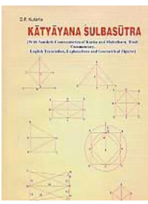
Ancient Indian Mathematics



By Noyal francis - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=49835812

We will discuss (Note that this is not an exhaustive outline)

- Prehistory: The Indus (or Harappan) civilisation ~2500 BCE-1700 BCE
- Vedic period (~800–200 BCE) Śulba Sūtras
- Jain mathematics (~500 BCE – 200 CE)
- Aryabhata (600 CE)
  - Trigonometry - The Sine Table in the oral tradition.
- Zero
- Mathematics in Sanskrit poetry.

China, India, and Islamic World

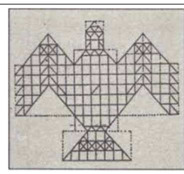
Difficulties Studying Math History

- Large (in space and time).
- Not homogeneous (for instance, different languages)
- Western centered approach
  - tendency to see them as alien or exotic
  - accounts from historians or writers with a variety of agendas.
- Lack of documents (destroyed by climate, wars, fires, and people)

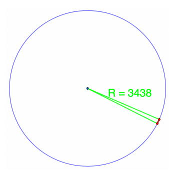
Some reasons for doing mathematics

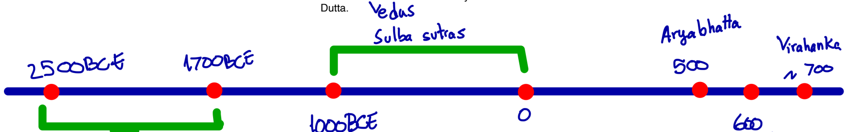
- Astronomy - including computations of the calendar, astrology and cosmology.)
- Religious: Calculating the direction of Mecca for the Islamic world, constructing altars in India.
- Measuring time.
- Land surveying
- Estimating areas and volumes
- Taxation and division of states.
- Teaching numeracy to an elite.
- Math for the sake of it.

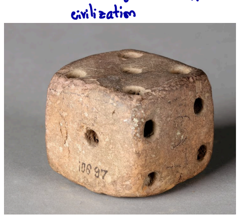
Indian Subcontinent Very Rough Chronology



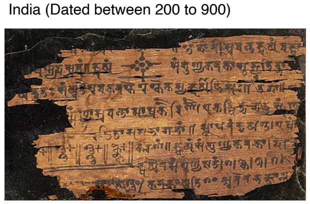
Vakrapaksa-syenacit. First layer of construction (after Baudhayana) - Mathematics in Ancient India by A. Dutta.







One of the Bakhshali manuscripts. Earliest written use of zero as a numeral in India (Dated between 200 to 900)



National Geographic, Public domain, via Wikimedia Commons

The Indus (or Harappan) civilisation

The Indus (or Harappan) civilisation

- from 2500 BCE to 1700 BCE.
- literate, written script with around 500 characters not yet deciphered.
- **a quite advanced and accurate, uniform system of weights and measures.**
- unclear what caused the decline in the Harappan civilisation.
  - a change in climatic patterns or a climatic disaster such flooding or severe drought?
  - disease spread by epidemic?
  - invasion of Indo-Aryans peoples from the north?
- eventually the Indo-Aryans peoples from the north spread over the region.

Image credit: Wikipedia

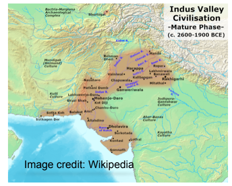


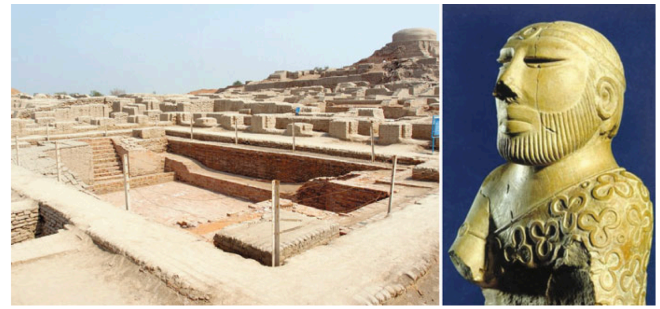
Image credit: Wikipedia

Indus Valley Weights= https://tamilandvedas.com/2019/09/

The study of mathematical astronomy in India goes back to at least the **third millennium BC** and **mathematics and geometry** must have existed to support this study in these ancient times.

Terracotta Dice (Harapan Civilization)
-2600/-1900

Lahore Museum
Lahore, Pakistan



Mohenjo-Daro. Exvacated ruins of one of the largest Settlements of the ancient Indus Valley Civilisations [Photo: Saqib Qayyum, 2014]; stone statue of a 'Priest-King', found in 1927 AD in Mohenjo-Daro (National Museum, Karachi, Pakistan) [Photo: Mamoon Mangal]

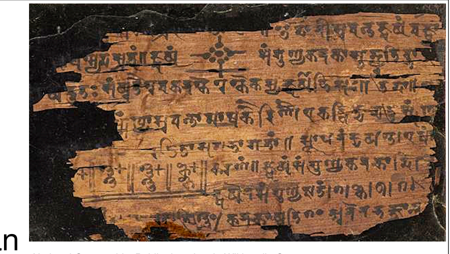
Number systems in India

Did the concept of zero appear first as a number or as a numeral? (Recall 1922 is a number, 0, 1,2...9 are the numerals in our decimal number system) Give reasons for your answer.

1. Why is important the occurrence of zero as a numeral (or place holder) in the Bakhshali manuscript? 2. What is the approximate date of the Bakhshali manuscript?

https://youtu.be/pV_gXGTuWxY

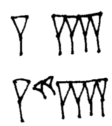
Bakhshali manuscript



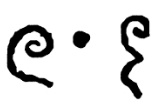
National Geographic, Public domain, via Wikimedia Commons

- mathematical text
- written on birch bark
- found in present-day Pakistan
- written in Sanskrit
- perhaps the oldest extant manuscript in Indian mathematics.
- It is a list of rules and examples (use of arithmetic, algebra, geometry, and mensuration)
- contains the earliest known Indian use of a zero numeral
- Dated between 200 to 900 (dates are debated, different sections probably have different dates)

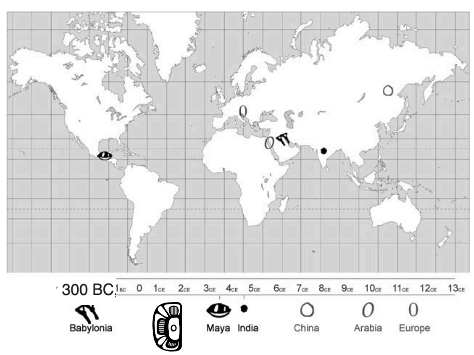
Zero as a numeral (dates are not completely certain)



The number 605 in Khmer numerals, (a date that corresponds to AD 683) one of the earliest known uses of zero as a numeral



https://commons.wikimedia.org/wiki/File:Khmer_Numerals_-_605_from_the_Sambor_inscriptions.jpg



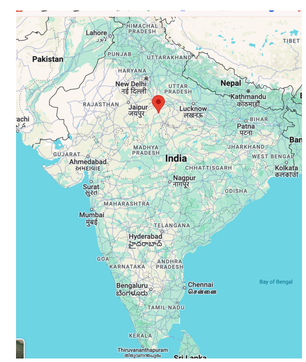
From A BRIEF HISTORY OF ZERO by Kristen McQuillin, July 1997 (revised January 2004)

https://mt.mediatinker.com/blog/archives/008821.html

Zero in Gwalior

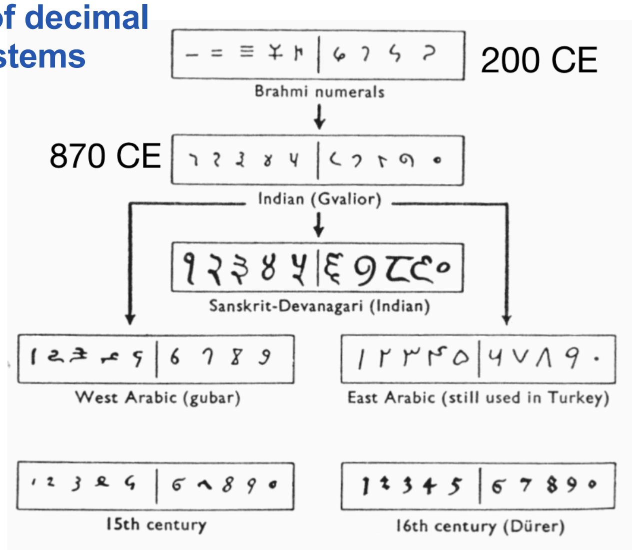
- The city of **Gwalior** in **India** is located on the main rail line south from Delhi, just a bit below Agra, the site of the Taj Mahal.
- In Gwalior there is a **medieval fort**.The fort, remarkable not only for its size and beauty.
- In a **temple** inside the fort, there is a **tablet** recording the establishment of a small 9th century Hindu temple on the eastern side of the plateau
- This tablet, which is dated by who wrote the text, records the **oldest "0"** in India for which one can assign a definite date.

... the town gave in perpetual endowment ... for a daily gift of 50 garlands of flowers ...



https://www.ams.org/publicoutreach/feature-column/fcarc-india-zero

The evolution of decimal number systems

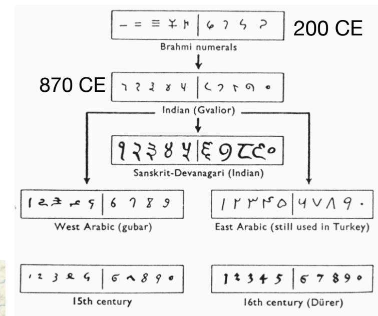


Higgins, Peter. Number Story: From Counting to Cryptography, Gottingen: Copernicus, 2008. P.189.

The evolution of decimal number systems

Image Credit: Khan Academy

https://www.math.uci.edu/~ndonalds/math184/india.pdf



Higgins, Peter. Number Story: From Counting to Cryptography, Gottingen: Copernicus, 2008. P.189.

The evolution of decimal number systems

- There were many ways of naming numbers in Sanskrit. One system recorded in the early **Vedas**

1. eka,
2. dva,
3. tri,
4. catur,
5. panca,
6. sad,
7. sapta,
8. asta,
9. nava,
10. dasa.

Hindu–Arabic numeral system

European (descended from the West Arabic)	0	1	2	3	4	5	6	7	8	9
Arabic-Indic	٠	١	٢	٣	٤	٥	٦	٧	٨	٩
Eastern Arabic-Indic (Persian and Urdu)	۰	۱	۲	۳	۴	۵	۶	۷	۸	۹
Devanagari (Hindi)	०	१	२	३	४	५	६	७	८	९
Tamil		௧	௨	௩	௪	௫	௬	௭	௮	௯

https://en.wikipedia.org/wiki/File:Arabic_numerals-en.svg

This Hindu system for writing whole numbers involved symbols that gradually changed as they migrated, one variant ending up as our Western symbols 0, 1, 2, . . . , 9. Later Islamic writers extended them to include decimal fractions, thus creating what is now justly known as the Hindu–Arabic system of numeration.

In parallel, other systems coexisted.

- Another system records numbers as the result of a calculation: for example, trisapta represents 21.
- Ptolemy's table of chords gives the chords and their sixtieths for all the arcs from 1 ◦ to 180◦ . Most likely, Ptolemy knew about the zero.

Jain Mathematics

Jainism

- Significant between 300BCE to 400 CE
- **Strong interest in large numbers for religious purposes.**
- Mention three classes of numbers: finite, countable infinite and uncountable infinite.
- Example:
  - Each individual has infinity capacity for liberation and goodness.
  - There are infinite number of souls
- Time has no beginning and no end
- Religious texts discuss ideas of countability.
- Idea of numbers linked to religion.

Empty mind before creating something.

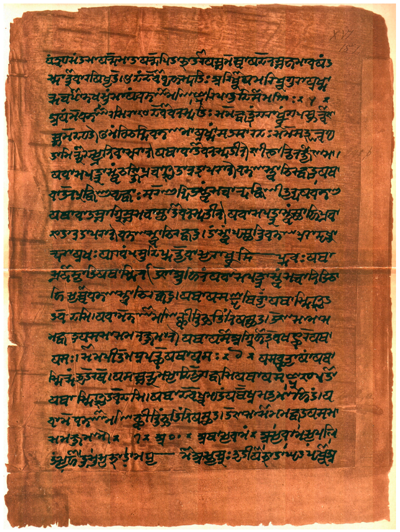
For the Jainism, the universe was born out of nothingness.

Vedas and Sulbasutras

Vedas and Sulbasutras

Vedas

- oldest scriptures of Hinduism, compiled **oral** wisdom starting around 2000 BCE.
- . written in early Sanskrit
- gave instructions for religious purposes.
- veda means knowledge
- main source of our knowledge of early mathematics in India.
- most important for Math History:
  - calendars
  - astronomy,
  - Sulba-Sutras.

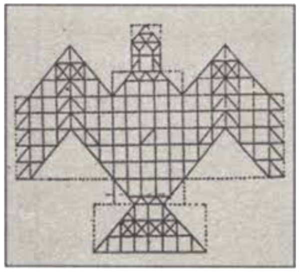


A page from the Atharvaveda.- Image of Codex Cashmiriensis folio 187a from Atharva-Veda Saṁhitā second half, by William Dwight Whitney and Charles Rockwell Lanman

Vedas and Sulba sutras

Sulba-sutra (**sutra** means rule, **sulba** means string or chord)

- The main sulbasutras were written between **800 and 200 BCE**.
- appendices to the **Vedas**
- Rituals involved the construction of sacrificial fire altars, measurements of these altars were required to be very precise for a ritual sacrifice to be successful
- explained how to **make shapes of various kinds (**altars, fireplaces, etc) of given **area**.
- **stakes and marked cords** were used to make right-angled triangles
- **all** that is known about Vedic mathematics is contained in the Sulba Sutras.
- each sutra is named after the priest-scholar that wrote it.



Vakrapaksa-syenacit. First layer of construction (after Baudhayana) - Mathematics in Ancient India by A. Dutta.

Mathematics was studied in families. It was passed down from generation to generation. A family will have a library of mathematical texts that they copied and re-copied. It is likely that in times of political uncertainty this tradition was broken.

More about the Sulba sutras

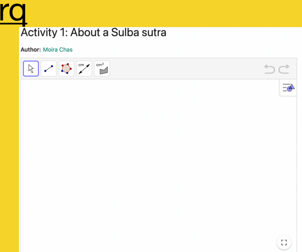
- No proofs or explanations why a statement holds.
- Some constructions are precise and other are approximate.

Like the crest of a peacock, like the gem on the head of a snake, so is mathematics at the head of all knowledge.

—Vedanga Jyotisa (~ 500 BCE)

Sulba sutras Activity 1

https://www.geogebra.org/m/gc5setrq



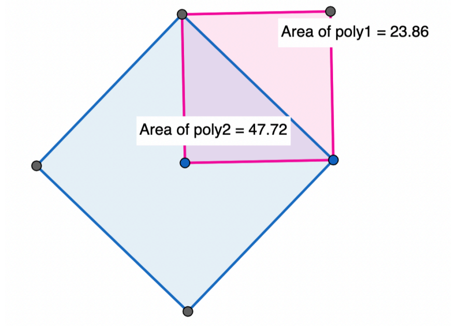
1. Construct a square (To do it, first click on the Regular Polygon tool, then click twice on the plane to create the endpoints of an edge of the square, finally, click "ok" if the number of sides offered is four, otherwise change it and then click OK)

2. Using the Regular Polygon tool again construct a new square such that one of its sides is one of the diagonals of the first square.

3. Using the area tool, compute the area of the two squares.

4.What is the relation between the areas of the two squares? If you have time, find an explanation for your answer. (answer in the platform)

(Note: You can only move the first two points you constructed, which originally are blue. The rest of the construction depends on these two points).

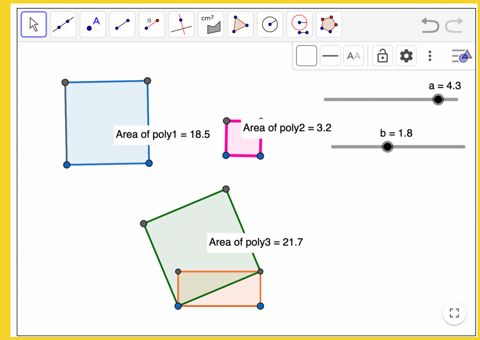


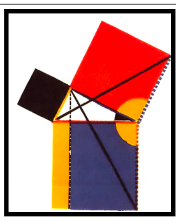
The rope which is stretched across the diagonal of a square produces an area double the size of the original square.

Activity 2: Explain why the area of the rectangle you constructed is the sum of the areas of the given squares.

“The rope which is stretched along the length of the diagonal of a rectangle produces an area which the vertical and horizontal sides make together.”

https://www.geogebra.org/m/ztcsqdum

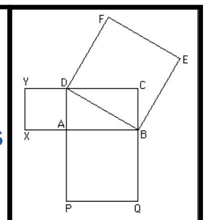




Euclid’s elements, 300BCE
(Interpreted by Oliver Byrne in 1800)

One of the sutras in the Baudhāyana sūtras compiled (perhaps) between 800BCE and 400BCE.

"The rope which is stretched along the length of the diagonal of a rectangle produces an area which the vertical and horizontal sides make together”



The Katyayana form of Pythagoras's theorem

https://mathshistory.st-andrews.ac.uk/HistTopics/Indian_sulbasutras/

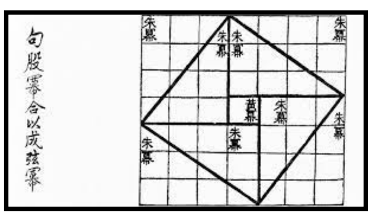


Diagram added by Zhao Shuang to the Zhoubi Suanjing that can be used to prove the Pythagorean Theorem

A diagram proof of the Pythagorean Theorem from Zhoubi Suanjing (周髀算經) (1046–256 BCE).

The Katyayana Sulbasutra gives the following approximation to √2:

Increase a unit length by its third and this third by its own fourth less the thirty-fourth part of that fourth

How many decimals (after the decimal period) of this approximation match the real value of √2?

The Katyayana Sulbasutra gives the following approximation to $\sqrt{2}$:

The measure is to be increased by its third and this [third] again by its own fourth less the thirty-fourth part [of that fourth]; this is [the value of] the diagonal of a square [whose side is the measure].

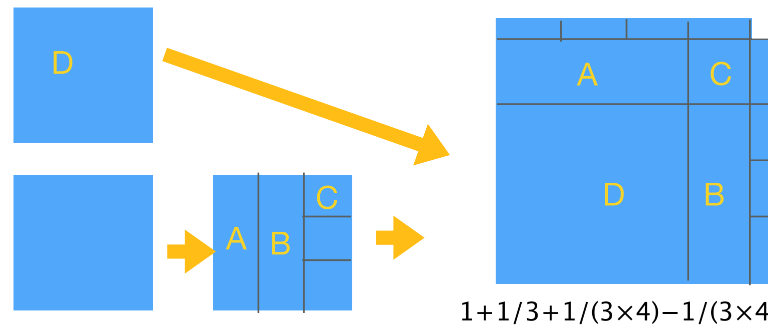
$$=1+1/3+1/(3\times4)-1/(3\times4\times34) = 577/408 \sim 1.4142156863$$

$$\sqrt{2} = 1.414213562\ \dots$$

How was the approximation of √2 found? A conjecture

We do not know, but here is a possible explanation.

- Start with two squares, each of area 1.
- We are going to cut up and arrange one of the squares around the other, to make a third square. Ideally, this square should have area 2.

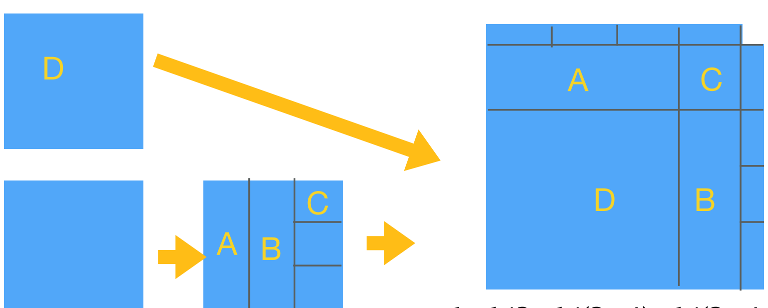


$1+1/3+1/(3\times4)-1/(3\times4\times34)$

How was the approximation of √2 found? A conjecture

We do not know, but here is a possible explanation.

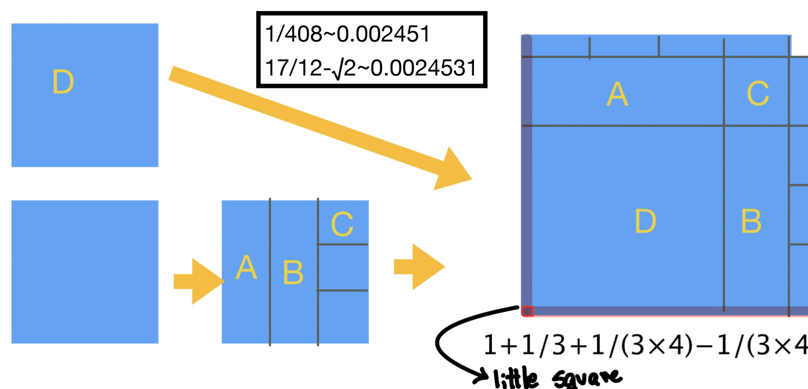
- Cut off a strip of width x on the left hand side and bottom to fill in the missing part which has area $(1/12)^2$
- Then $2.x.(1+1/3+/12)= (1/12)^2$.
- $x=1/(3\times4\times34)$



$1+1/3+1/(3\times4)-1/(3\times4\times34)$

How was the approximation of √2 found? A conjecture

- Then $2.x.(1+1/3+/12)= (1/12)^2$ -> $x= 1/408 = 1/(3\times4\times34)$
- But there is a little square of side x overlapping. Thus, x satisfies $2.x.(1+1/3+/12) - x^2 = (1/12)^2$, gives $x=17/12-\sqrt{2}$



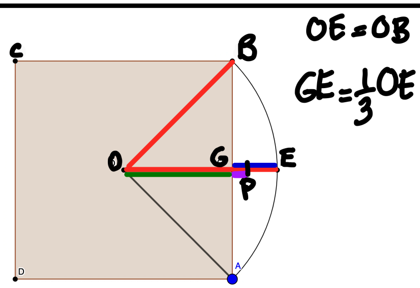
https://mathshistory.st-andrews.ac.uk/HistTopics/Indian_sulbasutras/

What is the value of “π” in the construction "circling the square”?

If it is desired to transform a square into a circle,

- *[a cord of length] half the diagonal [of the square] is stretched from the centre to the east [a part of it lying outside the eastern side of the square];*
- *with one-third [of the part lying outside] added to the remainder [of the half diagonal], the [required] circle is drawn.*

https://www.maa.org/press/periodicals/convergence/more-classroom-activities-based-on-ancient-indian-rope-geometry-transforming-a-square-into-a-circle



Aryabhata

Heading

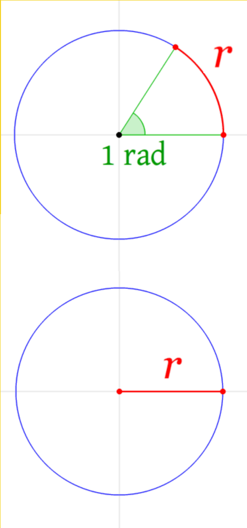
- Math motivations: Need vs curiosity
- Fifth postulate in Euclid’s elements
- Inca
  - Kipus (to represent numbers)
- Mayan
  - Base 20 number system
  - Method similar to modular arithmetic to move between dates in different calendars.
- China
  - Nine chapters of the Mathematical Art
  - Continuing board (solve linear equations, polynomial, magic squares)
  - Volume of solids (sphere, and slanted pyramid)
- Greece
  - Zeno’s paradoxes.
  - Archimedes apro of π
  - Euclids Elemtens and its method.
    - Infinitude of prime numbers.
    - Pythagorean theorem.
  - The three impossible problems
  - Platonic solid
  - Golden ratio
  - Circumference of the earth (Erathostenes)
  - Diophantus
- India
  - √2 Aprox
  - 0! (as place holder and NUMBER)
- Mesopotamia
  - √2 Aprox
  - Base 60 different but arrive to us.
  - Method of false position
  - Area of quadrilaterals.
- Egypt
  - apro of π
  - multi.
  - Rhind papyrus
  - Rosetta Stone

Aryabhata

- 500 CE
- Wrote the book Âryabhâtiya
  - Compilation of known and new results about astronomy and computational issues
  - one of the principal sources for our decimal place-value system, including the use of the zero.
  - this text survives in part because it was subject of many **commentaries**.

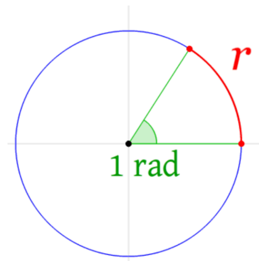
Aryabhata wrote: “*a place should be 10 times the previous place”*

Recall that a radian is a unit of angular measure. Write down a definition of radian. (Hint: see below)



By Lucas Vieira - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=25112326

In a circle of radius 1, an arc of length 1 determines an angle of 1 radian.



In a circle of radius r, the arc of length r determines an angle of 1 radian.

Early trigonometry in India
The Sine function

greek: trigōnon — triangle; metron — measure

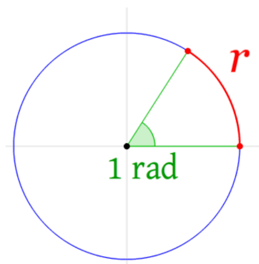
- Mainly used for **astronomy**, for computations (using combinations of circles) to predict positions of the planets.
- Use of a function Sine (similar to "our" sine) which related angular and linear measures.
- A circle of fixed radius R = **3438** was considered.

Sine(angle) = 3438 x sine(angle)

Why 3438?
How did the Sine appeared?

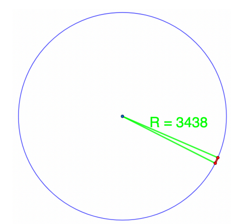
Early trigonometry in India
The Sine function

2π rad	—	360x60 min = 21,600 min
1 rad	—	(360x60)/(2π) ~ 3438 min



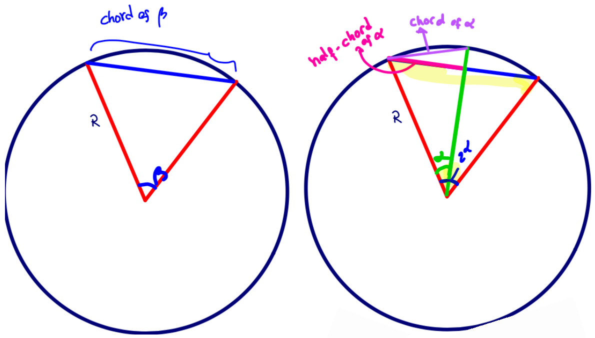
If r=1, the arc of length 1 radian determines a central angle of 1 radian.

- The unit of measure for angles and length were minutes.
- The **circumference of a circle** of radius R = **3438** and a certain approximation of π was 21,600 minutes. (21,600=360 x 60.)



If R = 3438, the arc of length 1 minute determines a central angle of 1 minute.

Early trigonometry in India
The Sine function



ß central angle

Half chord of α =

(1/2) chord(2α) =

R sin(α)

Large numbers and verses

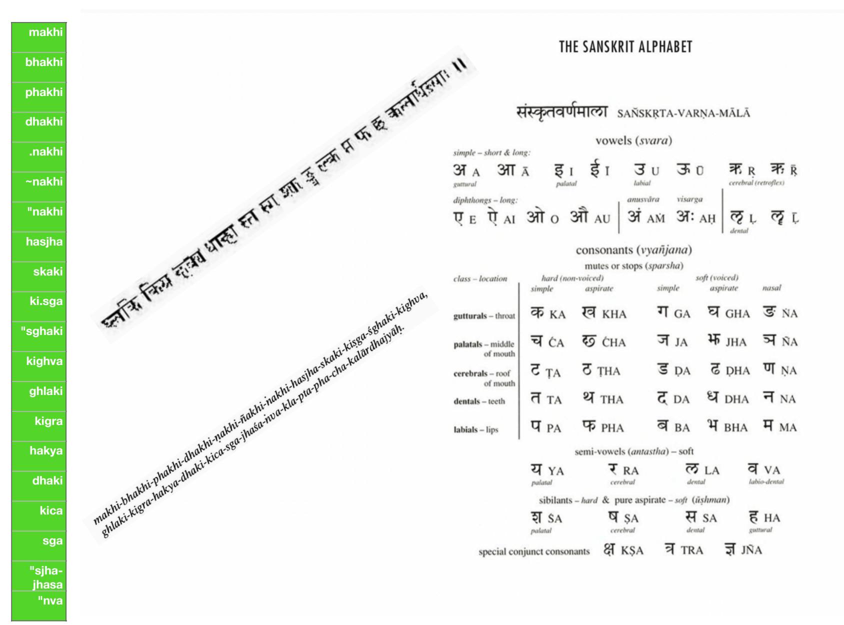
- Indian mathematicians transmitted information **orally**, usually in **verse form in Sanskrit.**
- As in Ptolemaic astronomy, in early Indian astronomy the **first task** of computing celestial positions is to **determine the mean position of a celestial body at a desired time.**
- These mean **motions** are presented in the form of simple **proportions** between the **large integer numbers of cycles** completed in a certain period of time.
- Consequently, the author needs a way of **verbally expressing such large integer numbers—about ten decimal digits long—in verse form.**

Mathematics in India by Kim Plofker

Early trigonometry in India
The Sine function

- A "table" of Sine was in was usually presented as a list ***in verse*** of
  - **twenty-four Sines** in the first quadrant
  - at **intervals of 3 3/4 ◦ (225 minutes)**,
  - a **rule** for linear interpolation to find Sines and arcs

No explanation of how it was derived geometrically



मखि भखि फखि धखि णखि ञखि ङखि हस्झ स्ककि किष्ग श्घकि किघ्व |
घ्लकि किग्र हक्य धकि किच स्ग झश ङ्व क्ल प्त फ छ कला–अर्ध–ज्यास् ||

makhi bhakhi phakhi dhakhi ṇakhi ñakhi
ṅakhi hasjha skaki kiṣga ghakhi kighva |
ghlaki kigra hakya dhaki kica
sga śjha ṅva kla pta pha cha
kala-ardha- jyāḥ ||

Aryabhata’s poem-table of sines

Words from verse
makhi
bhakhi
phakhi
dhakhi
.nakhi
~nakhi
"nakhi
hasjha
skaki
ki.sga
"sghaki
kighva

Words from verse
ghlaki
kigra
hakya
dhaki
kica
sga
"sjha-jhasa
"nva
kla
pta
pha
cha

Complete the table -Teams of 3 or 4

https://docs.google.com/spreadsheets/d/1m_RaOfagWoNvBvlbHTtqDe66UCucl4lhmmyXSvWFVW8/

- Open the link on the course website or use QR code
- Make a copy of the spreadsheet and share it with me.
- Fill the gaps (in yellow)!

Classified consonants

ka	1
kha	2
ga	3
gha	4
“na	5
ca	6
cha	7
ja	8
jha	9
~na	10
.ta	11
.tha	12
.da	13
.dha	14
.na	15
ta	16
tha	17
da	18
dha	19
na	20
pa	21
pha	22
ba	23
aha	24
ma	25

Unclassified consonants

ya	30
ra	40
la	50
va	60
“sa	70
.sa	80
sa	90
ha	100

Vowels

a	1
i	100
u	100^2
.r	100^3
.l	100^4
e	100^5
ai	100^6
o	100^7
au	100^8

Word				Value
makhi	khi=kha x i=200	ma=25		225
bhakhi	khi=kha x i=200	bha=24		224
phakhi				
dhakhi	khi=kha x i=200	dha=19		219
.nakhi	khi=kha x i=200	.na=15		215
~nakhi				
"nakhi	khi=kha x i=200	"na=5		205
hasjha	ha=100	sa=90	jha=9	199
skaki	ki=100	sa=90	ka=1	191
ki.sga	ki=100	.sa=80	ga=3	183
"sghaki				
kighva	ki=100	va=60	gha=4	164
ghlaki	ki=100	gha=4	la=50	154
kigra				
hakya	ha=100	ya=30	ka=1	131
dhaki	dha=19	ki=ka x i=100		119
kica				
sga	sa=90 g	a=3		93
"sjha- jhasa	"sa=70	jha=9		79
"nva	"na=5	va=60		65
kla	ka=1	la=50		51
pta	pa=21	ta=16		37
pha				
cha	cha=7			7

Examples

	Decomposition	Computation	Value
ka	ka	1	1
ki	ka x i	1x100	100
gu	ga x u	3x10000	30000
gnu	(g+na) x u	(20+3)x10000	230000
khyughr	(gha x .r) + (ya + kha) x u	(4x100^3)+(30+2)x100^2	4,320,000

Aryabhata “sine” table

Word from verse	Number from verse	Number from formula	Angle (in minutes)	Angle (in degrees)	Angle (in radians)	Total from verse	Total for verse/3438	Number/ (180*60/Pi)	Actual sine	sin(real)-(Number/ (180*60/Pi)-sin
	0		0	0	0.000	0	0.00000	0.00000	0.00000	0.0000000
makhi	225	225.00	225	4	0.065	225	0.06545	0.06545	0.06540	-0.0000467
bhakhi	224	224.00	450	8	0.131	449	0.13060	0.13061	0.13053	-0.0000826
phakhi	222	222.00	675	11	0.196	671	0.19517	0.19519	0.19509	-0.0000957
dhakhi	219	219.02	900	15	0.262	890	0.25887	0.25889	0.25882	-0.0000715
.nakhi	215	215.07	1125	19	0.327	1105	0.32141	0.32143	0.32144	0.0000080
~nakhi	210	210.16	1350	23	0.393	1315	0.38249	0.38252	0.38268	0.0001654
"nakhi	205	204.31	1575	26	0.458	1520	0.44212	0.44215	0.44229	0.0001386
hasjha	199	197.56	1800	30	0.524	1719	0.50000	0.50004	0.50000	-0.0000368
skaki	191	189.92	2025	34	0.589	1910	0.55556	0.55560	0.55557	-0.0000262
ki.sga	183	181.45	2250	38	0.654	2093	0.60878	0.60883	0.60876	-0.0000676
"sghaki	174	172.17	2475	41	0.720	2267	0.65939	0.65944	0.65935	-0.0000978
kighva	164	162.12	2700	45	0.785	2431	0.70710	0.70715	0.70711	-0.0000425
ghlaki	154	151.35	2925	49	0.851	2585	0.75189	0.75195	0.75184	-0.0001062
kigra	143	139.91	3150	53	0.916	2728	0.79348	0.79354	0.79335	-0.0001897
hakya	131	127.85	3375	56	0.982	2859	0.83159	0.83165	0.83147	-0.0001798
dhaki	119	115.22	3600	60	1.047	2978	0.86620	0.86627	0.86603	-0.0002397
kica	106	102.07	3825	64	1.113	3084	0.89703	0.89710	0.89687	-0.0002265
sga	93	88.48	4050	68	1.178	3177	0.92408	0.92415	0.92388	-0.0002723
"sjha	79	74.49	4275	71	1.244	3256	0.94706	0.94713	0.94693	-0.0002019
"nva	65	60.17	4500	75	1.309	3321	0.96597	0.96604	0.96593	-0.0001139
kla	51	45.58	4725	79	1.374	3372	0.98080	0.98088	0.98079	-0.0000898
pta	37	30.79	4950	83	1.440	3409	0.99156	0.99164	0.99144	-0.0001930
pha	22	15.86	5175	86	1.505	3431	0.99796	0.99804	0.99786	-0.0001785
cha	7	0.86	5400	90	1.571	3438	1.00000	1.00007	1.00000	-0.0000737
			5625	94	1.636	3438	1.00000	1.00007	0.99786	-0.0022147

Aryabhata “sine” table

Word from verse	Number from verse	Number from formula	Angle (in minutes)	Angle (in degrees)	Angle (in radians)
	0		0	0	0.000
makhi	225	225.00	225	4	0.065
bhakhi	224	224.00	450	8	0.131
phakhi	222	222.00	675	11	0.196
dhakhi	219	219.02	900	15	0.262
.nakhi	215	215.07	1125	19	0.327
~nakhi	210	210.16	1350	23	0.393
"nakhi	205	204.31	1575	26	0.458
hasjha	199	197.56	1800	30	0.524
skaki	191	189.92	2025	34	0.589
ki.sga	183	181.45	2250	38	0.654

Aryabhata's explanation of his system for denoting numbers

Classified consonants

ka	1
kha	2
ga	3
gha	4
“na	5
ca	6
cha	7
ja	8
jha	9
~na	10
.ta	11
.tha	12
.da	13
.dha	14
.na	15
ta	16
tha	17
da	18
dha	19
na	20
pa	21
pha	22
ba	23
aha	24
ma	25

Unclassified consonants

ya	30
ra	40
la	50
va	60
“sa	70
.sa	80
sa	90
ha	100

Vowels

a	1
i	100
u	100^2
.r	100^3
.l	100^4
e	100^5
ai	100^6
o	100^7
au	100^8

- The classified consonants [starting] from k [are encoded] in the square [places],
- the non-classified consonants, [starting from] y which is [equal to] n˙m, to the non-square [places].
- Nine vowels [are assigned] to the square and non-square [places] in a double nine-tuple of zeros, and [beyond] the square [places] ending with nine.

Mathematics in India by Kim Plofker

Aryabhata sine table

Three interpretations of

प्रथमाच्चापज्यार्धाद्यैरूनं खण्डितं द्वितीयार्धम् ।
तत्प्रथमज्यार्धांशैस्तैस्तैरूनानि शेषाणि ॥ १२ ॥

[1] K. S. SHUKLA: Āryabhaṭīya of Āryabhaṭa, Critically Edited with Introduction, English Translation, Commentary and Indexes, Indian National Science Academy, New Delhi 1976.

No explanation of how it was derived geometrically. However…

$S_n = R\sin(n\alpha)$ $\alpha = 225'$ $n = 1, 2, \ldots 24$

$R = 3438$

$d_n = S_n - S_{n-1}$ $n \geq 2$

(1) The first Rsine divided by itself and then diminished by the quotient gives the second Rsine-difference. The same first Rsine diminished by the quotients obtained by dividing each of the preceding Rsines by the first Rsine gives the remaining Rsine-differences.

(2) "The first Rsine divided by itself and then diminished by the quotient will give the second difference. For computing any other difference, (the sum of) all the preceding differences is divided by the first Rsine and the quotient is subtracted from the preceding difference. Thus, all the remaining differences (can be calculated)."[1]

$d_2 = S_1 - \frac{S_1}{S_1}$

$d_{n+1} = S_1 - \frac{S_1 + S_2 + \cdots + S_n}{S_1}$ $n \geq 2$

$R.\sin(n\alpha)$, $n=1$

(3) "The first Rsine divided by itself and then diminished by the quotient gives the second Rsine-difference. To obtain any other Rsine-difference, divide the preceding Rsine by the first Rsine and multiply the quotient by the difference between the first and second Rsine-differences and subtract the resulting product from the preceding Rsine-difference."

d_2 $S_1 = 225$

$d_2 = S_1 - \frac{S_1}{S_1}$

$d_{n+1} = d_n - \frac{S_n}{S_1}(d_1 - d_2)$

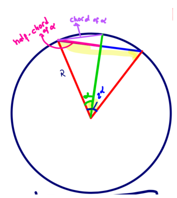
(Note $d_1 - d_2 = 2R\sin\alpha(1 - \cos\alpha)$)

Apparently, Aryabhata did not use his own rule! It is conjectured that he copied the values, probably from Ptolemy.

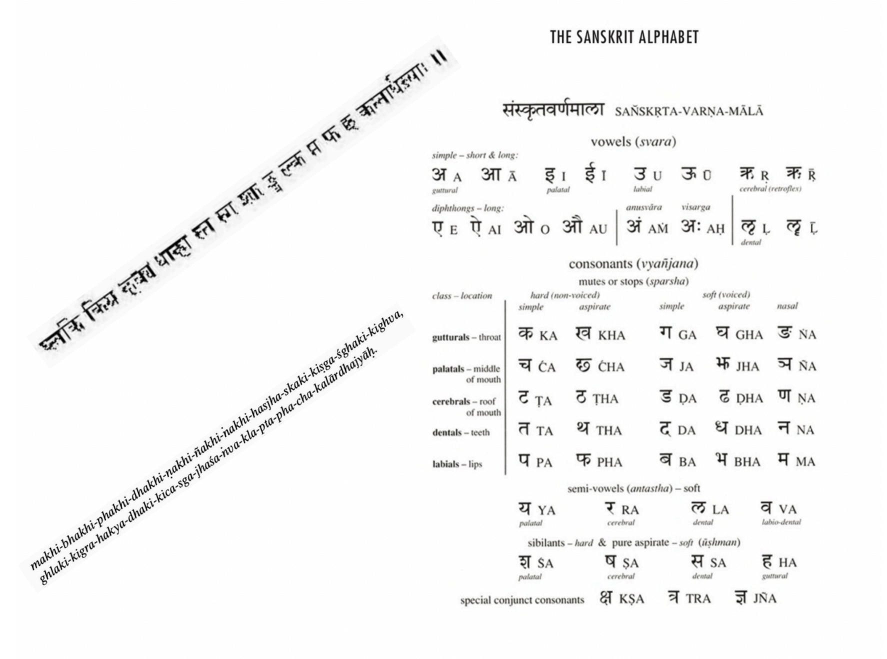
Extra, extra! probably he did not copy from Ptolemy

Origin of the word "sine"

- **jya** The mathematical technical term **jya¯** (bowstring) appears originally to have meant "**chord**."
- **ardhajya** Later the term **ardhajya** "half-chord," denoted the Sine.
- **jya** People dropped the "**ardha**" and kept "**jya**."
- **jiba** When Arabic writers translated his works from Sanskrit into Arabic, they referred it as **jiba**.
- **jb** Since in Arabic writings, vowels are omitted, it was abbreviated as **jb**.
- **jaib** Later writers substituted **jb** with "**jaib"**, meaning "pocket" or "fold (in a garment)".
- **sinus** Later in the 12th century, these writings were translated from Arabic into Latin, the Arabic **jaib** was replaced with its Latin counterpart, **sinus**, which means "**cove**" or "**bay"** which is the word that arrived to us.



Sanskrit Poetry



Between 600 and 800 CE, in India, a poet named Virahanka gave a rule for counting certain variations in a given rhythm of Sanskrit poetry.

The basic units in Sanskrit poetry are ***mora*** or "syllabic instant" which can be either

- **Short**, lasting 1 beat, denoted by S.
- **Long**, lasting 2 beats, denoted by L.

Thus, we have

- exactly one verse type lasting one beat - S
- exactly two verses types lasting two beats - SS and L
- exactly three verses types lasting three beats: SSS, SL and LS

Let's write down verses (in English!) reinterpreting the rules as follows: Each

- **mora** (or "syllabic instant") is a **word**,
- **beat** is a **syllable** of the word

Air	Cane	Dusk	Gold	Leaf	Oak	Rock	Steel
Ant	Cat	Dust	Grain	Lee	Onyx	Rook	Stone
Ash	Clay	Elm	Green	Leigh	Ox	Rose	Storm
Bay	Cliff	Fawn	Grey	Loch	Park	Rue	Suede
Beach	Cloud	Fern	Grove	Lynx	Peach	Rush	Teak
Bear	Clove	Field	Hart	Lys	Pearl	Rye	Teal
Bee	Coast	Fig	Hawk	Mace	Pike	Sage	Thyme
Beech	Colt	Fin	Haze	Maize	Pine	Sand	Tide
Birch	Cove	Finch	Heath	Mare	Pink	Sea	Tree
Bird	Crane	Fir	Ice	Mars	Plum	Shade	Vale
Black	Dale	Fjord	Ink	Marsh	Pond	Shell	Vine
Blaze	Dawn	Flame	Jade	Mauve	Quartz	Shore	Wave
Blue	Day	Flax	Jay	May	Quince	Silk	West
Blush	Dean	Flint	Jett	Mist	Rain	Sky	Wild
Branch	Dell	Fox	Jewel	Moon	Ram	Smoke	Wind
Breeze	Dill	Frost	Kale	Moss	Ray	Snow	Wolf
Brick	Doe	Gale	Koi	Neve	Reed	Sol	Wood
Brook	Dove	Gem	Lake	Night	Reef	Spring	Wren
Brown	Drake	Glen	Larch	Nile	Ridge	Spruce	Yew
Bud	Dune	Glow	Lark	North	Roc	Star	Yves

Each of the words we use is:

- **S**hort, lasting 1 syllable (one beat), denoted by **S**. (for instance: "green")
- **L**ong, lasting 2 syllables (2 beats), denoted by **b**. (for instance,"purple")

able	narrow
butter	orange
cable	purple
drama	quibble
even	ripple
follow	silly
gallop	teacher
happy	untrue
icy	very
jingle	water
kitten	x-ray
little	yellow
maple	zebra

Examples of one and two syllables words (for inspiration)

ant	no
bike	ox
can	pig
dog	queen
eat	run
fish	six
good	the
have	up
ice	van
jump	was
king	
live	yes
mom	zoo

Find verses of 3 syllables state their type.
Example: Math is great- SSS

Let's write down verses (in English!) reinterpreting the rules as follows: Each

- **mora** (or "syllabic instant") is a **word**,
- **beat** is a **syllable** of the word

Air	Cane	Dusk	Gold	Leaf	Oak	Rock	Steel
Ant	Cat	Dust	Grain	Lee	Onyx	Rook	Stone
Ash	Clay	Elm	Green	Leigh	Ox	Rose	Storm
Bay	Cliff	Fawn	Grey	Loch	Park	Rue	Suede
Beach	Cloud	Fern	Grove	Lynx	Peach	Rush	Teak
Bear	Clove	Field	Hart	Lys	Pearl	Rye	Teal
Bee	Coast	Fig	Hawk	Mace	Pike	Sage	Thyme
Beech	Colt	Fin	Haze	Maize	Pine	Sand	Tide
Birch	Cove	Finch	Heath	Mare	Pink	Sea	Tree
Bird	Crane	Fir	Ice	Mars	Plum	Shade	Vale
Black	Dale	Fjord	Ink	Marsh	Pond	Shell	Vine
Blaze	Dawn	Flame	Jade	Mauve	Quartz	Shore	Wave
Blue	Day	Flax	Jay	May	Quince	Silk	West
Blush	Dean	Flint	Jett	Mist	Rain	Sky	Wild
Branch	Dell	Fox	Jewel	Moon	Ram	Smoke	Wind
Breeze	Dill	Frost	Kale	Moss	Ray	Snow	Wolf
Brick	Doe	Gale	Koi	Neve	Reed	Sol	Wood
Brook	Dove	Gem	Lake	Night	Reef	Spring	Wren
Brown	Drake	Glen	Larch	Nile	Ridge	Spruce	Yew
Bud	Dune	Glow	Lark	North	Roc	Star	Yves

Each word is either:

- Short, lasting 1 syllable (one beat), denoted by S. (for instance: "green")
- Long, lasting 2 syllables (2 beats), denoted by L. (for instance,"purple")

able	narrow
butter	orange
cable	purple
drama	quibble
even	ripple
follow	silly
gallop	teacher
happy	untrue
icy	very
jingle	water
kitten	x-ray
little	yellow
maple	zebra

ant	no
bike	ox
can	pig
dog	queen
eat	run
fish	six
good	the
have	up
ice	van
jump	was
king	
live	yes
mom	zoo

Find verses of 4 syllables and state their type.
Example: The kitten sleeps - SLS

Let's write down verses (in English!) reinterpreting the rules as follows: Each

- **mora** (or "syllabic instant") is a **word**,
- **beat** is a **syllable** of the word

Air	Cane	Dusk	Gold	Leaf	Oak	Rock	Steel
Ant	Cat	Dust	Grain	Lee	Onyx	Rook	Stone
Ash	Clay	Elm	Green	Leigh	Ox	Rose	Storm
Bay	Cliff	Fawn	Grey	Loch	Park	Rue	Suede
Beach	Cloud	Fern	Grove	Lynx	Peach	Rush	Teak
Bear	Clove	Field	Hart	Lys	Pearl	Rye	Teal
Bee	Coast	Fig	Hawk	Mace	Pike	Sage	Thyme
Beech	Colt	Fin	Haze	Maize	Pine	Sand	Tide
Birch	Cove	Finch	Heath	Mare	Pink	Sea	Tree
Bird	Crane	Fir	Ice	Mars	Plum	Shade	Vale
Black	Dale	Fjord	Ink	Marsh	Pond	Shell	Vine
Blaze	Dawn	Flame	Jade	Mauve	Quartz	Shore	Wave
Blue	Day	Flax	Jay	May	Quince	Silk	West
Blush	Dean	Flint	Jett	Mist	Rain	Sky	Wild
Branch	Dell	Fox	Jewel	Moon	Ram	Smoke	Wind
Breeze	Dill	Frost	Kale	Moss	Ray	Snow	Wolf
Brick	Doe	Gale	Koi	Neve	Reed	Sol	Wood
Brook	Dove	Gem	Lake	Night	Reef	Spring	Wren
Brown	Drake	Glen	Larch	Nile	Ridge	Spruce	Yew
Bud	Dune	Glow	Lark	North	Roc	Star	Yves

Each word is either:

- Short, lasting 1 syllable (one beat), denoted by S. (for instance: "green")
- Long, lasting 2 syllables (2 beats), denoted by L. (for instance,"purple")

able	narrow
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icy	very
jingle	water
kitten	x-ray
little	yellow
maple	zebra

ant	no
bike	ox
can	pig
dog	queen
eat	run
fish	six
good	the
have	up
ice	van
jump	was
king	
live	yes
mom	zoo

Find verses of 5 syllables and state their type.
Example: My students are smart - SLSS

How many *types* of verses of one syllable are there? How many of two? three? four? five?

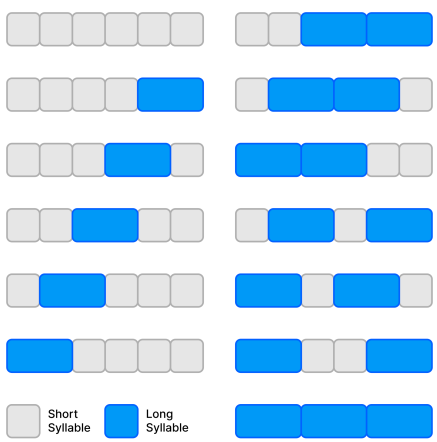
Let's write down verses (in English!) reinterpreting the rules as follows: Each

- **mora** (or "syllabic instant") is a **word**,
- **beat** is a **syllable** of the word

Each word is either:

- Short, lasting 1 syllable (one beat), denoted by S. (for instance: "green")
- Long, lasting 2 syllables (2 beats), denoted by L. (for instance,"purple")

Thirteen ways of arranging long and short syllables in a cadence of length six. Eight end with a short syllable and five (F5) end with a long syllable.



By Romain - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=114415748

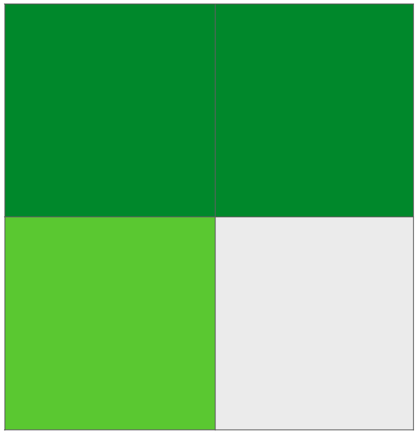
# of Rythms of N syllables composed by short and long words.

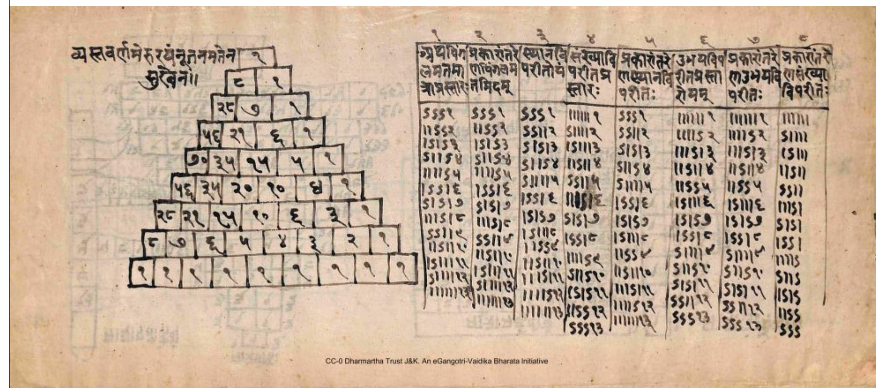
N	# of Rythms of N
1	1
2	2 (ss, l)
3	3(sss,sl,ls)
4	5(ssss, ll, sls, ssl, lss)
5	8 (sssss, lls, lsl, sll, lsss, slss, ssls, sssl)
6	
7	

Types of words we can use for verses

- Short, lasting 1 syllable, denoted by l. (for instance: "green")
- Long, lasting 2 syllables, denoted by S. (for instance,"purple")

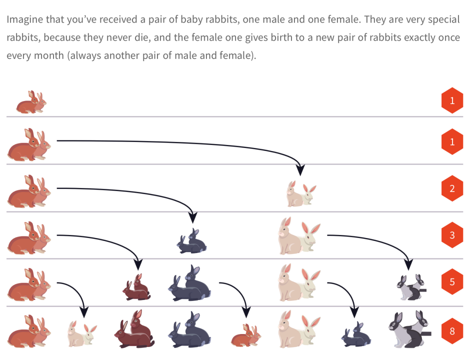
Why the Fibonacci numbers count the number of rhythms of n syllables?

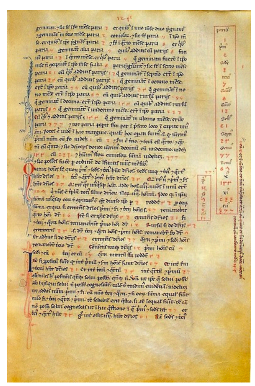




Fibonacci numbers in the Liber Abaci (1202)

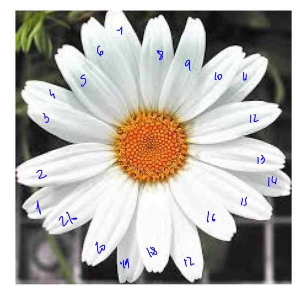
"A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?"

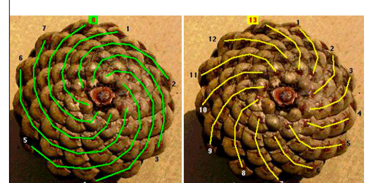


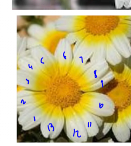


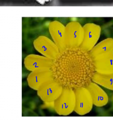
https://mathigon.org/course/sequences/fibonacci

Fibonacci numbers appear when Nature tries to pack things in an efficient way.









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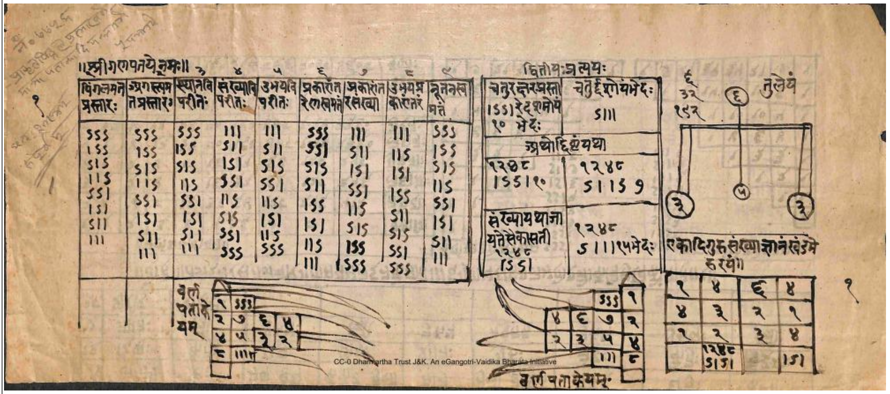
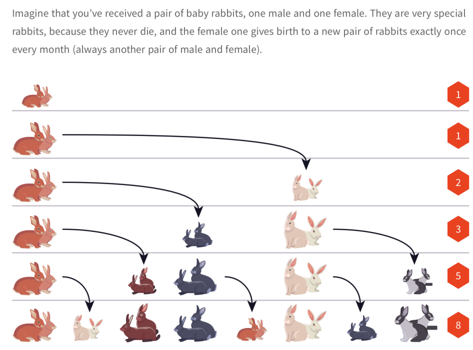
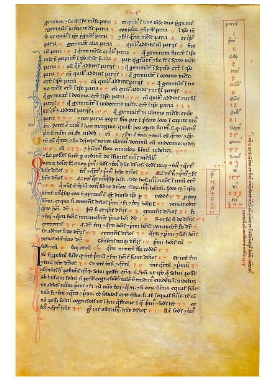


Prakrit Pingala Prastara; Raghunath Temple, Jammu,J&K;
https://archive.org/details/PrakritPingalaPrastaraVarnaMatraPatakadiYantrani775GhaAlm4Shlf3DevanagariAlankarShastra/page/n3/mode/2up

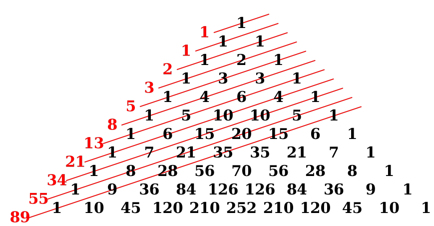
Fibonacci numbers in the Liber Abaci (1202)

"A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?"





https://mathigon.org/course/sequences/fibonacci



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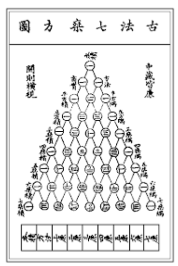
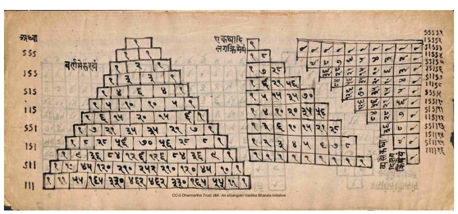
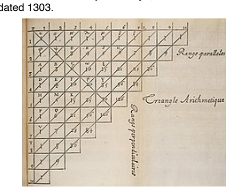


Yang Hui's triangle in Jade Mirror of the Four Unknowns, a mathematical work by Zhu Shijie, dated 1303.

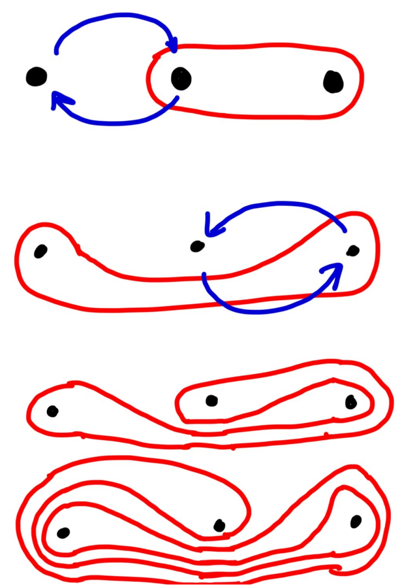


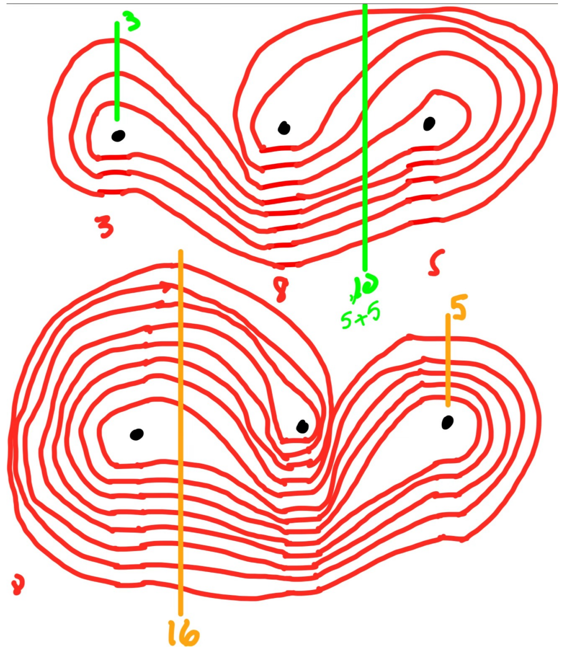
Pingala binomial coefficients triangle. https://archive.org/details/PrakritPingalaPrastaraVarnaMatraPatakadiYantrani775GhaAlm4Shlf3DevanagariAlankarShastra/page/n3/mode/2up

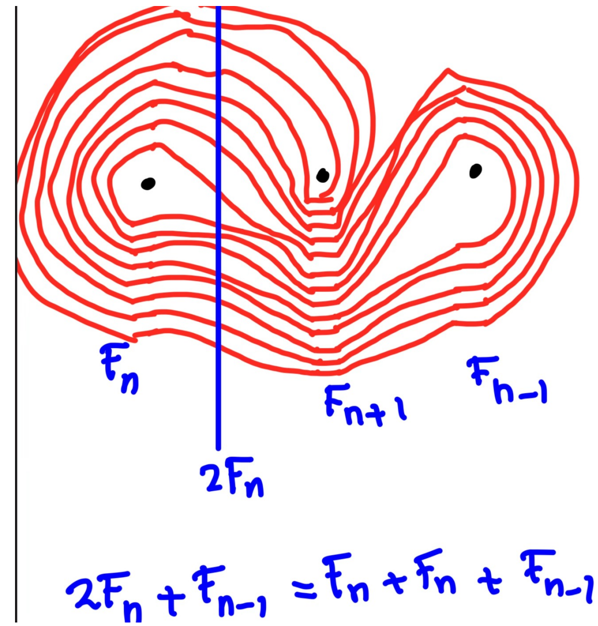


Pascals Pascal Triangle Blaise Pascal - Cambridge University Library, https://cudl.lib.cam.ac.uk/view/PR-CCB-00013-00024/5

Pingala (~250 BCE) classifying poetic meters of long and short syllables, presents the Mount Meru (known now as Pascal Triangle)







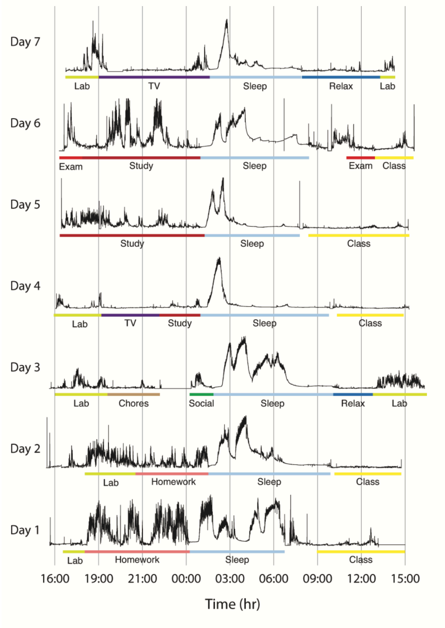
Prince and princess playing chess 19th century

The game of chess was born in India during the Gupta dynasty in the 6th century. Today, more than 1500 years later, it is played in 172 countries. In this exhibit, curators from Salar Jung Museum, Hyderabad take us on a tour of the story of chess. https://artsandculture.google.com/exhibit/a-game-of-thrones-how-chess-conquered-the-world/JwIiNlxUQVZ2Kg

The ingenious method of expressing every possible number using a set of ten symbols (each symbol having a place value and an absolute value) emerged in India. The idea seems so simple nowadays that its significance and profound importance is no longer appreciated. Its simplicity lies in the way it facilitated calculation and placed arithmetic foremost amongst useful inventions. the importance of this invention is more readily appreciated when one considers that it was beyond the two greatest men of Antiquity, Archimedes and Apollonius

Laplace

From Eric Mazur Slides



doi: 10.1109/TBME.2009.2038487

https://mazur.harvard.edu/sites/projects.iq.harvard.edu/files/mazur/files/talk_2060.pdf