerals***, and a set of ***rules*** for writing this numerals to represent ***numbers***.



Number systems

A ***number system*** consist on a set of symbols, called ***numerals***, and a set of ***rules*** for writing this numerals to represent ***numbers***.

We are going to consider four characteristics of number systems

- **Additive**
- **Ciphered or alphabetic**
- **Multiplicative**
- **Positional**

Note:
characteristics ≠ classification:

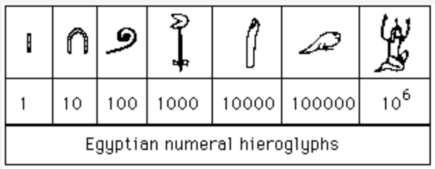
Additive number systems

Number systems

A ***number system*** consist on a set of symbols, called ***numerals***, and a set of ***rules*** for writing this numerals to represent ***numbers***.

We are going to consider four characteristics of number systems

- **Additive:** The value of a number is the sum of the values of the numerals.
- **Ciphered or alphabetic**
- **Multiplicative**
- **Positional**



Egyptian Additive Number System

Additive: The value of a number is the sum of the values of the numerals.

Numeral	Hieroglyph	Meaning
1	|	A vertical stroke
10	∩	A cattle hobble, (device used to tie animals' legs)
100		A coiled rope
1000		A lotus flower
10000		A bent finger
100000		A tadpole or frog
1000000		A figure with raised arms, sometimes interpreted as a god or a person marveling at the large number,

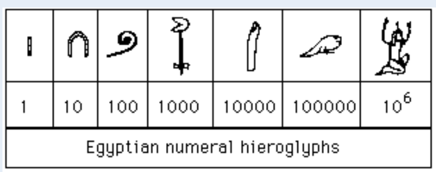
Design found in several places in Africa, for instance carved in wooden doors in Nigeria. in fabrication of baskets in Egypt Also in burial sites in **Ancient Egypt.**

Images credits: https://mathshistory.st-andrews.ac.uk/HistTopics/Egyptian_numerals/

An additive number system: Egyptian Hieroglyphs

numerals:

- based on a **scale** of 10
- used as far back as 3400 B.C.E.
- mostly for inscription in stones



1. Write the number 752 in Egyptian hieroglyphics.

2 Express number on the left in Hindu-Arabic numerals.

Images credits: https://mathshistory.st-andrews.ac.uk/HistTopics/Egyptian_numerals/

An additive number system: Egyptian Hieroglyphs

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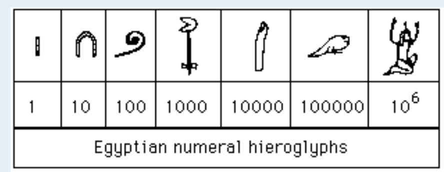
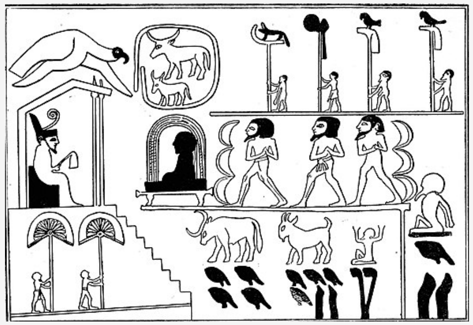


Example

Educated guess: what are the rules of the Egyptian hieroglyphic system?
Hint: One rule is related to the number of times a numeral can be repeated.

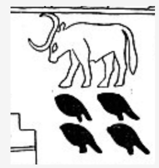
Ceremony in which captives and plunder are presented to Egyptian King Narmer (c. 31st century BCE)



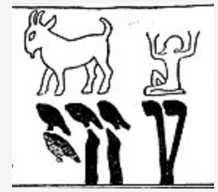
Narmer Macehead (drawing). The design shows captives being presented to Pharaoh Narmer enthroned in a naos. Ashmolean Museum, Oxford.

Decipher with your team 1, 2 and 3 (on the left). Each member of the team writes down their answer individually. You have 7 minutes.

2



1



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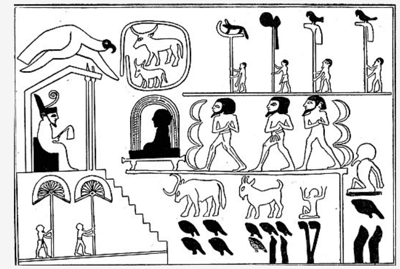
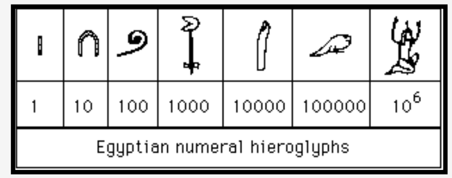
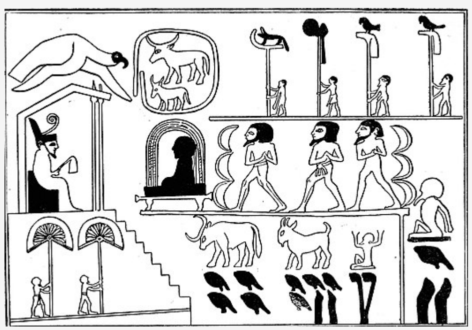


Narmer Macehead (drawing). The design shows captives being presented to Pharaoh Narmer enthroned in a naos. Ashmolean Museum, Oxford.



Ceremony in which captives and plunder are presented to Egyptian King Narmer (c. 31st century BCE)

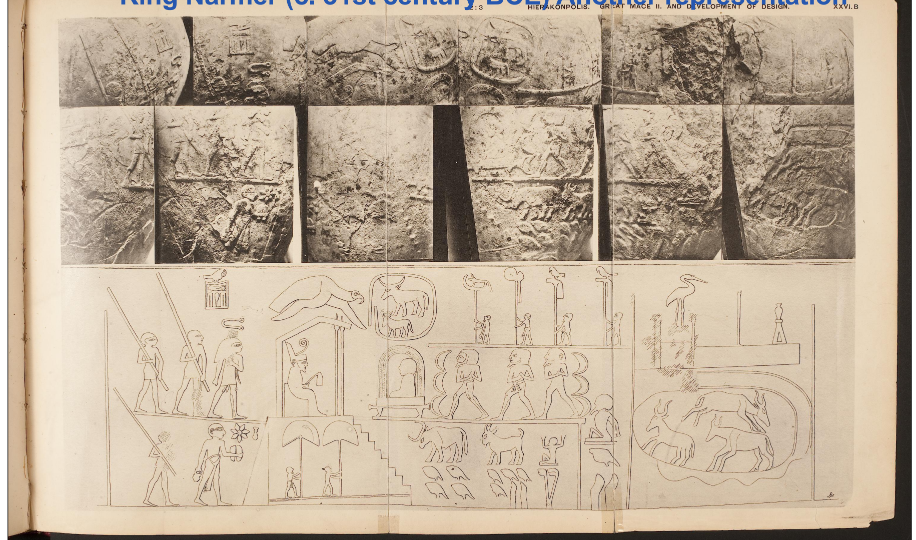
Here is a good picture of this source
https://digi.ub.uni-heidelberg.de/diglit/quibell1900bd1/0038/image



Narmer Macehead (drawing). The design shows captives being presented to Pharaoh Narmer enthroned in a naos. Ashmolean Museum, Oxford.

The scene depicts a ceremony in which captives and plunder are presented to King Narmer, who is enthroned beneath a canopy on a stepped platform. He wears the Red Crown of Lower Egypt, holds a flail, and is wrapped in a long cloak. To the left, Narmer's name is written inside a representation of the palace facade (the *serekh*) surmounted by a falcon. At the bottom is a record of animal and human plunder; 400,000 cattle, 1,422,000 goats, and 120,000 captives

Ceremony in which captives and plunder are presented to Egyptian King Narmer (c. 31st century BCE) Another representation



An additive system invented by your instructor

value	1	5	25	125
numerals	a	b	c	d

1. Express abbcdd in Hindu-Arabic numerals.
2. Express 106 in this additive system

Rules:
- Numerals are written from left to right, from the numeral with smallest value to the numeral with largest value. (abbcdd)
- The number of numerals used must be the smallest possible (for instance, we should write "b" instead of "aaaaa")

Ciphered or alphabetic number systems

Number systems

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A number system can be.

- **Additive**:
- **Ciphered or alphabetic**: Numerals design 1, 2,..9, and the powers of 10 (or, more generally, some base) but also to the multiples of this powers. Example: Greek Alphabetic
- **Multiplicative**
- **Positional**:

Letter	Value	Letter	Value	Letter	Value
α alpha	1	ι iota	10	ρ rho	100
β beta	2	κ kappa	20	σ sigma	200
γ gamma	3	λ lambda	30	τ tau	300
δ delta	4	μ mu	40	υ upsilon	400
ε epsilon	5	ν nu	50	φ phi	500
ϛ digamma	6	ξ xi	60	χ chi	600
ζ zeta	7	ο omicron	70	ψ psi	700
η eta	8	π pi	80	ω omega	800
θ theta	9	Q koppa	90	ϡ sampi	900

Table from https://online.math.uh.edu/Math2303-unpaid/ch1/s12/index.html

A ciphered number system: Greek Alphabetic

Rule: Numeral in ascending value, from right to left. Repetitions?

1. Write the number 752 in Greek numerals
2. Translate σπγ to Hindu-Arabic.

Numerals

Letter	Value	Letter	Value	Letter	Value
α alpha	1	ι iota	10	ρ rho	100
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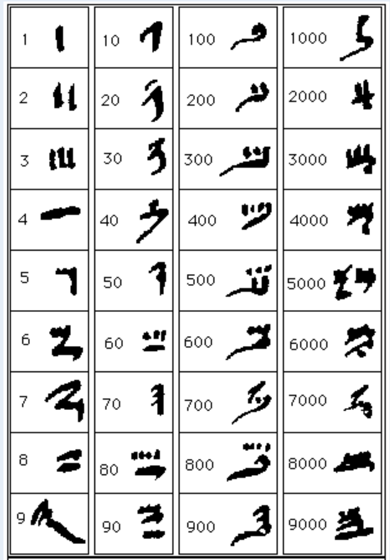
Hieratic script is the cursive form of hieroglyphic. It was used for administrative and literary purposes.

Looking at the hieratic numerals below, which type of number system (additive, multiplicative, ciphered, positional) do they suggest? Explain your reasoning.



Hieratic numerals

https://mathshistory.st-andrews.ac.uk/HistTopics/Egyptian_numerals/

Greek alphabetic numerals

Letter	Value	Letter	Value	Letter	Value
α alpha	1	ι iota	10	ρ rho	100
β beta	2	κ kappa	20	σ sigma	200
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ζ zeta	7	ο omicron	70	ψ psi	700
η eta	8	π pi	80	ω omega	800
θ theta	9	Q koppa	90	sampi	900

Does the Greek alphabetic system count as additive? Explain why or why not.

Greek alphabetic system

- **Ciphered or alphabetic**: Numerals design 0, 1, and the powers of 10 (or, more generally, some base) but also to the multiples of this powers. Example: Greek Alphabetic

Letter	Value	Letter	Value	Letter	Value
α alpha	1	ι iota	10	ρ rho	100
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Alphabetic Greek: For the numbers 1000 to 9000, they wrote: ‘α,’β, ‘ɣ…’Ɵ (For instance, ’β represents 2000)

10000 was written $\overset{\alpha}{M}$

There were rules for numbers up to 640,000,and even larger

Multiplicative number systems

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A number system can be.

- **Additive**:
- **Ciphered or alphabetic**:
- **Multiplicative**: There are two sets of numerals, the elements of one set represent digits and the elements of the other set represent position. If necessary, a digit and a position symbols are used together, and the values of numerals are multiplied. Finally, all the products are added.
- **Positional**

Number	Symbol
0	零
1	一
2	二
3	三
4	四
5	五
6	六
7	七
8	八
9	九
10	十
100	百
1000	千

A multiplicative system

Traditional Chinese numerals

Write the numbers below (in traditional Chinese numerals) in Hindu-Arabic numerals.

(a) 八十三 (b) 四百七十 (c) 二萬九千五 (d) 五千六百三十四

Burton, David M. "The history of mathematics: An introduction." (1985)

1	一	10	十
2	二	100	百
3	三	1000	千
4	四	10,000	萬
5	五		
6	六		
7	七		
8	八		
9	九		

Burton, David M. "The history of mathematics: An introduction." (1985)

Positional number systems

Number systems

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Four characteristics of number systems

- **Additive:**
- **Ciphered or alphabetic**
- **Multiplicative**
- **Positional:** The value of each numeral depends on its position. The system consists of a **base** (a natural number greater than one) and a **set of numerals** representing the numbers from zero to one less than the base. The numbers from zero to the base minus one are the digits in the system.

Example

$345 = 3.\ 10^2 + 4.10 + 5$

$5 = (101)_2 = 1.\ 2^2 + 0.2^1 + 1.2^0$

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- **Multiplicative**
- **Positional**

Note:
characteristics ≠ classification:

Examples of a Positional Systems Around the World

- **Binary**
- **Hindu-Arabic ("ours")**
- **Mayan**
- **Babilonian (Mesopotamian)**
- **Chinese Rod Number System (different from the Traditional Chinese number system we discussed before)**

- **Positional**: The value of each numeral depends on its position. The system consists of a base (a natural number greater than one) and a set of numerals representing the numbers from zero to one less than the base. The numbers from zero to the base minus one are the digits in the system.

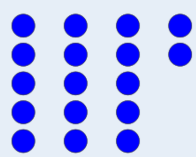
Important statement for Positional Number Systems

Integer division:

Given two integers **a** and **b**, with b > 0, there exist unique integers **q** and **r** such that **a = b.q + r** and 0 ≤ **r** < **b**

- a is called the *dividend*,
- b is called the *divisor*,
- q is called the *quotient,*
- r is called the *remainder.*

In this figure, a=17. What are the values of b, q, and r?



This statement answers the question: *What is the maximum number of times* **b** *"enters" into* **a**, *and what is remaining after this maximum number of* **b** *is subtracted from* **a**?

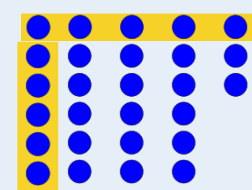
Note: The result works for a, b integers, b ≠ 0, but we will only work with positive numbers.

Important statement for Positional Number Systems

Integer division: Given two integers **a** and **b**, with b > 0, there exist unique integers **q** and **r** such that **a = b.q + r** and 0 ≤ **r** < **b**,

In this figure, a=27 (the total umber of blue dots). What are the values of b, q, and r?

- **a** is called the *dividend*,
- **b** is called the *divisor*,
- **q** is called the *quotient,*
- **r** is called the *remainder.*



This statement answers the question: *What is the maximum number of times* **b** *"enters" into* **a**, *and what is remaining after this maximum number of* **b** *is subtracted from* **a**?

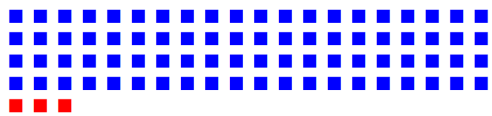
Note: The result works for a, b integers, b ≠ 0, but we will only work with positive numbers.

Division Algorithm

Theorem (Integer or Euclidean Division) For each pair a and b of integers, a positive there exists unique integers q and r such that

- $a = q.b+r$
- $0 \leq r < b$.

Example: If a=83, b=20, then q=4 and r=3



$83 = 4 \times 20 + 3$

From base 10 to base b≠10

We are given number N (in base 10).
Suppose N has the form $N=(u,v)_b$,
then we have N=u.b +v, with $0 \leq u, v < b$.
In this case, to write N in base b need to find u and v.

Find u and v when N = 100 and b = 11. Give your answer as an ordered pair (u,v)

Recall: Given two integers **a** and **b**, with b > 0, there exist unique integers **q** and **r** such that **a = b.q + r** and $0 \leq$ **r** < **b**

From a base b≠10 to base 10.

If $N=(18,6)_b$ then N=18.b +6
For instance, if the base b is 20, then
$N=(18,6)_{20}$ =18.20 +6 =376

Analogously, if $N=(15, 0, 10)_b$
then N=15. b^2 +0.b +10.

(N without parenthesis is assumed to be in base 10)

Find the quotient and remainder of dividing 445 by 20.
That is, determine q and r such that 445 = q·20 + r, where $0 \leq r < 20$.

Use your result to write 445 in base 20.

Recall: Given two integers **a** and **b**, with b > 0, there exist unique integers **q** and **r** such that **a = b.q + r** and $0 \leq$ **r** < **b**

Find the quotient and remainder of dividing 445 by 20.
That is, determine q and r such that $445 = q \cdot 20 + r$, where $0 \le r < 20$.

Use your result to write 445 in base 20.

$445 = 22 \cdot 20 + 5 = (1 \cdot 20+2)20+5 = 1 \cdot 20^2+2 \cdot 20+5$

Answer: $445 = (1,2,5)_{20}$.

Recall: Given two integers **a** and **b**, with $b > 0$, there exist unique integers **q** and **r** such that **a = b.q + r** and $0 \le r < b$

Express the number 752 in base 20.

Hint: Recall how we found that 445 can be written as $(1,2,5)_{20}$

Recall: Given two integers **a** and **b**, with $b > 0$, there exist unique integers **q** and **r** such that **a = b.q + r** and $0 \le r < b$

Express the number 752 in base 20.

Hint: Recall how we found that 445 can be written as $(1,2,5)_{20}$

$$752 = 37 \cdot 20 + 12$$
$$= (1 \cdot 20 + 17) \cdot 20 + 12$$
$$= 1 \cdot 20^2 + 17 \cdot 20 + 12$$

Answer: $752 = (1,17,12)_{20}$.

Recall: Given two integers **a** and **b**, with $b > 0$, there exist unique integers **q** and **r** such that **a = b.q + r** and $0 \le r < b$

Example: Express 445 in base 20.

1. Integer division of N by b:
   Find q and r such that:
   445 = 445 = 22 . 20 + 5 so q=22 and r=5
2. Check: Is $q < 20$ or $q \ge 20$?
   - **$q = 22 \ge 20$**, so continue dividing.
3. Integer division of q by b:
   Find q_1 and r_1 such that:
   22 = 1. 20 + 2
4. Check: Is $q_1 < 20$ or $q_1 \ge 20$?
   - **$q_1 = 1 < 20$**, so we are done.
5. Final representation in base 20:
   $N = (q_1, r_1, r)_{20}$
   $445 = (1, 2, 5)_{20}$

Expressing a Number N in Base b

1. Integer division of N by b:
   Find q and r such that:
   $N = q \cdot b + r$, where $0 \le r < b$.
2. Check: Is $q < b$ or $q \ge b$?
   - If $q < b$, you're done: $N = (q, r)_b$.
   - If $q \ge b$, continue dividing:
3. Integer division of q by b:
   Find q_1 and r_1 such that:
   $q = q_1 \cdot b + r_1$.
4. Check: Is $q_1 < b$ or $q_1 \ge b$?
   - If $q_1 < b$, you're done: $N = (q_1, r_1, r)_b$.
   - If $q_1 \ge b$, repeat the process.
5. Next slide

Expressing a Number N in Base b

5. Write the number in base b:
- Start with **N** and repeatedly replace each quotient **q** with its own quotient and remainder until the final quotient is smaller than **b**.
- You will obtain
  $N = q_n \cdot b^n + r_n \cdot b^{n-1} + r_{n-1} \cdot b^{n-2} + \ldots + r_1 \cdot b + r_0$.
- This process builds the number as
- $\mathbf{N = (q_n, r_n, r_{n-1}, \ldots, r_1, r_0)_b}$, where:
  - q_n is the last quotient (the first **q** that is smaller than **b**).
  - $r_0, r_1, \ldots, r_n$ are the remainders found at each step.

$(q_n, r_n, r_{n-1}, \ldots, r_1, r_0)_b$ represents N in base b.

Express the number 752 in base 20.

Hint: Recall how we found that 445 can be written as $(1,2,5)_{20}$

Recall: Given two integers **a** and **b**, with $b > 0$, there exist unique integers **q** and **r** such that $\mathbf{a = b.q + r}$ and $0 \le \mathbf{r} < \mathbf{b}$

Express the number 752 in base 20

To express **752** in base **20**, follow the steps:

1. Integer division of 752 by 20: $752 = 37 \cdot 20 + 12$.
   - $q=37$ and $r = 12$.
2. Check: Is $q < 20$ or $q \ge 20$?
   - $\mathbf{q = 37 \ge 20}$, so continue dividing.
3. Integer division of 37 by 20: $37 = 1 \cdot 20 + 17$.
   - $q_1=1$ and $r_1 = 17$.
4. Check: Is $q_1 < 20$ or $q_1 \ge 20$?
   - $\mathbf{q_1 = 1 < 20}$, so we are done.
5. Final representation in base 20: $752 = (1, 17, 12)_{20}$

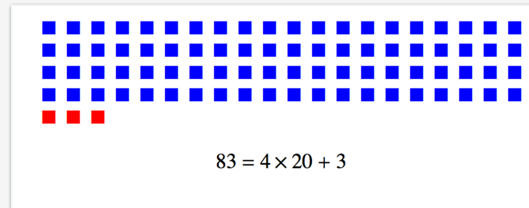
Express 20 in base 20.

Division Algorithm

Theorem (Integer or Euclidean Division) For each pair a and b of integers, a positive there exists unique integers q and r such that

- a = q.b+r
- $0 \leq r < b$.

Example: If a=83, b=20, then q=4 and r=3



To “translate”

From base 10 to base b≠10: integer division

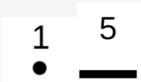
From base b≠10 to base 10: replace

Two examples of positional number systems: Maya and Babylonia

A positional system in base 20

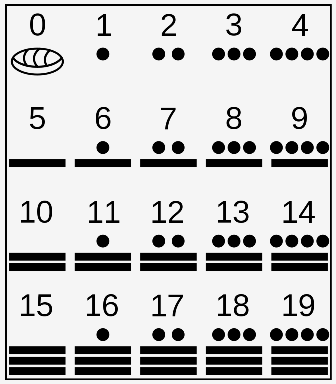
Mayan (in Mesoamerica)

Two special numerals

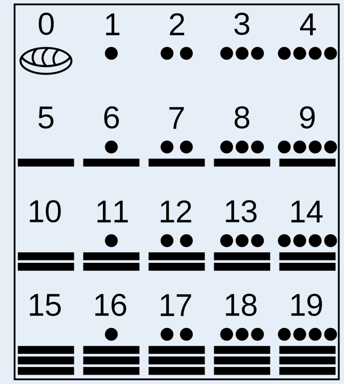


Most likely, these two numerals are from an older additive number system.

All the Mayan numerals



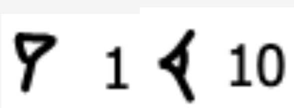
Express the number 752 Mayan number system.
Recall that 752 = $(1, 17, 12)_{20}$
Submit a photo of your answer.



A positional system in base 60

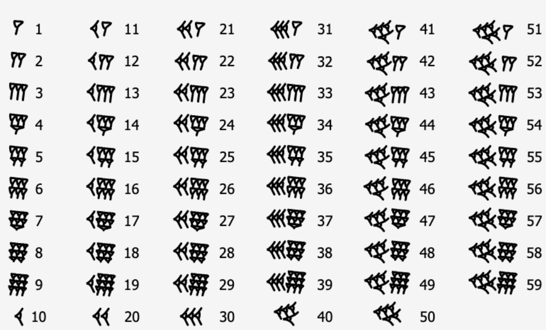
Mesopotamian

Two special numerals



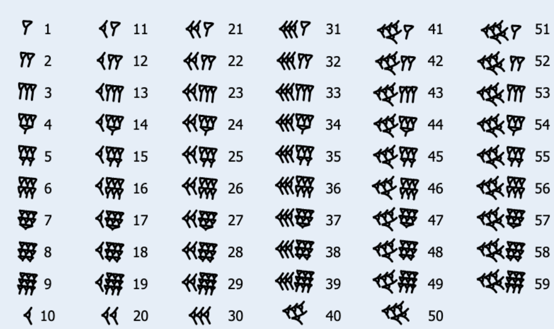
Most likely, these two numerals are from an older additive number system.

All the Mesopotamian numerals



Express the number 752 cuneoiform number system.
In Wooclap, express 752 in base 60.
For instance, 70 can be expressed as (1,10)_60

Numerals



Review and comparison

Note:
There are two different concepts, number and representation of number (in symbols or words).

Numbers and numerals are also different concepts.

Complete the table

Mayan	*Hindu-Arabic ("ours") and Rod numerals*	*Roman*	**Egyptian hieroglyphics**	**Babylonian Cuneiform**	**Traditional Chinese**	**Greek alphabetic**	**Moira's Multiplicative system**
	25						
		DCCXIX					
	45625						

Questions for Group Discussion:
Why do you think the Mayans used base 20?
Why do you think the Babylonians used base 60?

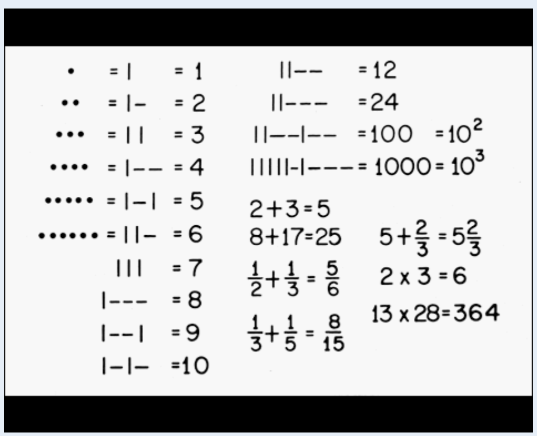
Is the Roman number system positional? Why or why not?

Pierre-Simon Laplace ~ 1800

The ingenious method of expressing every possible number using a set of ten symbols (each symbol having a place value and an absolute value) emerged in India. The idea seems so simple nowadays that its significance and profound importance is no longer appreciated. Its simplicity lies in the way it facilitated calculation and placed arithmetic foremost amongst useful inventions. the importance of this invention is more readily appreciated when one considers that it was beyond the two greatest men of Antiquity, Archimedes and Apollonius.

One of the images from the Golden Record launched in 1977is shown below.
Can you relate it to the topic we are studying, number systems?

Images on the Golden Record



Source: https://voyager.jpl.nasa.gov/galleries/images-on-the-golden-record/
Image credit: Francis Drake

For more info: https://voyager.jpl.nasa.gov/golden-record/

Goal: Write a number **a=0.205** between 0 and 1 in base 60. We want numbers c_1, c_2, c_3... such that

$$0.205 = c_1/60 + c_2/60^2 + c_3/60^3 + \ldots$$

1) Each c_i is an integer
2) $0 \le c_i < 60$

Step 1: $c_1 = \lfloor 60 \cdot 0.205 \rfloor = \lfloor 12.3 \rfloor = 12$

$a_1 = 60a - c_1 = 12.3 - 12 = 0.3$

Step 2: $c_2 = \lfloor 60 a_1 \rfloor = \lfloor 60 \cdot 0.3 \rfloor = \lfloor 18 \rfloor = 18$

$a_2 = 60 a_1 - c_2 = 60 \cdot 0.3 - 18 = 0$

Stop: If we reach i such that $a_i = 0$ the process ends.

Note: $\lfloor x \rfloor$ denotes the largest integer smaller or equal than x.

Goal: Write a number **a** between 0 and 1 in base 60. We want numbers c_1, c_2, c_3... such that

$$a = c_1/60 + c_2/60^2 + c_3/60^3 + \dots$$

10 Each c_i is an integer
2) $0 \le c_i < 60$

Step 1: $c_1 = \lfloor 60 \cdot a \rfloor$

$a_1 = 60a - c_1$

Step 2: $c_2 = \lfloor 60 a_1 \rfloor$

$a_2 = 60a_1 - c_2$

Step n: $c_n = \lfloor 60 a_{n-1} \rfloor$,

$a_n = 60a_{n-1} - c_n$)

Stop: If we reach i such that $a_i = 0$ the process ends.

Note: $\lfloor x \rfloor$ denotes the largest integer smaller or equal than x.

Convert these fractions into base 60 (sexagesimal). Then, show what the result would look like in cuneiform.
Answer in Wooclap using base-60 notation.
Upload a screenshot of your cuneiform work.

A. 1/20
B. 21/80
C. (Optional) 1/7