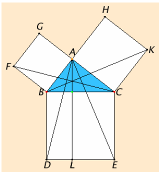
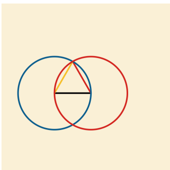
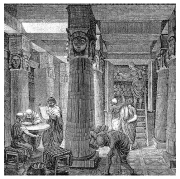
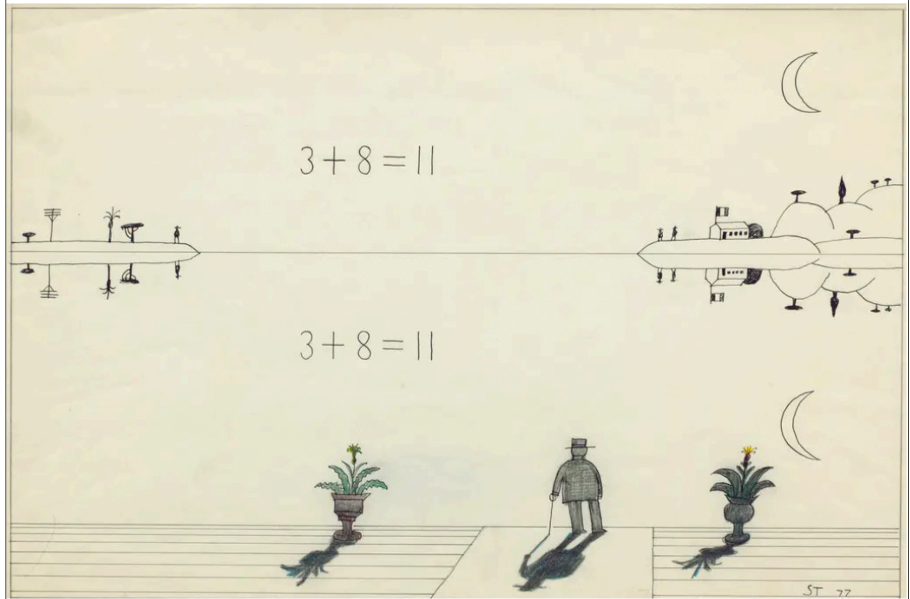


Euclid's Elements



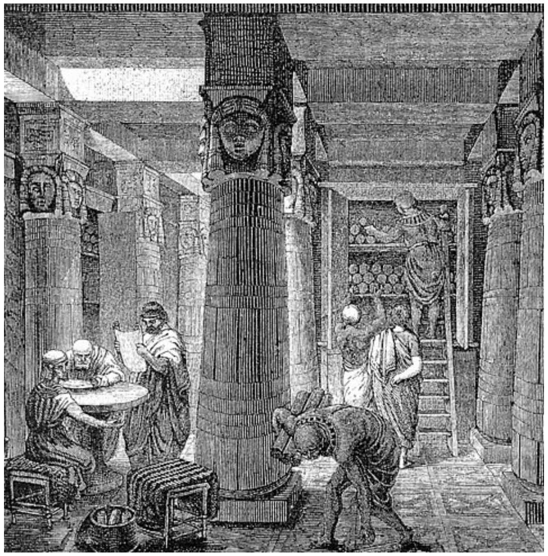
Steinberg

Alexandria

Map of Ancient Hellenic Mathematics Encyclopedia Britannica

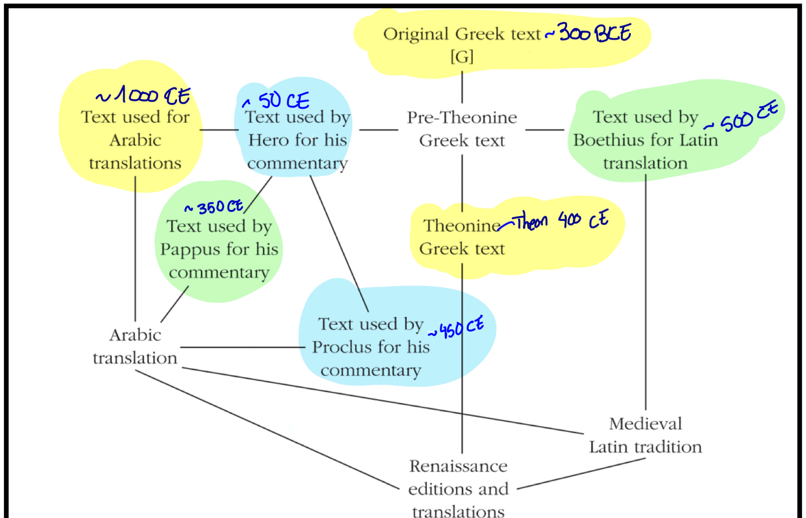
It is likely that Euclid worked and taught there

About 500,000 volumes



Nineteenth-century artistic rendering of the Library of Alexandria by the German artist O. Von Corven, based partially on the archaeological evidence available at that time

Euclid's Elements Early impact



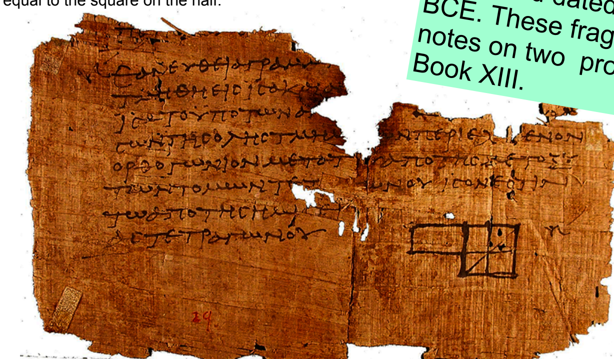
The medieval and Renaissance tradition of Euclid's *Elements* (adapted from Murdoch (1971))

Diagram from the book "Ancient Mathematics" by Serafina Cuomo

One of the Oldest Fragment of Euclid's Elements, dated from 1st century CE,

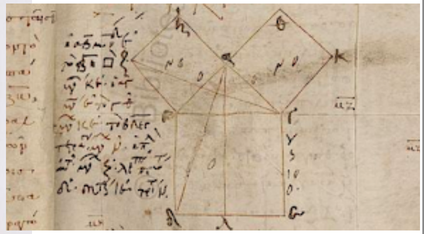
Proposition II.5: If a straight line is cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole, together with the square on the straight line between the points of the section, is equal to the square on the half.

Also there are fragments found in potsherds discovered in Egypt and dated from 225 BCE. These fragments contain notes on two propositions from Book XIII.



http://www.math.ubc.ca/~cass/Euclid/papyrus/

The 9th-century Vatican manuscript, Vat. gr. 190 Euclid's work possibly without major adulteration (in particular, without Theon's note).



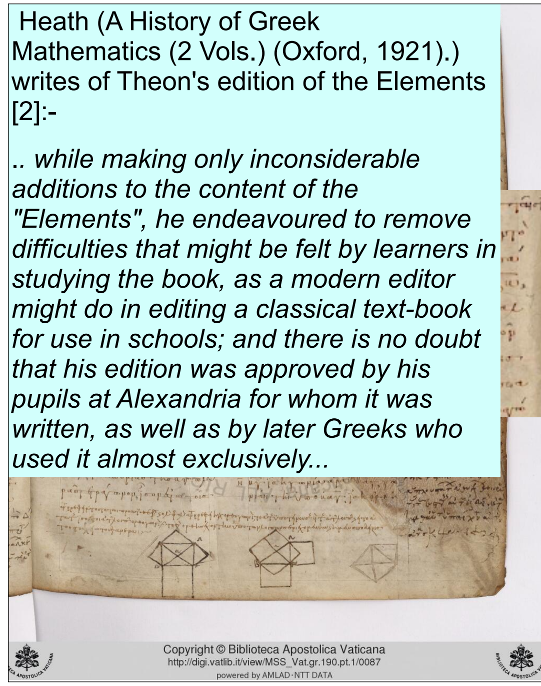
Book I proposition 47

The Pythagorean Theorem

https://digi.vatlib.it/view/MSS_Vat.gr.190.pt.1/0087

Heath (A History of Greek Mathematics (2 Vols.) (Oxford, 1921).) writes of Theon's edition of the Elements [2]:-

.. while making only inconsiderable additions to the content of the "Elements", he endeavoured to remove difficulties that might be felt by learners in studying the book, as a modern editor might do in editing a classical text-book for use in schools; and there is no doubt that his edition was approved by his pupils at Alexandria for whom it was written, as well as by later Greeks who used it almost exclusively...



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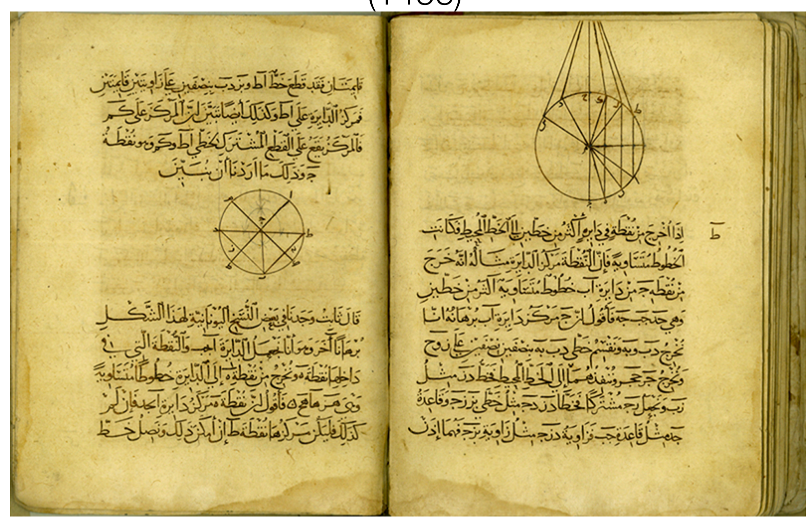
Book I proposition 47

The Pythagorean Theorem

https://digi.vatlib.it/view/MSS_Vat.gr.190.pt.1/0087

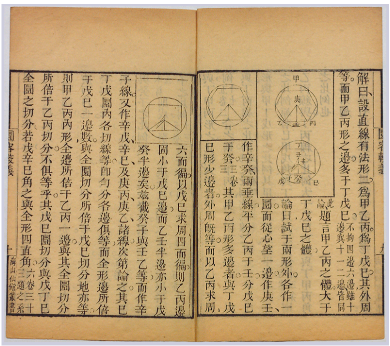
An illumination from a manuscript based on Adelard of Bath's translation of the *Elements*, *circa* 1309–1316; Adelard's is the oldest surviving translation of the *Elements* into Latin, done in the 12th-century work and translated from Arabic. (Wikipedia)

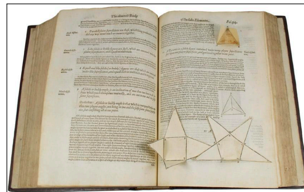
Early Translation of Euclid's Elements into Arabic (1466)

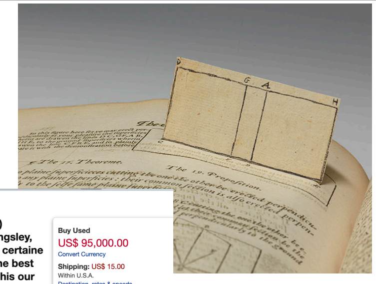


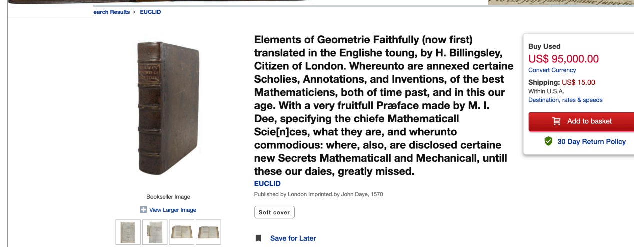
https://www.maa.org/press/periodicals/convergence/mathematical-treasure-early-translation-of-euclids-elements-into-arabic

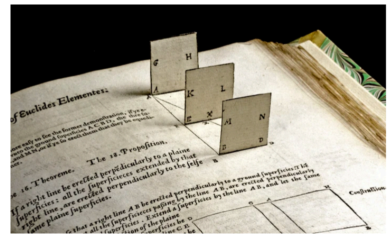
The Italian Jesuit Matteo Ricci (left) and the Chinese mathematician Xu Guangqi (right) published the Chinese edition of Euclid's Elements (幾何原本) in 1607. (Wikipedia)











https://www.c82.net/euclid/

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The First Six Books of the Elements of Euclid in which coloured diagrams and symbols are used instead of letters for the greater ease of learners. By Oliver Byrne.

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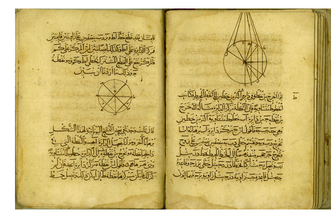
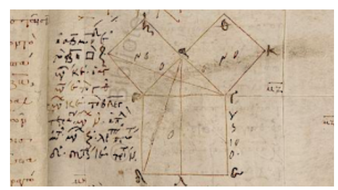
Euclid's element's

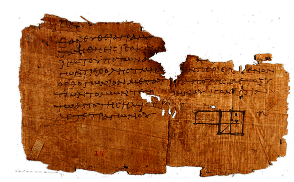




The 9th-century Vatican manuscript, Vat. gr. 190 Euclid's work possibly without major adulteration (in particular, without Theon's note).

were not created overnight

were not created by Euclid alone.



Euclid's Elements

- Definitions
- Postulates and common notions
- Propositions and constructions

"Euclid alone has looked on Beauty bare."

BY EDNA ST. VINCENT MILLAY

Euclid alone has looked on Beauty bare.
Let all who prate of Beauty hold their peace,
And lay them prone upon the earth and cease
To ponder on themselves, the while they stare
At nothing, intricately drawn nowhere
In shapes of shifting lineage; let geese
Gabble and hiss, but heroes seek release
From dusty bondage into luminous air.
O blinding hour, O holy, terrible day,
When first the shaft into his vision shone
Of light anatomized! Euclid alone
Has looked on Beauty bare. Fortunate they
Who, though once only and then but far away,
Have heard her massive sandal set on stone.

https://www.poetryfoundation.org/poems/148566/euclid-alone-has-looked-on-beauty-bare

Definitions

Definition 1.
A ***point*** is that which has no part.

Definition 2.
A ***line*** is breadthless length.

Definition 3.
The ***ends*** of a line are points.

Definition 4.
A ***straight line*** is a line which lies evenly with the points on itself.

First four definitions of Book 1 of Euclid's Elements.

Order the four definitions from easiest to understand to hardest. Hint: If you didn't know what the objects being defined are, would you be able to draw them just from the definition? Why?

Definition 1: A **point** is that which has no part.

Definition 2: A **line** is breadthless length.

Definition 10: When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right**, and the straight line standing on the other is called a **perpendicular** to that on which it stands.

Definition 15: A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

- It has been suggested that the definitions were added to the Elements sometime after Euclid wrote them.
- Another possibility is that they are actually from a different work, perhaps older.

Axioms, postulates and common notions

Common notions

- Things equal to the same thing are also equal to one another.
- And if equal things are added to equal things then the wholes are equal.
- And if equal things are subtracted from equal things then the remainders are equal.
- And things coinciding with one another are equal to one another.
- And the whole [is] greater than the part

The Greek text of J.L. Heiberg translated by Richard Fitzpatrick https://farside.ph.utexas.edu/Books/Euclid/Elements.pdf

Postulates

- Let it have been postulated to draw a straight-line from any point to any point.
- And to produce a finite straight-line continuously in a straight-line.
- And to draw a circle with any center and radius.
- And that all right-angles are equal to one another.
- And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).

The Greek text of J.L. Heiberg translated by Richard Fitzpatrick https://farside.ph.utexas.edu/Books/Euclid/Elements.pdf

Euclid’s five postulates

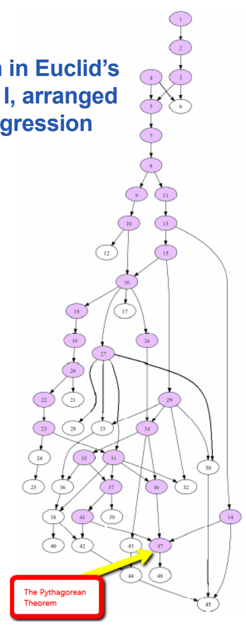
1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines intersect each other on that side if extended far enough.

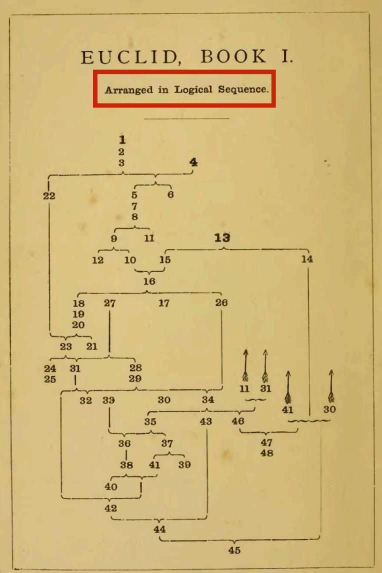
The first four postulates were straightforward; they were considered "truths". (Recall Aristotle's definition: "An axiom is a statement worthy of acceptance.")

The fifth postulate isn't simple or straightforward; it feels more like a proposition or theorem than an axiom.

. Eyer's graph of Euclid's Book I
https://www.maa.org/book/export/html/590371

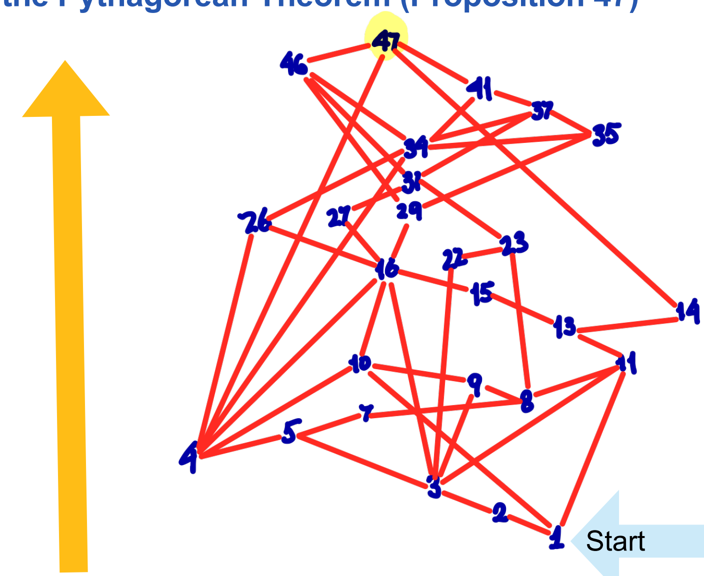
The proposition in Euclid’s Elements Book I, arranged in logical progression





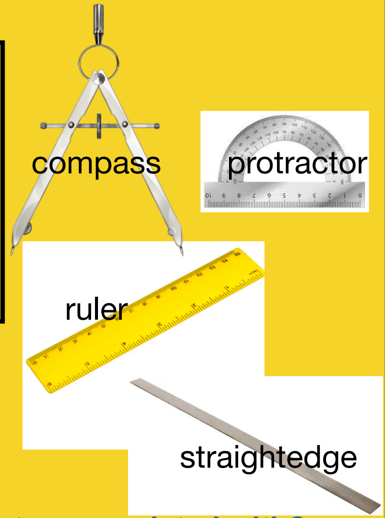
From Euclid and His Modern Rivals By Lewis Carroll

The structure of the proportions in Euclid’s Elements leading to the Pythagorean Theorem (Proposition 47)



The first three postulates in the Elements

- Postulate I asserts that it is possible to draw a segment through any two given points.
- Postulate II says that any segment can be extended to a longer segment.
- Postulate III states that it is possible to construct a circle with any given center and radius.



These first three postulates are associated with the tools that are used to implement them on a piece of paper.

Which tool is each of these postulates associated with? (Select the option with only the essential features)

Definition 15: A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

Postulate 4 states "*That all right angles are equal one another.*" Isn't it true that all right angles are equal to each other? Why does it have to be a postulate? In other words, why it cannot be assumed?

Postulate 4.
That all right angles equal one another.

Postulate 5'. (Playfair)
For any given point not on a given line, there is exactly one line through the point that does not meet the given line.

Byrne's version

If two ſtraight lines ( ) meet a third ſtraight line ( ) ſo as to make the two interior angles (and) on the ſame ſide leſs than two right angles, theſe two ſtraight lines will meet if they be produced on that ſide on which the angles are leſs than two right angles.

Postulate 5.
That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Euclid's Elements

Postulates

1. Let it have been postulated† to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).‡

Common Notions

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.†
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part.

- Each book starts with a list of **definitions** of the mathematical terms which will be used in that book.
- After the definitions in Book I there are five "**postulates**" and five "**common notions**."
- Most of Euclid's postulates are simple statements of *intuitively obvious* facts about plane and space.
- The common notions are also intuitively obvious facts, most of them about properties of equality. These common notions are not specific to geometry

We are going to refer to the whole set of assumptions (postulates together with common notions) as **axioms**

In general, an *axiom* is a statement which is regarded as being accepted, or self-evidently true.

You
what is an axiom?

ChatGPT
An axiom is a statement considered self-evidently true without needing proof.

You
Is an axiom true in mathematics?

ChatGPT
Yes, an axiom is considered true within the context of mathematics.

You
NO!

ChatGPT
Yes, that's correct. My apologies for the oversight. Axioms are assumptions or postulates that are accepted as true without requiring proof within a particular mathematical system or framework. They serve as the foundation upon which mathematical theories are built. Thank you for the correction!

Euclid's elements	Today's math
Definitions	**Undefined** terms Definitions
Common notions Postulates **(self evident truths)**	Axioms **(assumptions wisely chosen)**
Propositions Constructions	Propositions Constructions

Postulate 1.
To draw a straight line from any point to any point.

Postulate 2.
To produce a finite straight line continuously in a straight line.

Postulate 3.
To describe a circle with any center and radius.

Postulate 4.
That all right angles equal one another.

Postulate 5.
That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Compare the Postulates and the Common Notions, how do they differ? What do the Postulates have in common with each other? What do the Common Notions have in common with each other?

Common notion 1.
Things which equal the same thing also equal one another.

Common notion 2.
If equals are added to equals, then the wholes are equal.

Common notion 3.
If equals are subtracted from equals, then the remainders are equal.

Common notion 4.
Things which coincide with one another equal one another.

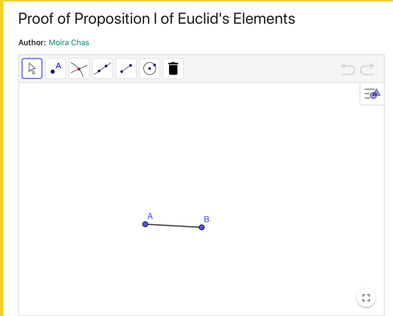
Common notion 5.
The whole is greater than the part.

Recall that Euclid's Elements start with ten axioms (five postulates and five common notions). Ten is not a large number. Is that important? Would it be better to have less or more axioms? Explain the reasons for our answer.

Construction of an equilateral triangle

Note: In the proof of the propositions, I will separate statements by bullet points so we can isolate them and, hopefully, understand the proof better.

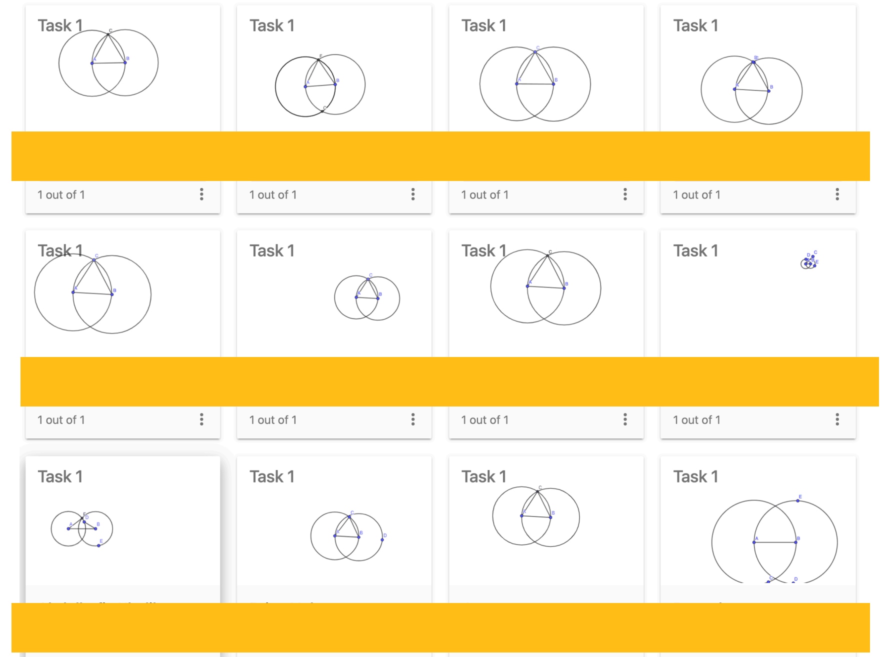
Let's concentrate in understand the steps of the proof first, and later, let's try to achieve a more global understanding.



https://www.geogebra.org/classroom/z6ecau5x

Geogebra Classroom Code. Z6EC AU5X

1. On the Geogebra window (above this text) make the construction of an equilateral triangle by following the instructions.
2. Explain why the triangle you constructed is equilateral (in complete sentences, in Wooclap)
3. (If you have time) Compare the instructions given in 1. with this one , which is a translation of the original Euclid's Element. What are the differences?)



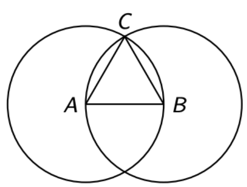
Are you sure that the two circles always intersect? Why?

Proposition 1: To construct an equilateral triangle on a given finite straight-line

First part of the proof: describe the construction.

1. Let AB be the given line.
2. Draw a circle, center A, distance AB. [Post. 3]
3. Draw a circle, center B, distance AB. [Post. 3]
4. Let C be a point where the circles intersect.
5. Draw straight lines AC and BC. [Post. 1]

ABC is the required triangle.



Modern version Of Euclid's Elements

Text and figure by Patrick Maher

Def 15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.

Postulates

1. Let it have been postulated† to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).‡

Common Notions

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.†
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part.

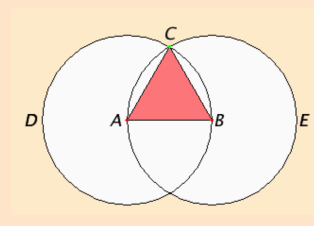
Translation by R. Fitzpatrick

Euclid's Elements

Proposition I:To construct an equilateral triangle on a given finite straight-line

Direct translation of the proof

Let AB be the given finite straight line.



It is required to construct an equilateral triangle on the straight line AB.

Describe the circle BCD with center A and radius AB. Again describe the circle ACE with center B and radius BA. Join the straight lines CA and CB from the point C at which the circles cut one another to the points A and B. I.Post.3 I.Post.1

Now, since the point A is the center of the circle CDB, therefore AC equals AB. Again, since the point B is the center of the circle CAE, therefore BC equals BA. I.Def.15

But AC was proved equal to AB, therefore each of the straight lines AC and BC equals AB.

And things which equal the same thing also equal one another, therefore AC also equals BC. C.N.1

Therefore the three straight lines AC, AB, and BC equal one another.

Therefore the triangle ABC is equilateral, and it has been constructed on the given finite straight line AB. I.Def.20

Q.E.F.

Text and figure by David Joyce https://mathcs.clarku.edu/~djoyce/elements/bookI/propI1.html

Postulates

1. Let it have been postulated† to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).‡

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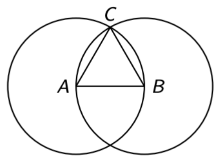
15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.

20. And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having three unequal sides.

Translation by R. Fitzpatrick

Euclid's Elements

Proposition I:To construct an equilateral triangle on a given finite straight-line



Second part: Prove that the construction does what was required.

1. $AB = AC$ [Def. 15]
2. $AB = BC$ [Def. 15]
3. $AC = BC$ [C.N. 1]
4. Hence ABC is an equilateral triangle [Def. 20] and it is on the given line AB. Q.E.F.

Text and figure above by Patrick Maher

Postulates

1. Let it have been postulated† to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
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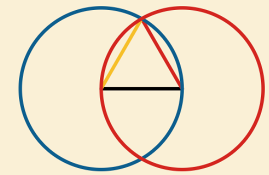
Translation by R. Fitzpatrick

Euclid's Elements

Proposition I:To construct an equilateral triangle on a given finite straight-line

PROPOSITION I. PROBLEM.

O*N a given finite ſtraight line* ( ) *to deſcribe an equilateral triangle.*



Deſcribe and (poſtulate 3.); draw and (poſt. 1.). then will be equilateral.

For = (def. 15.);
and = (def. 15.),
∴ = (axiom. 1.);

and therefore is the equilateral triangle required.

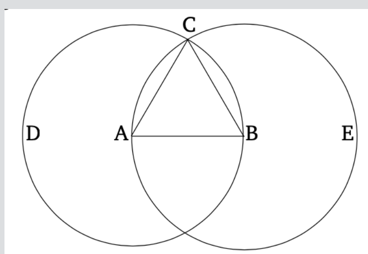
Q. E. D.

Byrne's Euclid

Euclid's Elements

Proposition I:To construct an equilateral triangle on a given finite straight-line

- Let AB be the given finite straight-line.
- So it is required to construct an equilateral triangle on the straight-line AB.
- Let the circle BCD with center A and radius AB have been drawn [Post. 3],
- Let the circle ACE with center B and radius BA have been drawn [Post. 3].
- And let the straight-lines C A and C B have been joined from the point C, where the circles cut one another,† to the points A and B (respectively) [Post. 1].



Postulates

1. Let it have been postulated† to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).‡

Common Notions

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.†
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part.

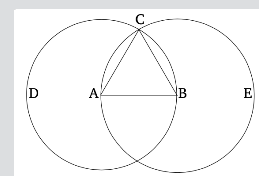
15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.

Translation by R. Fitzpatrick

Euclid's Elements

Proposition I:To construct an equilateral triangle on a given finite straight-line

- (CONT.) And since the point A is the center of the circle CDB, AC is equal to AB [Def. 1.15].
- Again, since the point B is the center of the circle CAE, BC is equal to BA [Def. 1.15].
- But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [C.N. 1].
- Thus, CA is also equal to CB. Thus, the three (straight- lines) CA, AB, and BC are equal to one another.
- Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do.



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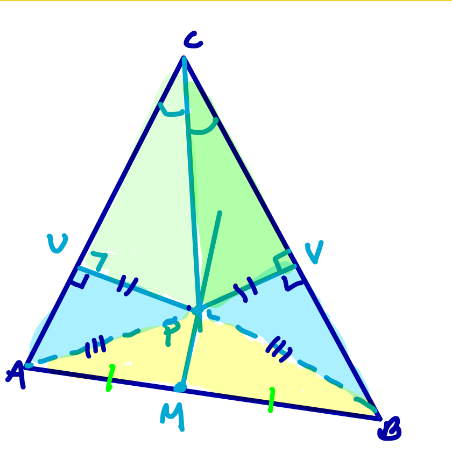
Translation by R. Fitzpatrick

Are all triangles isosceles?



Let M be the midpoint of AB.

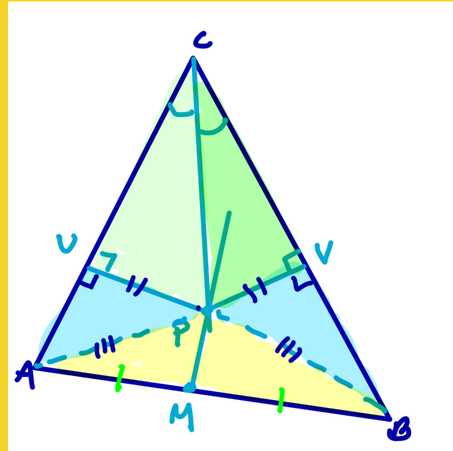
Let P be the intersection point of the perpendicular to AB through M and the angle bisector at C.

Let U be the perpendicular to AC through P.

Let V be the perpendicular to BC through P.

Claim: The following pairs triangles are congruent:

- AMP and BMP
- PCV and PCU
- APU and APV



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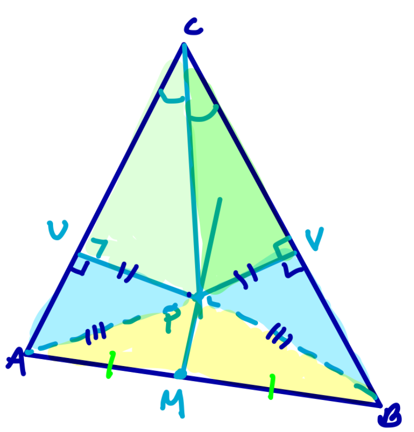

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Claim: The following pairs triangles are congruent:

- AMP and BMP (By SAS)
- PCV and PCU (By SAS)
- APU and APV (By ASA)

Let M be the midpoint of AB.

Let P be the intersection point of the perpendicular to AB through M and the angle bisector at C.

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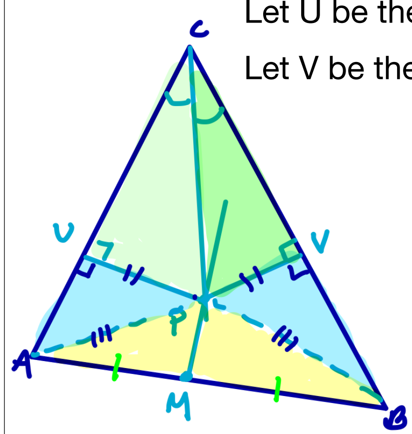
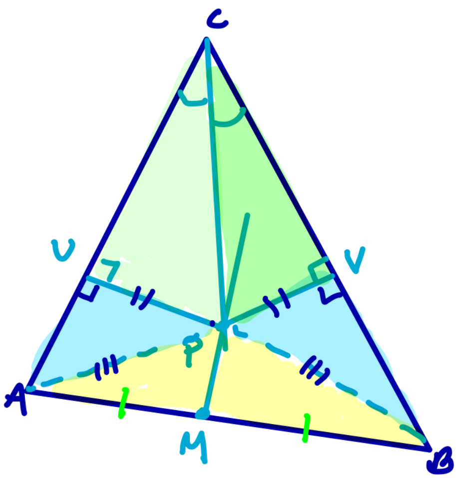
Let V be the perpendicular to BC through P.



Claim: The following pairs of triangles are congruent:

- PCV and PCU.
- AMP and BMP
- APU and APV

All triangles are isosceles?
Corollary. Every triangle is equilateral.
Proof. The previous argument proceeded from an arbitrary vertex of the triangle, and so any pair of adjacent sides in the triangle are congruent. So all three are congruent, and therefore it is equilateral.

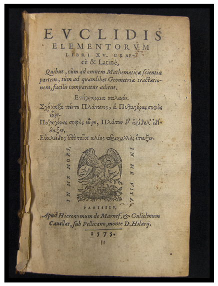


Tuesday March 19

- Annotated bibliography: https://www.math.stonybrook.edu/~moira/courses/mat336-sp2024/bib.html
- Review of last week

Review of last week

- Euclid ~ 300 BCE - Alexandria
- Euclid's Elements
  - product of many years of work of many mathematicians.
  - strong impact, still relevant on the math we study here and now.



Printed edition of Euclid's *Elements* in both Greek and Latin 1573/1574

Describe the most important aspects of Euclid's Elements.

Euclid's Elements

- A book about geometry written around 300 BCE, which played (and still does) a fundamental role in the development of mathematical thought.
- Organized into mathematical statements or theorems called propositions.
- Propositions are arranged in a logical progression, where each step is justified by earlier results and axioms.

Describe the most important aspects of Euclid's Elements.

Euclid's elements	Today's math
Definitions	**Undefined** terms Definitions
Common notions Postulates **(self evident truths)**	Axioms **(assumptions wisely chosen)**
Propositions Constructions	Propositions Constructions

In logical progression: Proposition N is proved using some of the following: postulates, common motions, definitions, and Proposition 1, 2,… Proposition N-1

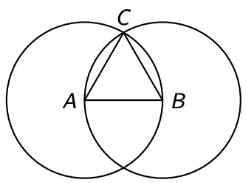
Proposition 1 of Euclid's Elements

Proposition 1: (*in modern language*) Given a line segment, it is possible to construct an equilateral triangle using the line segment as one of its sides.

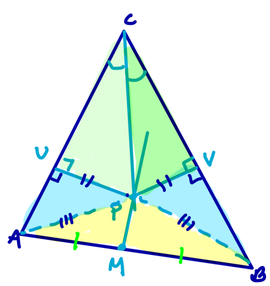
First part of the proof: describe the construction.

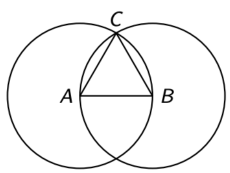
1. Let AB be the given line.
2. Draw a circle, center A, distance AB. [Post. 3]
3. Draw a circle, center B, distance AB. [Post. 3]
4. Let C be a point where the circles intersect.
5. Draw straight lines AC and BC. [Post. 1]

ABC is the required triangle.



What, if anything you deduce from this?





Second part: Prove that the construction does what was required.

6. $AB = AC$ [Def. 15]
7. $AB = BC$ [Def. 15]
8. $AC = BC$ [C.N. 1]
9. Hence ABC is an equilateral triangle [Def. 20] and it is on the given line AB. Q.E.F.

Proposition I:To construct an equilateral triangle on a given finite straight-line

On the left is the second part of the proof of proposition I, proving that the triangle is indeed equilateral.

Text and figure above by Patrick Maher

Postulates

1. Let it have been postulated† to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
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15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.

20. And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having three unequal sides.

Translation by R. Fitzpatrick

Book VII. Some Definitions

1. A ***unit*** is (that) according to which each existing (thing) is said (to be) one.

2. A ***number*** is a multitude composed of units.

3. A number is ***a part* of a number**, the less of the greater, when **it measures the greater;**

11. A ***prime number*** is that which is measured by a unit alone.

12. Numbers ***relatively prime*** are those which are measured by a unit alone as a common measure.

13. A ***composite number*** is that which is measured by some number.

Proposition IX.20 in Euclid's Elements states:

The (set of all) prime numbers is more numerous than any assigned multitude of prime numbers.

Express Proposition IX.20 in contemporary language

Euclid's Elements

Proposition IX.20: Prime numbers are more than any assigned multitude of prime numbers.

(Modern) Proof: Assume that p_1, p_2, and p_3 are prime.

Suppose that the primes A, B, C are 2, 3, and 5. Let N be as above and let G be the largest prime that divides N. What is G?

Euclid's Elements

This proposition is not used in the rest of the Elements.

Proposition IX.20: Prime numbers are more than any assigned multitude of prime numbers.

- Let *A, B,* and *C* be the assigned prime numbers.
- I say that there are more prime numbers than *A, B,* and *C.*
- Take the least number *DE* measured by *A, B,* and *C.* Add the unit *DF* to *DE.*
- Then *EF* is either prime or not.
- Let it, first of all, be prime. Thus, the (set of) prime numbers A, B, C, EF, (which is) more numerous than A, B, C, has been found.
- And so let EF not be prime.
- Thus, it is measured by some prime number [Prop. VII.31]. Let it be measured by the prime (number) G.

- I say that G is not the same as any of A, B, C. For, if possible, let it be (the same).
- And A, B, C (all) measure DE.
- Thus, G will also measure DE.
- And it also measures EF.
- (So) G will also measure the remainder, unit DF, (despite) being a number [Prop. 7.28]. The very thing (is) absurd. Thus, G is not the same as one of A, B, C.
- And it was assumed (to be) prime. Thus, the (set of) prime numbers A, B, C, G, (which is) more numerous than the assigned multitude (of prime numbers), A, B, C, has been found. (Which is) the very thing it was required to show.

Euclid's Elements

This proposition is not used in the rest of the Elements.

Consider the primes A=7, B=11 and C=13. Let E =A.B.C+1 and let G be the smallest prime that divides E. What is G?

Table 3

P1	P2	P3	P1.P2.P3+1	FACTORS	FACTORS	FACTORS	FACTORS	FACTORS
2	3	5	31	31				
3	5	7	106	2	53			
5	7	11	386	2	193			
7	11	13	1002	2	3	167		
11	13	17	2432	2	19			
13	17	19	4200	2	3	5	7	
17	19	23	7430	2	5	743		
19	23	29	12674	2	6337			
23	29	31	20678	2	7	211		
29	31	37	33264	2	3	7	11	
31	37	41	47028	2	3	3919		
37	41	43	65232	2	3	151		
41	43	47	82862	2	13	3187		
43	47	53	107114	2	7	1093		
47	53	59	146970	2	3	5	23	71
53	59	61	190748	2	43	1109		

Euclid's Elements: What is an axiom system?

Full length portraits of Euclid and Ptolemy, each holding a compass; a sphere on a staff is between them.
- Cardano, Girolamo, 1501-1576,

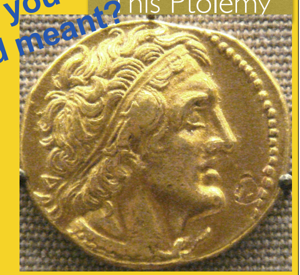
According to Proclus, Ptolemy (the pharaoh) once asked Euclid if there was not a shorter road to the knowledge of geometry than by the study of the Elements, and Euclid replied:

There is no royal road to geometry.

What do you think Euclid meant?

The Philosophers (Ptolemy and Euclid with Their Pupils) Pietro della Vecchia - 1600

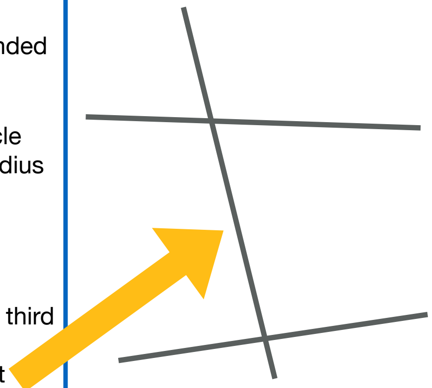
This Ptolemy



Tetradrachm of Ptolemy I, British Museum, London

Euclid's five postulates

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight lines segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines intersect each other on that side if extended far enough.



Proof of Pythagorean Theorem

What is the statement of the Pythagorean Theorem?

What is the statement of the Pythagorean Theorem?

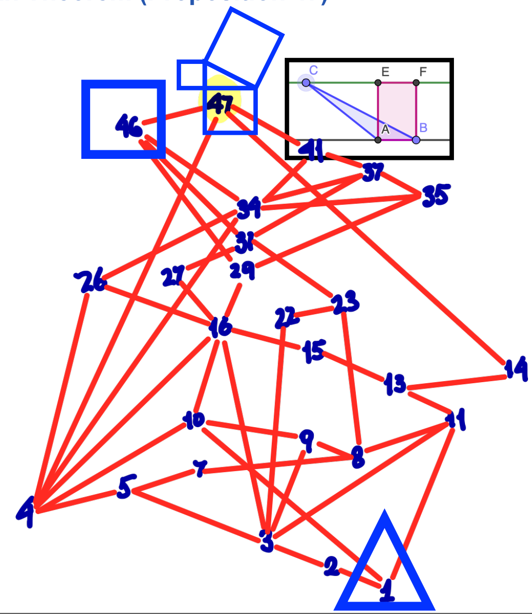
In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

Where does the first axiomatic proof of the Pythagorean Theorem we know of appear?

1 Babylonian clay tablets	5 Plato's writings
2 Egyptian papyrus	6 Euclid's Elements
3 Indian sutras	7 The Nine Chapters of the Mathematical Arts (A a classical Chinese mathematics book, dating back to around 200 BC)
4 Pythagoras writings	8 None of the above.

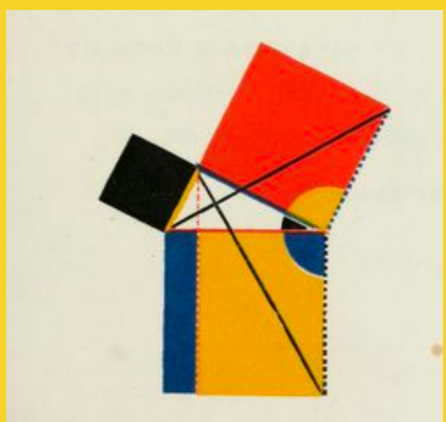
The structure of the proportions in Euclid's Elements leading to the Pythagorean Theorem (Proposition 47)

The proposition in Euclid's Elements Book I, arranged in **logical progression**

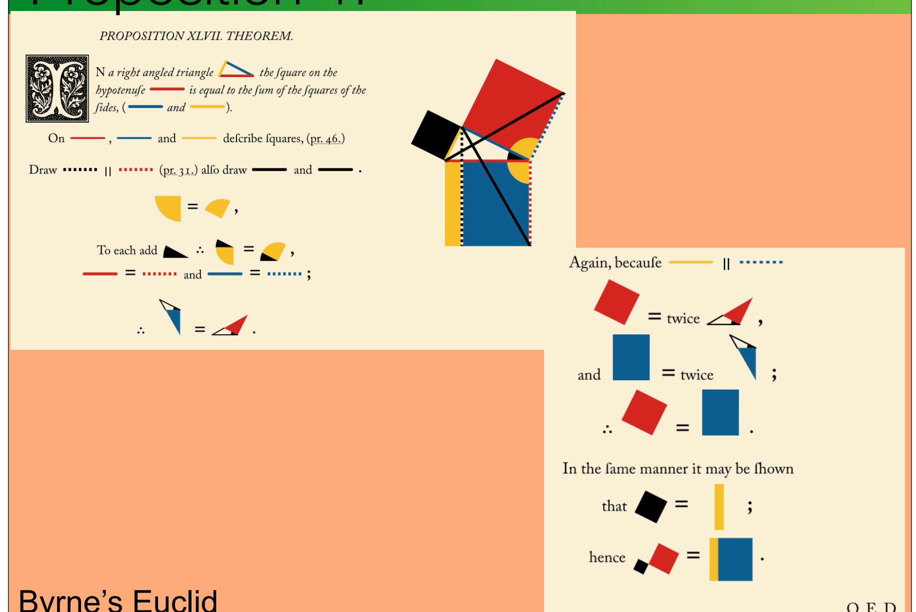


Describe the the proof of the Pythagorean theorem we just discussed.

Later



Proposition 47



PROPOSITION XLVII. THEOREM.

IN a right angled triangle the fquare on the hypotenufe is equal to the fum of the fquares of the fides, (and).

On , and defcribe fquares, (pr. 46.)

Draw || (pr. 31.) alfo draw and .

= ,

To each add ∴ = ,

= and = ;

∴ = .

Again, becaufe ||

= twice ,

and = twice ;

∴ = .

In the fame manner it may be fhown

that = ;

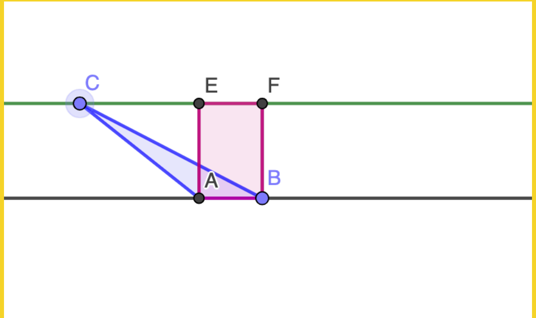
hence = .

Q. E. D.

Byrne's Euclid

What is the ratio of the areas of the triangle ABC and the rectangle EFBA? Does this relation change if you drag point C along the green line? Why or why not?

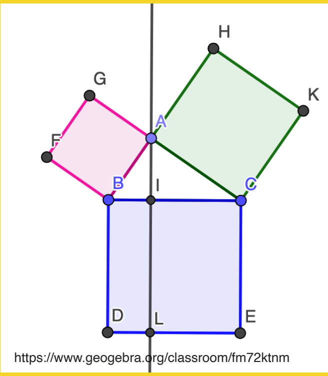
One Step in the Proof of Pythagoras' Theorem



https://www.geogebra.org/m/xyfcawtm

---

1. **Geogebra** (QR Code) https://www.geogebra.org/m/hgrzqsv6
2. **Slido** Describe the pattern you observed in the Geogebra app we just worked on, in such a way that somebody who has not seen the figure can understand what you are saying. For instance, you can start by “Consider a right triangle ABC. Suppose that BAC is the right angle…



Exploration of Euclid's proof of the Pythagorean Theorem

Lect 1

entering the code at www.geogebra.org/classroom

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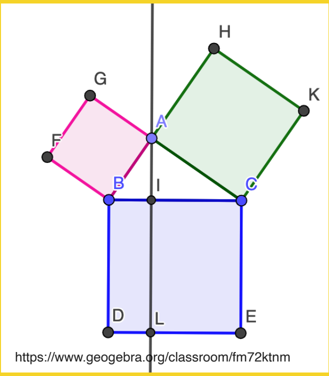
2. **Slido** Describe the pattern you observed in the Geogebra app we just worked on, in such a way that somebody who has not seen the figure can understand what you are saying. For instance, you can start by “Consider a right triangle ABC. Suppose that BAC is the right angle…



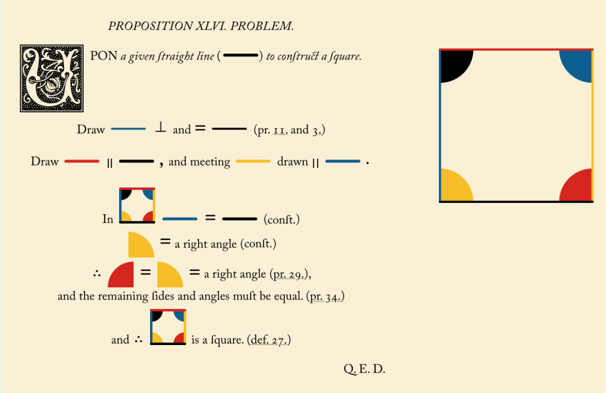
Exploration of Euclid's proof of the Pythagorean Theorem

https://www.geogebra.org/m/hgrzqsv6

Proposition 46:

To describe a square on a given straight line.

Similar to construction of equilateral triangle.



PROPOSITION XLVI. PROBLEM.

UPON *a given ſtraight line* ( ) *to conſtruct a ſquare.*

Draw ⊥ and = (pr. 11. and 3.)

Draw || , and meeting drawn || .

In = (conſt.)

= a right angle (conſt.)

∴ = = a right angle (pr. 29.),

and the remaining ſides and angles muſt be equal. (pr. 34.)

and ∴ is a ſquare. (def. 27.)

Q. E. D.

Translation by R. Fitzpatrick

Postulates

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5. And the whole [is] greater than the part.

Proposition 47

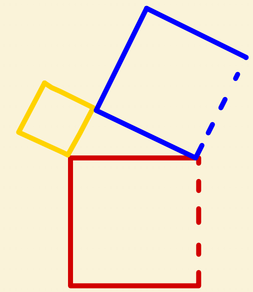
IN a right angled triangle the ſquare on the hypotenuſe is equal to the ſum of the ſquares of th ſides, (and).

Check: http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI47.html

Proposition 47

IN a right angled triangle the ſquare on the hypotenuſe is equal to the ſum of the ſquares of th ſides, (and).

On , and deſcribe ſquares, (pr. 46.)



Byrne's Euclid

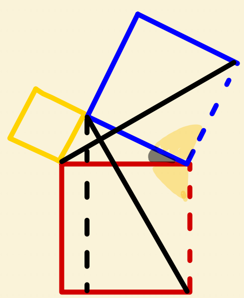
Proposition 47

IN a right angled triangle the ſquare on the hypotenuſe is equal to the ſum of the ſquares of th ſides, (and).

On , and deſcribe ſquares, (pr. 46.)

Draw || (pr. 31.) alſo draw and .

= ,



Byrne's Euclid

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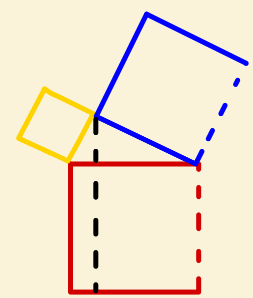
Draw || (pr. 31.)

.



Observe the doted black line.

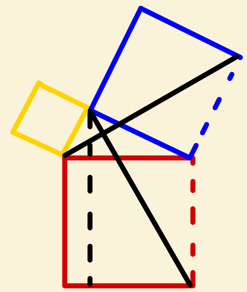
Byrne's Euclid

Proposition 47

IN *a right angled triangle the ſquare on the hypotenuſe is equal to the ſum of the ſquares of the ſides,* (*and*).

On , and deſcribe ſquares, (pr. 46.)

Draw || (pr. 31.) alſo draw and .



Byrne's Euclid

Proposition 47

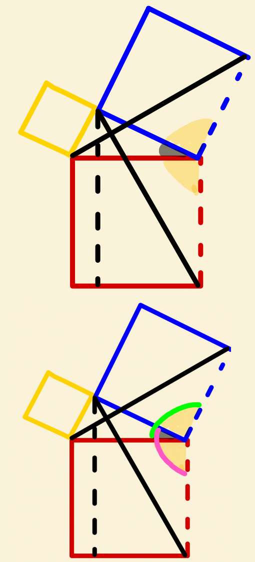
IN *a right angled triangle the ſquare on the hypotenuſe is equal to the ſum of the ſquares of the ſides,* (*and*).

On , and deſcribe ſquares, (pr. 46.)

Draw || (pr. 31.) alſo draw and .

= ,

To each add ∴ = ,



Byrne's Euclid

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= ,

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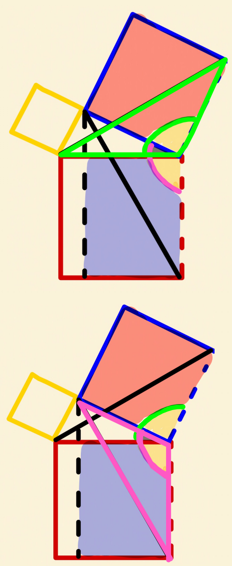
= and = ;

∴ = .

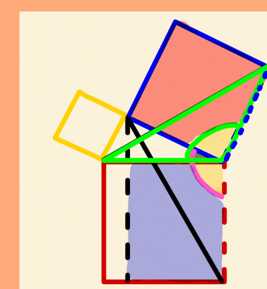


Byrne's Euclid

Proposition 47

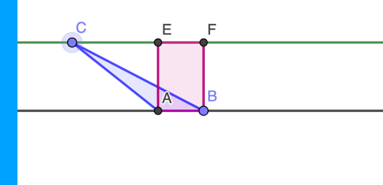
Again, becauſe ||

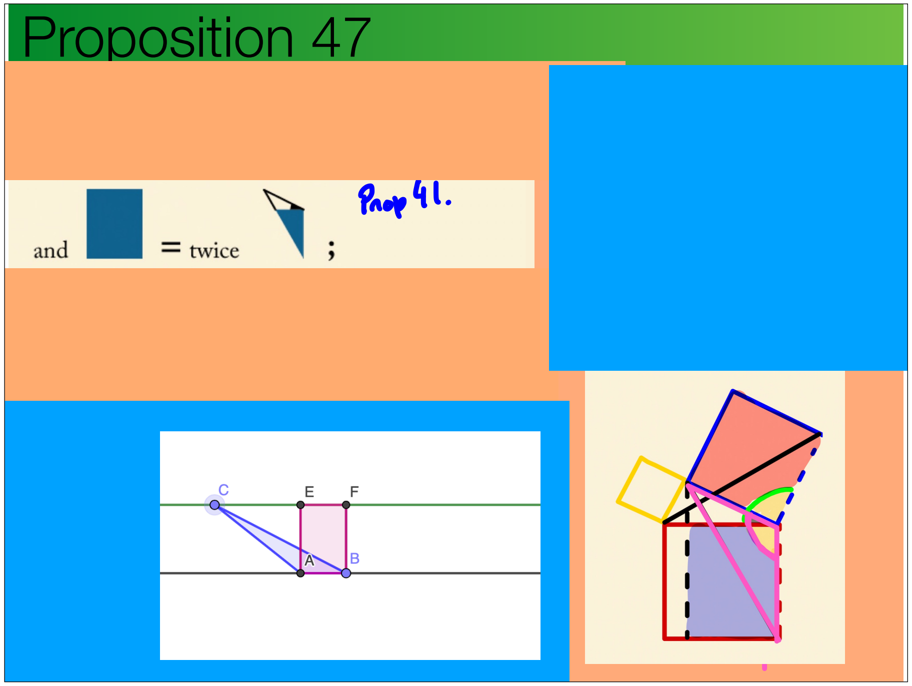
= twice ,

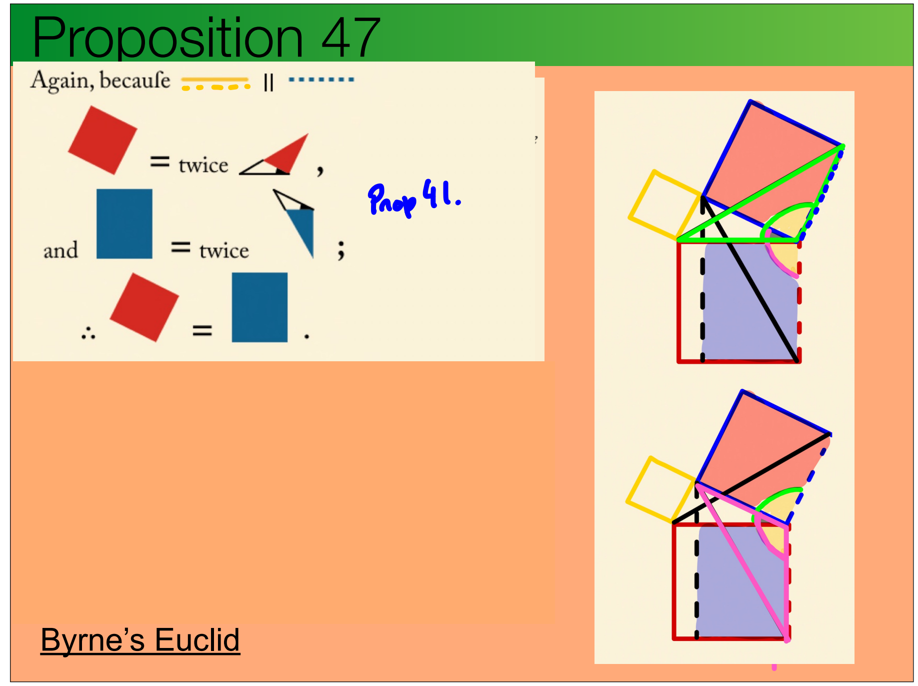


Proposition I.41
If a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle.

Recall




Proposition 47
and
= twice
;
Prop 41.
C
E
F
A
B


Proposition 47
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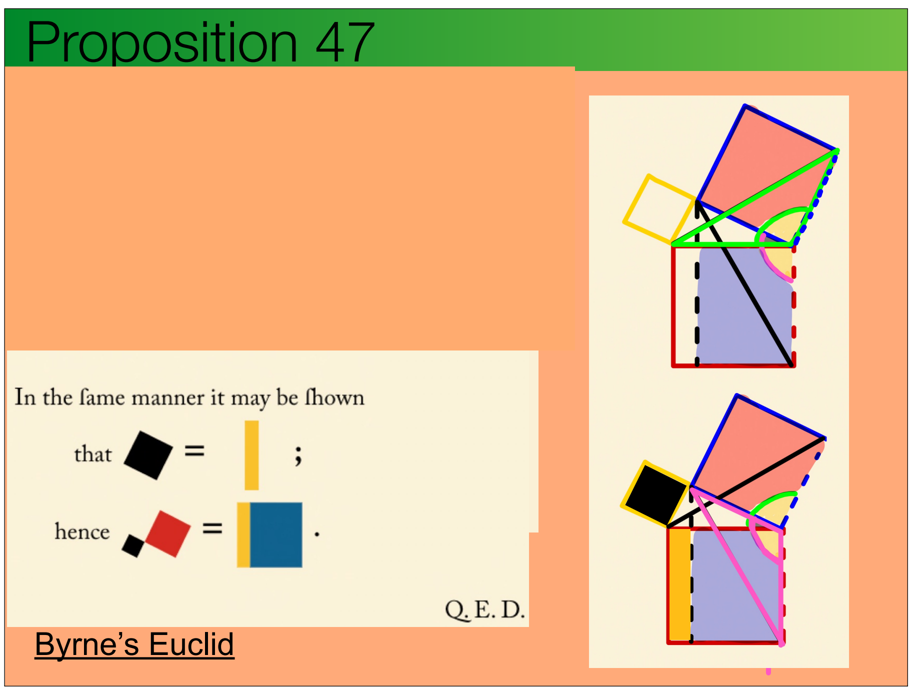
Byrne's Euclid


Proposition 47
In the fame manner it may be fhown
that
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hence
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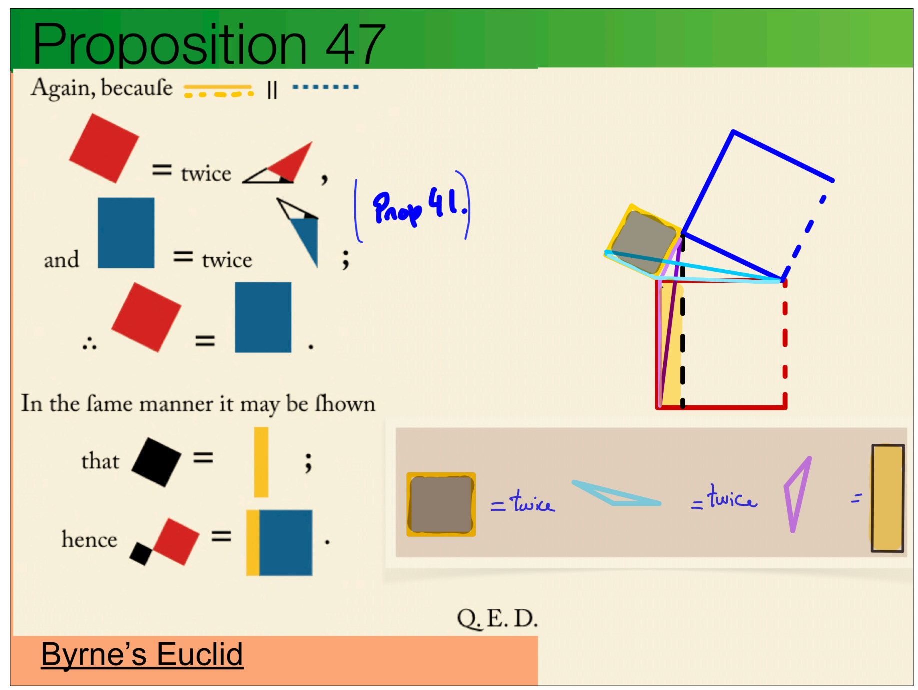
Q. E. D.
Byrne's Euclid


Proposition 47
Again, becaufe
= twice
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= twice
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(Prop 41.)
∴
=
.
In the fame manner it may be fhown
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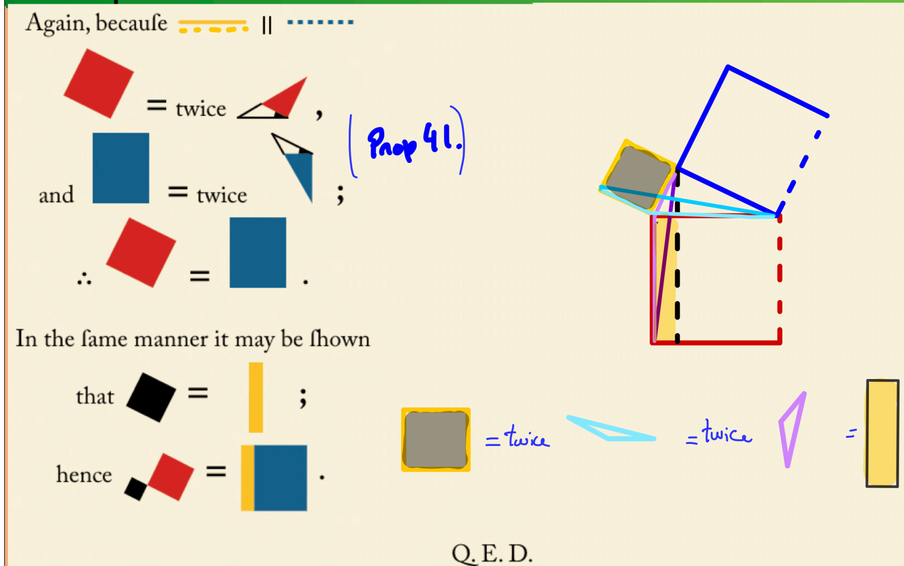
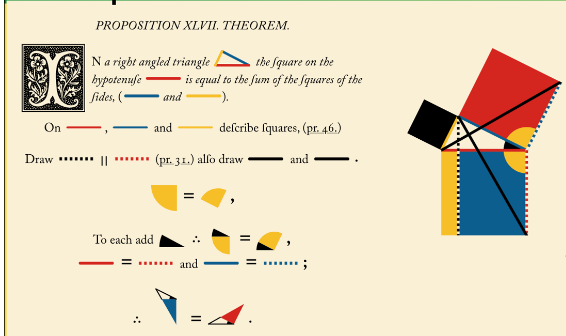
Byrne's Euclid

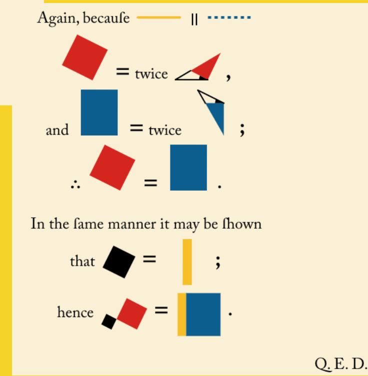
Proposition 47



Byrne's Euclid

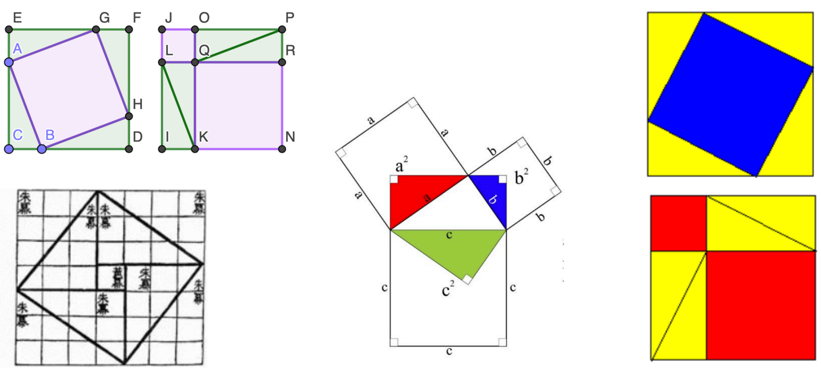
Proposition 47





Describe the proof of the Pythagorean theorem we just discussed.

Byrne's Euclid



The illustrations come from the site

https://www.cut-the-knot.org/pythagoras/

where you can find 122 proofs of the Pythagorean theorem.

Euclid's proof is axiomatic.

If some of the propositions in Euclid's Elements are added as axioms then…

1. **A contradiction can be deduced from the new (larger) set of axioms.**
2. **All the propositions in Euclid's Elements can be proven with the new (larger) the set of axioms**
3. **Some, but not all of the propositions proven in Euclid's Elements can be proven with the new (larger) set of axioms**

Euclid’s Elements. Select one true statement.

A. All the logical consequences of the postulates and common notions in Euclid’s Elements are stated and proven in the Elements. Hence, no **new propositions were proven after Euclid.**

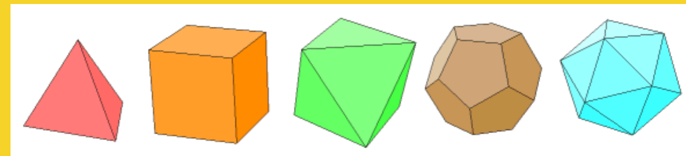
B. All the logical consequences of the postulates and common notions in Euclid’s Elements were either proven in the Elements or before the fifteenth century. **Since the fifteenth century, no new proportions were proved.**

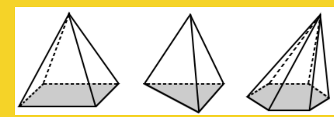
C. There are logical consequences of the postulates and common notions in Euclid’s Elements w**hich not have been proven.**

D. All the above statements are **false**.

Euler Characteristic (Polyhedral Formula) In a polyhedron, the following equation holds: V-E+F=2, where V is the number of vertices, E is the number of edges and F is the number of faces. **Descartes (~1600); Euler (~1700)**

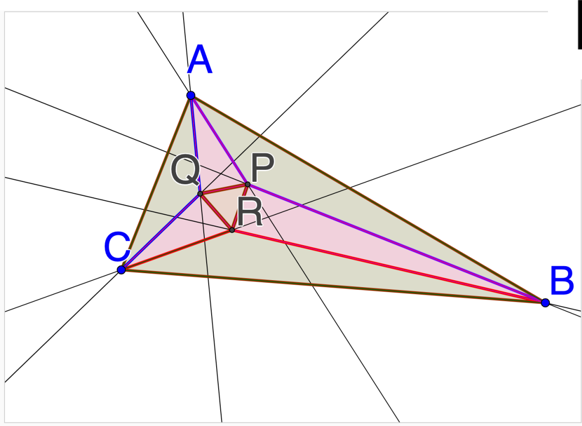
Choose a polyhedron and compute the number of vertices, edges and faces. Check that the Euler characteristic is two.





Morley's trisector theorem, 1899

https://ggbm.at/FRZ9Nfec



Summary