# Projective Geometry Seen in Renaissance Art

Word Count: 2655

Abstract: For the West, the Renaissance was a revival of long-forgotten knowledge from classical antiquity. The use of geometric techniques from Greek mathematics led to the development of linear perspective as an artistic technique, a method for projecting three dimensional figures onto a two dimensional plane, i.e. the painting itself. An "image plane" intersects the observer's line of sight to an object, projecting this point onto the intersection point in the image plane. This method is more mathematically interesting than it seems; we can formalize the notion of a vanishing point as a "point at infinity," from which the subject of projective geometry was born. This paper outlines the historical process of this development and discusses its mathematical implications, including a proof of Desargues' Theorem of projective geometry.

Outline:

I. Introduction
   A. Historical context; the Middle Ages and medieval art
   B. How linear perspective contributes to the realism of Renaissance art, incl. a comparison between similar pieces from each period.

C. Influence of Euclid

II. Geometry of Vision and Projection

A. Euclid's *Optics*

B. Cone of vision and the use of conic sections to model projection of an object onto a picture. Projection of a circle explained intuitively.

C. Brunelleschi's experiments and the invention of his technique for linear perspective

D. Notion of points at infinity as a vanishing point

III. Projective geometry and Desargues' Theorem

A. Definition of a projective plane and relation to earlier concepts discussed

B. History of early projective geometry and Desargues, Desargues Theorem; statement and proof of dim 3 case

IV. Conclusion

A. Perspective, and the study of mathematics and science as a whole, as a cultural shift toward an emphasis on the importance of the human experience by accuracy in portraying it.

Math Point: Desargues' Theorem and the basics of projective geometry.

- Books
  - *Perspectives on Projective Geometry: A Guided Tour Through Real and Complex Geometry* by Richter-Gebert
  - *The Geometrical Work of Girard Desargues* by Field and Gray
- Primary Sources
  - *Optics* by Euclid (translated by Burton)
  - *The Geometrical Work of Girard Desargues* by Field and Gray (contains both translations of Desargues and exposition written by Field and Gray)
- Secondary Sources
  - *Brunelleschi's mirror, Alberti's window, and Galileo's 'perspective tube'* by Edgerton (elaborates on Brunelleschi's experiments)
  - *The Geometrical Work of Girard Desargues* by Field and Gray (contains both translations of Desargues and exposition written by Field and Gray)

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I. Introduction

The European Middle Ages was a period of stagnation. While other regions such as Asia and Islamic empires continued the tradition of mathematics in the same spirit as the ancient Greeks, the western world was dormant. Due to complicated factors including the Bubonic plague and the dominance of the Roman Catholic Church, it was frozen in a culture of hierarchy and superstition. Intellectual activity, which largely consisted of dogmatic scholasticism, was restricted to the wealthy and clergy. Study of literature, history, and sciences was limited. Medieval art, such as the painting depicted below, was flat and disproportionate, and depicted religious matters rather than the physical world.

Figure 1

Berlinghiero, *Madonna and Child*, c. 1230s, (The Metropolitan Museum of Art).

This began to change at the dawn of the Renaissance, in approximately 1400. An explosion of intellectualism occurred due to the rise of cities such as Florence and the invention of the Gutenberg printing press. Ideas could be easily spread, as texts came to be written in vernacular, or common language, instead of purely in Latin. A revival of classical studies from architecture to philosophy caused many disciplines to flourish once again. As nobles and merchants accumulated wealth, they became patrons of emerging artistic talent, leading to a vibrant and diverse artistic culture. Geometric techniques such as perspective, scale, and shading led to an increase in realism; note the difference between Fig. 1 above and the following Renaissance era painting.

Figure 2

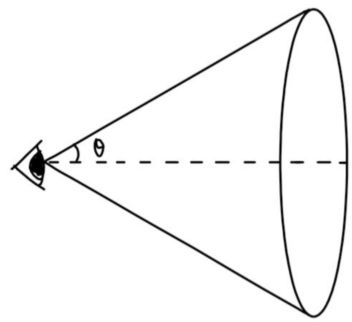
Giovanni Bellini, *Madonna and Child*, c. 1509, (Detroit Institute of Arts)

Bellini uses linear perspective to create an illusion of height and distance on a flat plane. Allow your eyes to travel from the animals in the immediate background, up the mountain, to the building in the distance. There is a clear ray of vision from the observer, past the foreground, to the objects in the distance. One can even perceive depth in the folds of the drapes. Renaissance painters in general obtained this effect through the application of techniques from projective geometry. This was founded on an understanding of optics and Euclidean geometry, as outlined by Euclid in his seminal work *Elements* and his lesser-known treatise *Optics*.

II. Geometry of Vision and Projection

Euclid's *Optics* explores optics from a geometric, rather than physiological lens. Much like *Elements,* it attempts to establish a deductive foundation for propositions which follow from a list of definitions and postulates. Postulate 2 states, "The form of the space included within our vision is a cone; with its apex in the eye and its base at the limits of our vision" (Burton 357). This cone of vision, illustrated by Fig. 3, has a semi-vertical angle of about 30°.

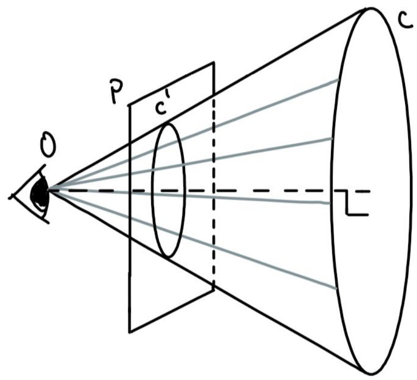
Figure 3




The cone of vision, shown here with semi-vertical angle θ.

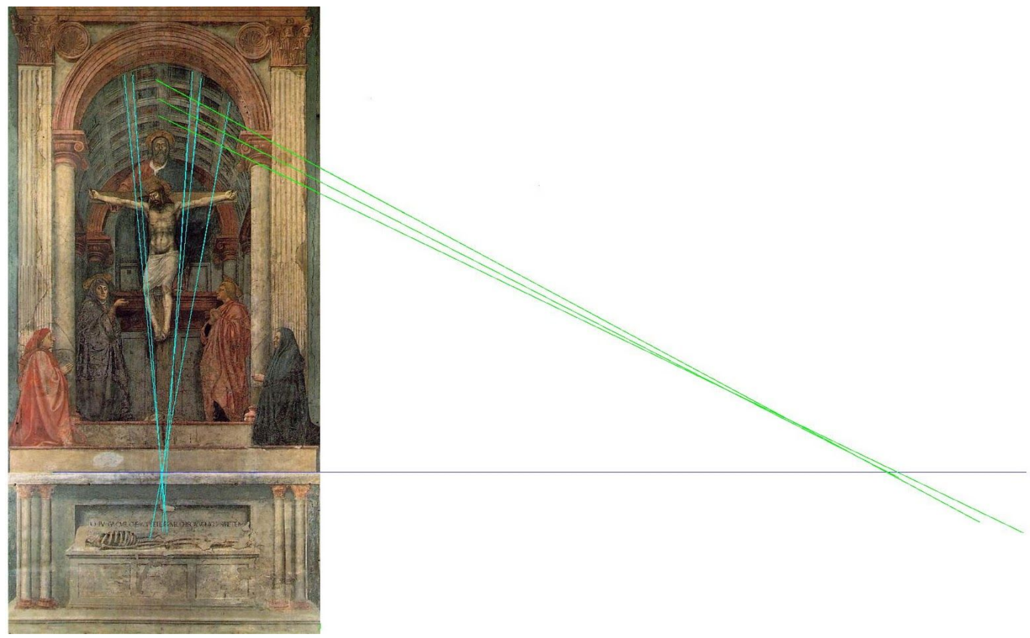
The cone of vision is made up of a continuous set of rays of vision. We know from modern science that each ray of vision from our eye at point O (for *oculus*) to a point F somewhere in our cone of vision is made up of a reflection of light from F to O. When painting a scene, then, one takes a cross section of these rays called the image plane, onto which the three dimensional scene is projected. The image of F under the projection is the point at which the ray of vision from O to F intersects the image plane. Intuitively, the image plane is the "paper," literally the plane onto which the picture is drawn. Recall that a conic section is a curve created by the intersection of a plane with two cones placed apex to apex. If we consider only one of these cones, identified with our cone of vision, this curve can be a circle, ellipse, or a parabola. This explains why the image of a conic section under a projective transformation is a conic section. For example, suppose we have a circle C, in a three dimensional space called the object space, in front of an observer with eye point O. We will describe how this circle is projected onto some image plane P, which is parallel to the plane containing C, as the observer moves. Suppose that the ray of vision intersecting the center of C is orthogonal to the plane containing C, as shown in Figure 4. Then the image of the circle upon projection onto P, denoted by C’, is a circle. After all, the set of rays of vision from O to each point in C is a cone, sliced by P which is parallel to the base.

Figure 4




As the observer moves downward in the object space, the angle will decrease, causing C’ to become elliptical. This occurs because P will intersect the cone of vision at an increasing angle as the observer moves, so it will no longer be parallel to the base of the cone and therefore the conic section created will be an ellipse. In fact, this will occur no matter which direction the observer moves in; C’ will become more elliptical the further the observer moves from the starting point (Treibergs). Additionally, the image of a line will be a line. In particular, if we imagine a plane containing the eye point O and the line, the intersection of this plane with the image plane will be the image of the line (Richter-Gebert). Therefore, the projection of a set of lines parallel to the observer’s line of sight will be a set of lines incident at a vanishing point V. To see why this is the case, we can imagine standing in the middle of an infinitely long straight road. Objects which are further away from us appear smaller, and as stated in *Optics* proposition 8, the relation between the size of an object and its distance is not proportional (Burton 358-359). Thus, the two sides of the road must eventually converge, as the distance between them becomes arbitrarily small at some point V on the horizon. This vanishing point V is referred to as a “point at infinity,” because we can think of it as the point at the end of a line of infinite length. So even though these lines are parallel in the object space, their projections will intersect "at infinity." Note that projection in this way does not preserve distances or angles (Richter-Gebert).

Figure 5

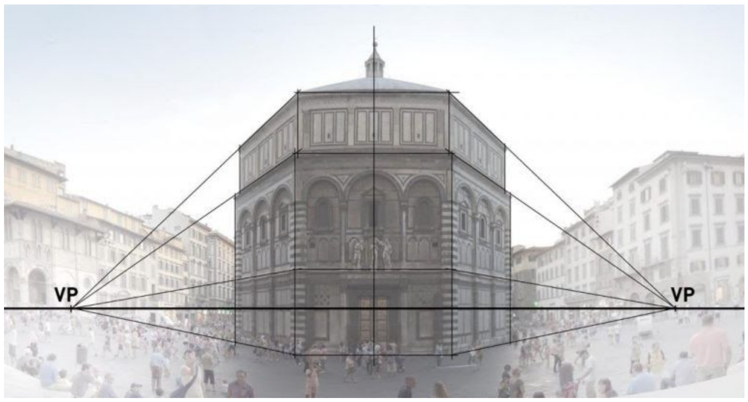


Masaccio, *Holy Trinity*, c. 1427 (Phillips)

A painting drawn in two point perspective. Notice how the realistic dome creates the illusion of depth. The blue and green lines, respectively, intersect at two vanishing points on the purple horizon line "at infinity."

The first to apply this theory to linear perspective in art was Filippo Brunelleschi, a Florentine artist and architect who is known for designing the dome of the Florence Cathedral ("The Art of Renaissance Science"). Brunelleschi conducted a series of experiments on linear perspective and used them to create the first known paintings in proper perspective. In approximately 1425, he drew an exact copy of the Florence Baptistery, an octagonal building in front of the Florence Cathedral (Figure 6). To verify its accuracy, he drilled a small hole in the center of the drawing and through a mirror. Brunelleschi peered through the hole in the drawing, facing away from him, while holding said mirror facing toward him at arm's length in front of the Baptistery. He then aligned the hole in the mirror with the central vanishing point. The drawing was reflected in the mirror, so that he could then move the mirror to the side and verify that the drawing exactly lined up with the actual dimensions of the Baptistery (Edgerton).

Figure 6




A view of the Florence Baptistery with oblique vanishing points labelled VP. The parallel lines are emphasized with sketch lines (ZT Tosha Art).

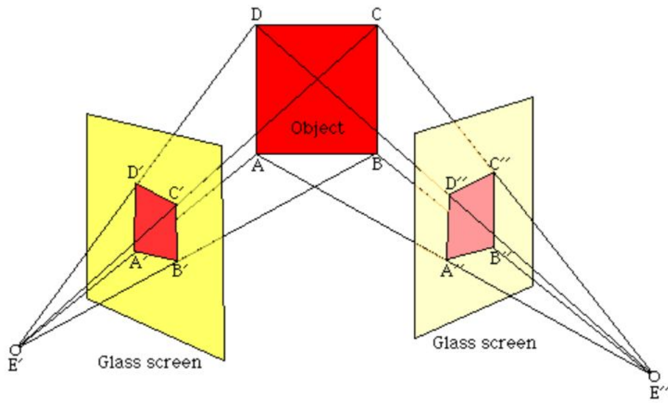
How did Brunelleschi obtain such accuracy? He realized exactly what we observed before: that parallel lines will converge to a vanishing point depending on the position of the observer, so to accurately represent the perspective of an object in a drawing, one must choose the proper vanishing points. The illustration in Figure 6 demonstrates two point projection. Note that the boundaries of the buildings in the background converge to a central vanishing point behind the Baptistery. Moreover, adjustment of the position of the horizon line can place a viewer at a different height in the scene; if the horizon line is above the subject, for example, the perspective is from above. These techniques and more were originated by Brunelleschi based on his experiments, and used to design the Church of Santo Spirito in 1428.

Figure 7

A perspective drawing by Brunelleschi of the Church of Santo Spirito, beside a photograph of the church today ("The Art of Renaissance Science").

Brunelleschi inspired a movement of linear perspective in art across nearly every artist of the Italian Renaissance. Alberti refined Brunelleschi's techniques in his treatise *Della pittura*; Alberti introduced the image plane as we described earlier. We know that a change in perspective changes the dimensions and shape of an object, however, Alberti raises a new question: which properties of a projection are invariant under a change in perspective (Treibergs)? In other words, if two artists produce a painting of the same subject from different perspectives, which properties of the two will remain the same?

Figure 8




A diagram depicting projections of the object ABCD from two different perspectives E' and E''. Projective geometry explores what the projections A'B'C'D' and A''B''C''D'' have in common (Treibergs).

## III. Projective Geometry and Desargues' Theorem

In mathematics, the study of these invariant properties is known as projective geometry. In projective geometry, the set of all vanishing points, or points at infinity, is known as the line at infinity. In the paintings above, the horizon line is the line at infinity. A plane equipped with a line at infinity such that there exist no parallel lines is called a projective plane. Formally, a projective plane is made up of points and lines with a relation called incidence such that the following three conditions hold:

i. For any two distinct points, there is a unique line incident with both of them.

ii. Any two distinct lines are incident at exactly one point.

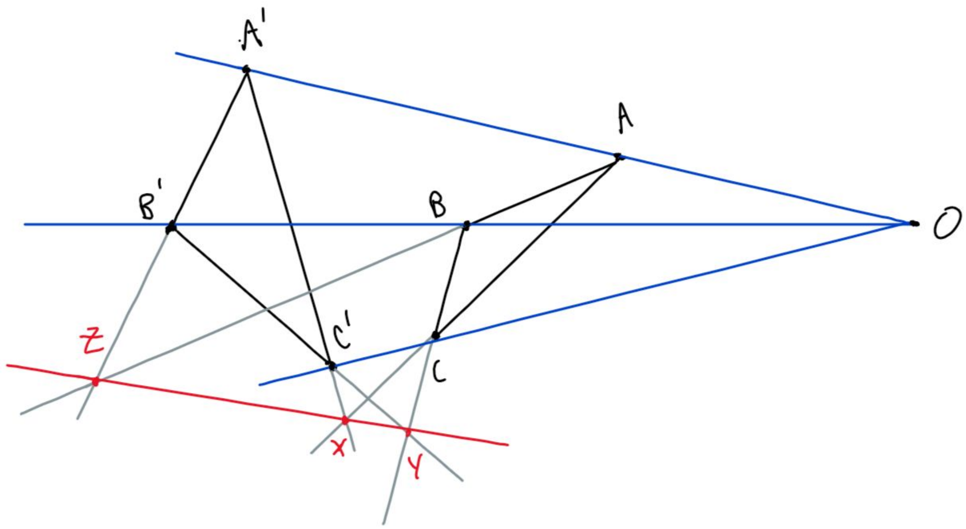
iii. There are four distinct points with no more than two of them incident with the same line.

We use the word "incidence" to describe a line passing through a point, or a point on a line. The first condition states that two points will always be joined by a line. The second ensures that there are no parallel lines; two lines will intersect at a unique point in the plane itself or at the line at infinity. The third condition ensures that trivial cases where most of the points are collinear, or incident with the same line, are excluded (Richter-Gebert).

Projective geometry was born out of the very principles used to create linear perspective during the Renaissance. Girard Desargues was the first mathematician to formalize this connection and is regarded as the inventor of projective geometry; he largely wrote of its applications such as painting in perspective, sundials, and cutting stone for buildings. Desargues formulated his arguments in the style of the Greek geometers, such as Euclid and Appolonius. He did not use numbers or coordinates and built his arguments purely synthetically, rendering his work extremely difficult to read. In fact, Desargues invented botanical terms to describe objects that already had names; in *The Rough Draft on Conics* (1638), he writes: “When through various points of a straight line there pass various other straight lines, in any manner, the line on which these points lie is called a Trunk. The points on the trunk through which other straight lines Knots pass in this manner are called Knots. Any other straight line which passes through one of the knots is called a Branch in relation to the trunk” (Field and Gray 71). Desargues’ entire body of work is written in this confusing style, and this is perhaps why much of his work has been lost, only accessible through secondhand accounts written by his students. A treatise called *Leçons de ténlbrès* has since been lost, but was republished in part in 1648 by a student of Desargues named Abraham Bosse (Fields and Gray 33). This treatise included Desargues’ most important theorem, called Desargues’ Theorem, which involves two triangles in perspective, and is stated as follows.

**Theorem:** Let ABC and A'B'C' be two triangles with distinct vertices, such that the lines AA' , BB' , and CC' are incident in the point O. In other words, these triangles are in perspective from the eyepoint O, with rays of vision AA', BB', and CC'. Let the points of intersection of the pair of lines AC and A'C' be denoted by X, the pair BC and B'C' by Y, and the pair AB and A'B' by Z. Then X, Y and Z are collinear.

Figure 9




Another way to state this theorem is as follows: Two triangles are in perspective from a point if and only they are in perspective from a line. We will give a spatial proof for this theorem if the object space is three dimensional, which is much more elegant than the two dimensional case. The two dimensional case requires theorems from Euclid's *Elements,* and others from antiquity, to be proved synthetically as Desargues would have done. This proof has been adapted from pages 271-272 of *Perspectives on Projective* G*eometry* by Jürgen Richter-Gebert.

**Proof:** To construct the above image, start with an eyepoint O, which will be contained in at least three arbitrary planes. Each of these three planes will contain a line, which we will call AA', BB', and CC' respectively. These are represented by the blue lines in Fig. 9, and can be considered to represent rays of vision. Imagine that the points O, A, B and C have been given some height, while all other points remain in the image plane P. Then the points A', B' and C' lie on some other plane P', and can be thought of as a projection of ABC from the plane P onto P'. The planes P and P' will meet in a line ℓ, which follows from the second axioms of a projective plane, since any two lines $p_1$ in P and $p_2$ in P' must be incident at a point. Recall our initial three planes which respectively contain the rays of vision AA' BB' and CC'. The four points A, A', B, and B' will lie on one of these planes. Then the intersection of this plane with the planes P and P' will be the point Z. The points X and Y are similarly constructed when we repeat this process for the sets of points A, A', C, C', and B, B', C, C'. The points X, Y, and Z will all lie on the line $\ell$, indicated by the red line in Fig. 9. Thus we have proved Desargues' Theorem in the case where the triangles lie in some three dimensional object space. □

Notice that this theorem holds no matter where the planes P and P' are positioned, which means that the collinearity property is independent of a change in perspective. In fact, collinearity of any set of points is preserved by any projective transformation (Richter-Gebert).

## IV. Conclusion

We have seen how the artistic techniques of the Renaissance have influenced the development of projective geometry. This was due to an increased focus on secular humanism; mathematical analysis of the world from a human perspective allowed artists and mathematicians to regard the human experience as one of beauty and mystery, which may even surpass a divine creator, the dominating power for centuries before. Furthermore, projective transformations show up in many areas of modern mathematics, including algebraic geometry and hyperbolic geometry. It is truly fascinating how such a rich field can emerge from the arts, a subject often seen as completely disjoint from mathematics. In reality, this cannot be further from the truth.

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