Word Count: 3035

Original Abstract:

Blaise Pascal was a French mathematician and philosopher. He was a child prodigy in mathematics and wrote an essay on conic sections at the age of 16. He made many contributions to mathematics, including creating an early calculator called the "Pascaline" and laying the groundwork for the formulation of calculus through his development of probability theory.

Maybe his most famous mathematical contribution is his *Treatise on the Arithmetical Triangle*, where he explains "Pascal's Triangle." Pascal's Triangle is a triangular system of numbers in which each number is the sum of the numbers above it. It is a pattern that is found in many areas of mathematics. Specifically, the number pattern in Pascal's Triangle is observed in binomial expansions and combinations.

There is, however, some confusion as to who really discovered this number pattern first. Multiple other mathematicians, such as Indian Halayudha, Persian Omar Khayyam, and Chinese Yang Hui, discussed this triangle in their own works throughout history.

Original Outline:

- Blaise Pascal Biography
  - Early/Personal Life – born in 1623; child prodigy in mathematics; essay on conic sections (*Essai pour les coniques*) in 1640; Christian theologian
  - Contributions to mathematics – built an early calculator (the Pascaline) for his father; developed probability theory; laid groundwork for Leibniz' formulation of calculus; studied "Pascal's Triangle"
- Pascal's Triangle
  - What is it? – triangular system of numbers in which each number is the sum of the numbers above it
  - Pascal's explanations/contributions – *Treatise on the Arithmetical Triangle*
- Other versions of Pascal's Triangle
  - Halayudha – it is believed that Indian mathematician Halayudha first described "Pascal's Triangle"
  - Khayyam Triangle – Persian mathematician Omar Khayyam explains binomial expansion in *The Difficulties of Arithmetic*
  - Yang Hui's Triangle – Chinese mathematician Yang Hui described "Pascal's Triangle" in *A Detailed Analysis of the Nine Chapters on the Mathematical Procedures*

    - Tartaglia’s Triangle – Italian mathematician Niccolò Fontana Tartaglia published the triangle in *General Treatise on Number and Measure*
- Applications
    - Binomial expansions – explain the relationship between Pascal’s Triangle and binomial expansions
    - Combinations – explain the relationship between Pascal’s Triangle and combinations (“n choose k”)

Math Point: combinations and binomial coefficients as explained in Pascal’s *Traité du triangle arithmetique*

Book:

Edwards, A. W. F. “Pascal's Work on Probability.” In *The Cambridge Companion to Pascal*, edited by Nicholas Hammond, 40–52. Cambridge: Cambridge University Press, 2003.

Primary Source:

Pascal, Blaise. *Traité du triangle arithmetique, avec quelques autres petits traitez sur la mesme matière*. Paris: Guillaume Desprez, 1665.

Secondary Source:

Cobeli, Cristian, and Alexandru Zaharescu. "Promenade around Pascal Triangle — Number Motives." *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, Nouvelle Série, 56 (104), no. 1 (2013): 73-98. Accessed November 18, 2020. http://www.jstor.org/stable/43679285.

Professor Chas

MAT 336.02

On Pascal's Triangle (3035 Words)

Few things have as much mathematical depth as the famous "Pascal's triangle." A triangular system of numbers in which each number is the sum of the two numbers above it, this figure has numerous applications. Its number patterns represent things such as the triangular numbers, the Fibonacci sequence, and the powers of two. The triangle can also be used to determine things like combinations and binomial coefficients.

Pascal's triangle gets its name from Blaise Pascal, a French mathematician and philosopher. He made many contributions to the field of mathematics, including creating an early calculator called the "Pascaline" and laying the groundwork for the formulation of calculus through his development of probability theory. One of his most well-known works is the *Traité du triangle arithmetique* ("Treatise on the Arithmetical Triangle"), in which he discussed probability problems and the arithmetical triangle.

However, while it bears his name, Pascal was not the first to study this triangular figure. Multiple other mathematicians, such as Pingala of India, Yang Hui of China, and Niccolò Tartaglia of Italy, discussed this triangle in their own works throughout history. They, similarly to Pascal, recognized the significance of the number patterns and thus used the figure in their own studies of combinatorics.

Regardless of who studied it first, the arithmetical triangle is known as Pascal's triangle in much of the West. His work with the figure contributed to the development of probability theory and even calculus. Nevertheless, the most notable aspect of the arithmetical triangle is not who discovered it first, but the patterns it presents and their many applications in mathematics.

Mathematics in Blaise Pascal's Life

Blaise Pascal was born in France in the early seventeenth century to a skilled mathematician, Etienne Pascal. His father (Etienne Pascal) homeschooled him and used a "'child-centered' approach" which "[favored] experimentation and discovery over rote learning."[1] This teaching strategy allowed Blaise Pascal to develop problem-solving skills, which would contribute to his interest in mathematics. Eventually, Pascal proved to be a child prodigy in mathematics. He discovered Pythagoras' Theorem on his own at age twelve[2] and published a short essay on conic sections by age sixteen.[3] At eighteen years old, Pascal began developing an early calculator-like machine that could add, subtract, multiply, and divide numbers "up to six figures long."[4]

Later in his life, Pascal began working with French mathematician Pierre de Fermat. They discussed a number of probability problems, which inspired Pascal to write his famous *Traité du triangle arithmetique*, in which he studies the arithmetic triangle (i.e., Pascal's triangle). This work is the foundation of much of Pascal's advances in probability theory, and it

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[1] Ben Rogers, "Pascal's Life and Times," in *The Cambridge Companion to Pascal*, ed. Nicholas Hammond (Cambridge: Cambridge University Press, 2003), 6.
[2] Ibid.
[3] A. W. F. Edwards, "Pascal's Work," in *The Cambridge Companion to Pascal*, ed. Nicholas Hammond (Cambridge: Cambridge University Press, 2003), 41.
[4] Ben Rogers, "Pascal's Life and Times," 8.

serves as a basis for later mathematical discoveries such as Isaac Newton's binomial theorem and Gottfried Wilhelm Leibniz's development of calculus.[5] Ultimately, its influence is "the justification for calling the arithmetical triangle 'Pascal's triangle.'"[6]

What is Pascal's Triangle?

Pascal's triangle is typically depicted as a triangular figure of numbers (see fig. 1), in which each number is the sum of the two numbers directly above it. However, Pascal illustrates it as an unbounded array in which "the number of each cell is equal to that of the cell which precedes it in its perpendicular rank, plus that of the cell which precedes it in its parallel rank"[7] (see fig. 2). In other words, the number in a given cell is equal to the sum of the number in the cell to the left of it and the number in the cell above it.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
⋮

Figure 1. Pascal's Triangle as depicted in "A Dozen Questions about Pascal's Triangle."[8]

[5] Cristian Cobeli and Alexandru Zaharescu. "Promenade around Pascal Triangle — Number Motives," *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, Nouvelle Série, 56 (104), no. 1 (2013): 74.

[6] A. W. F. Edwards, "Pascal's Work," 42.

[7] Blaise Pascal, *Traité du triangle arithmetique, avec quelques autres petits traitez sur la mesme matière* (Paris: Guillaume Desprez, 1665), 3.

[8] James Tanton, "A Dozen Question's about Pascal's Triangle," *Math Horizons* 16, no. 2 (2008): 5-30.

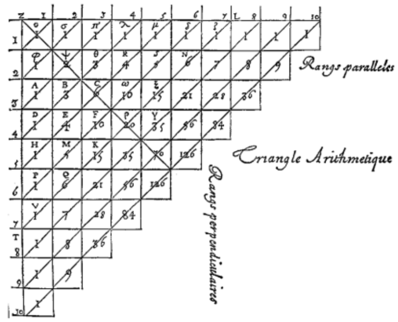

Figure 2. Pascal's Triangle as depicted in the *Traité du triangle arithmetique*.[9]

Considering figure 2, all the cells in both the first row and the first column have the number one in them, and there is an implied row and an implied column of zeroes before those first ones. The second row and second column list the "natural" numbers (1, 2, 3, …), the third row and third column list the "triangular" numbers (1, 3, 6, 10, 15, …), the fourth row and column list the "pyramidal" numbers (1, 4, 10, 20, 35, …), and so on.[10] Each numbered row and column present a pattern of numbers with some mathematical significance.

Similarly, the northeast diagonals have significance. First, the numbers in the cells on each northeast diagonal are symmetric about the main southeast diagonal. Second, when the numbers along a diagonal are added, each sum represents a power of two (e.g., $1+1=2^1$, $1+2+1=2^2$, $1+3+3+1=2^3$, and so on).[11] If the numbers in each cell along a diagonal are viewed as digits, each diagonal represents a power of eleven (e.g., $11^2=121$, $11^3=1331$, $11^3=14641$, and so on).[12] Even the Fibonacci sequence (i.e., 1, 1, 2, 3, 5, …) can be found in Pascal's triangle using specifically drawn diagonals (see fig. 3).

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[9] Blaise Pascal, *Traité du triangle arithmetique*, 5.
[10] Ibid, 10.
[11] TED-Ed, "The Mathematical Secrets of Pascal's Triangle – Wajdi Mohamed Ratemi," September 15, 2015, video, 4:49, https://youtu.be/XMriWTvPXHI.
[12] Ibid.

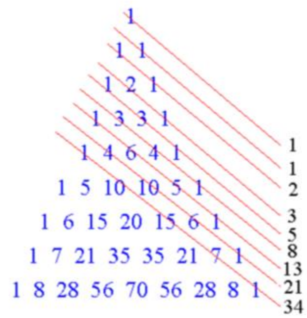

Figure 3. The Fibonacci sequence in Pascal's triangle.[13]

Applications of Pascal's Triangle

Clearly, there are many significant numerical patterns in Pascal's triangle. These sequences are applicable to numerous mathematical concepts and formulas. Perhaps the most common applications of the number patterns in the arithmetic triangle are determining combinations and binomial coefficients.

First, Pascal explains combinations in the following way: "If from four things expressed by these four letters, A, B, C, D, we permit to take from them, for example, any two, all the ways of taking from them two different in the four which are proposed, are called *Combinations*."[14] In general, combinations describe the number of different ways to choose some number of distinct objects from a given total. Today, they are presented as $\binom{n}{k}$, which can be read as "n choose k." Pascal, however, does not use this terminology in his *Traité du triangle arithmetique*. Instead, he uses the general format "combinations of k in n."

[13] "Pascal's Triangle and Its Relationship to the Fibonacci Sequence," accessed November 18, 2020, https://www.maplesoft.com/applications/view.aspx?SID=3617&view=html.
[14] Blaise Pascal, 11.

Considering the problem outlined above (choosing two objects from A, B, C, D), Pascal outlines the possible outcomes: he could choose "A and B, or A and C, or A and D, or B and C, or B and D, or C and D."[15] Thus, there are six possible ways to choose two objects from four, given that all four objects are distinct and not replaced. He continues by listing the possible ways to choose three objects from A, B, C, D, and concludes that "The number of combinations of 1 in 4 is 4. The number of combinations of 2 in 4 is 6. The number of combinations of 3 in 4 is 4. The number of combinations of 4 in 4 is 1."[16] From this example, Pascal derives a number of lemmas which he uses to explain how to find combinations using the arithmetic triangle.

Lemma I states that "[a] number does not combine at all in a smaller."[17] This implies that combinations can only consist of a smaller number in a larger number. Lemma II suggests that "generally any number is combined one time only in its equal,"[18] meaning the number of combinations of n in n is 1. Lemma III states that "generally the unit is combined in any number that it be as many times as it contains unity,"[19] which means that the number of combinations of 1 in n is n. Lastly, Lemma IV demonstrates that, given four numbers where each successive number is greater than the previous one, "the number of combinations of the first in the third, added to the number of combinations of the second in the third, equals the number of combinations of the second in the fourth."[20] In other words, given numbers 1, 2, 3, 4, for example, the number of combinations of 1 in 3 (i.e., 3) added to the number of combinations of 2 in 3 (i.e., 3) is equal to the number of combinations of 2 in 4 (i.e., 6).

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[15] Blaise Pascal, 11.
[16] Ibid, 12.
[17] Ibid.
[18] Ibid.
[19] Ibid.
[20] Ibid, 13.

Finally, Pascal solves Problem 1 of the section on combinations in two ways. Problem 1 is stated as "[t]wo numbers being proposed, to find how many times the one is combined in the other by the arithmetic Triangle,"[21] and Pascal specifically uses the numbers 4 and 6 to explain his solutions. First, he suggests taking "the sum of the cells of the fourth rank of the sixth triangle" (see fig. 4), which "will satisfy the question"[22] and result in the correct answer of 15. His second way involves taking the "[fifth] cell of the [seventh] base…because the numbers 5, 7 exceed by unity the given 4, 6"[23] (see fig. 5), which is also 15, thus solving the problem.

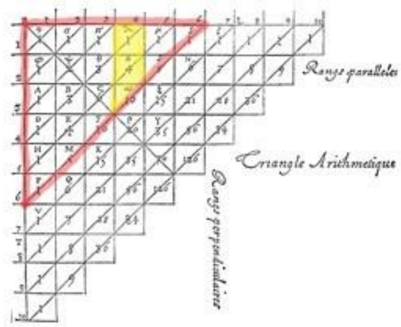

Figure 4. The red triangle outlines the sixth triangle, and the yellow column refers to one of the fourth ranks (the other is the fourth row, which is not highlighted) of that triangle. The numbers in the cells of the fourth rank (1, 4, 10) add up to 15.

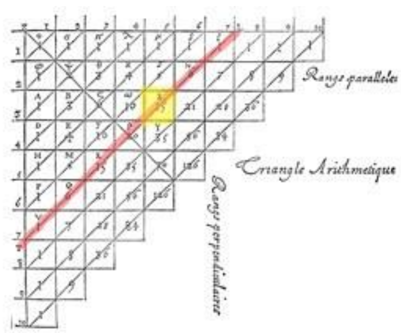

Figure 5. The red line refers to the seventh base, and the yellow box is the fifth cell of the seventh base. The number in this cell is 15.

[21] Blaise Pascal, 15.
[22] Ibid.
[23] Ibid.

Generally, his first solution explains that, given some numbers n and k, the number of combinations of k in n is equal to "the sum of the cells of the [k-th] rank of the [n-th] triangle."[24] In other words, take the triangle formed by the northeast diagonal in the n-th row and column, and add the numbers in the cells of the k-th row or column of that triangle. Pascal's second solution suggests that, given some numbers n and k, the number of combinations of k in n is equal to the number in the $(k+1)$-th cell of the $(n+1)$-th triangle base.

In his *Traité du triangle arithmetique*, Pascal also explains how to use the arithmetic triangle to determine degrees of binomials. Specifically, he states the problem as "it is necessary to find the square-square of $A+1$,"[25] which refers to finding the fourth degree of the binomial $(A+1)$. Pascal explains that one must take the numbers in the cell of the fifth triangle base (see fig. 6) and arrange them with the descending degrees of A as follows: $1A^4 + 4A^3 + 6A^2 + 4A + 1$. He plugs in $A = 1$ to prove this solution method, and so he gets $1 \cdot 1^4 + 4 \cdot 1^3 + 6 \cdot 1^2 + 4 \cdot 1 + 1$. As Pascal describes, this equation becomes $1 + 4 + 6 + 4 + 1$, which equals 16, and "indeed the square-square of 2 is 16."[26]

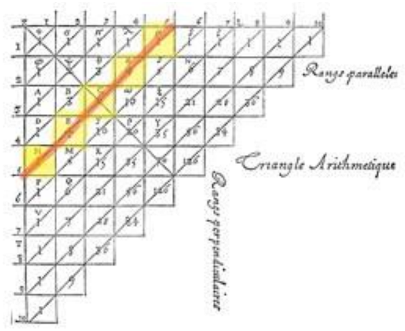

Figure 6. The red outlines the fifth triangle base, and the yellow highlights the cells of that base. The numbers in those cells, in order, are as follows: 1, 4, 6, 4, 1.

[24] Blaise Pascal, 15.
[25] Ibid, 24.
[26] Ibid.

In more general terms, the n-th degree of a binomial $(x+1)$ can be found using Pascal's triangle. The coefficients of each degree are the numbers in the cells of the $(n+1)$-th triangle base, and they are placed in front of $x^n + x^{n-1} + \cdots + x^0$.

Pascal also describes the solution to finding the fourth degree of the binomial $A+2$. He starts with the same polynomial $1A^4 + 4A^3 + 6A^2 + 4A + 1$, and then writes "the first four degrees of 2 under the numbers $4, 6, 4, 1$"[27] in the following way: $1A^4 + \frac{4}{2}A^3 + \frac{6}{4}A^2 + \frac{4}{8}A + \frac{1}{16}$. Finally, he multiplies "the numbers which correspond by one another"[28] (meaning 4 and 2, 6 and 4, 4 and 8, and so on) to get the equation $1A^4 + 8A^3 + 24A^2 + 32A + 16$ as the solution to $(A+2)^4$. He then proves this is the correct answer by setting A equal to 1. From there, he gets $1 + 8 + 24 + 32 + 16$, which equals 81, and "indeed 81 is the square-square of 3."[29]

Again, in more general terms, the n-th degree of a binomial $(x+k)$ can be found using Pascal's triangle. The same process used to find the n-th degree of a binomial $(x+1)$ is used to start, but then each term is multiplied by $k^0, k^1, \ldots, k^n$. So, cumulatively, to find the n-th degree of a binomial $(x+k)$ using Pascal's triangle, find the coefficients of each term using the $(n+1)$-th triangle base, then combine each with their respective term in $k^0x^n + k^1x^{n-1} + \cdots + k^nx^0$.

Other Versions of Pascal's Triangle

While Blaise Pascal is usually credited with the discovery of the arithmetic triangle in the United States, many other mathematicians from around the world discussed this figure prior to

[27] Blaise Pascal, 24.
[28] Ibid.
[29] Ibid, 25.

him. Among those mathematicians are Pingala of India, Yang Hui of China, and Niccolò Tartaglia of Italy.[30] They all also applied the numbers in the figure to combinatorics (the branch of mathematics dealing with combinations) like Pascal did.

To begin, it is currently understood that "[t]he connection between the arithmetical triangle and combinatorial problems was first made in India."[31] Pingala, an ancient Indian mathematician, is the earliest known person to study the arithmetical triangle as it relates to combinatorics. He is probably best known for his (circa 200 B.C.E.) book, the *Chandahsūtra*, in which he discussed the possible combinations that could be made from short and long syllables.[32] In this book, Pingala referred to the triangle as the "*meru prastāra*," or "the holy mountain."[33] He used the arithmetical triangle to determine the number of possible combinations of a meter with a given number of syllables, similarly to how Pascal determined combinations.

The arithmetical triangle was also seen in other Indian mathematicians' work throughout history. For example, in the tenth century, Halayudha explained how to draw the triangle as a matrix and use it to determine the number of possible combinations of a given number.[34] In 850, Mahāvīra listed the seventh and ninth rows of the arithmetical triangle when discussing combinations in his *Ganitasāngraha* ("Epitome of the Essence of Calculation").[35] Additionally,

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[30] Cristian Cobeli and Alexandru Zaharescu. "Promenade around Pascal Triangle," 74.
[31] A. W. F. Edwards, "The Arithmetical Triangle," in *Combinatorics: Ancient and Modern*, ed. Robin J. Wilson and John J. Watkins (Oxford: Oxford University Press, 2015), 168.
[32] Takanori Kusuba and Kim Plofker, "Indian Combinatorics," in *Combinatorics: Ancient and Modern*, ed. Robin J. Wilson and John J. Watkins (Oxford: Oxford University Press, 2015), 44.
[33] A. W. F. Edwards, "Arithmetical Triangle," 168.
[34] Ibid, 169.
[35] Ibid.

in 1068, Bhattotpala also explicitly used the arithmetic triangle and outlined Pingala's *meru prastāra* rule for combinations in his commentary.[36]

The arithmetical triangle was also seen in the work of ancient Chinese mathematicians. Notably, it appeared in Yang Hui's *Xiangjie jiu zhang suanfa* ("Detailed Explanations of the Nine Chapters on Mathematical Methods") of 1261, in which he discussed "algorithms for root extraction."[37] In 1303, Zhu Shijie emphasized the significance of the arithmetical triangle in his work by placing it at the front of his book, the *Siyuan yujian* ("Jade Mirror of Four Elements").[38] Also, in 1867, Li Shanlan published his *Duoji bilei* ("Analogical Categories of Discrete Accumulations"), in which he included an arithmetical triangle representation of the square binomial coefficients[39] (see fig. 7).

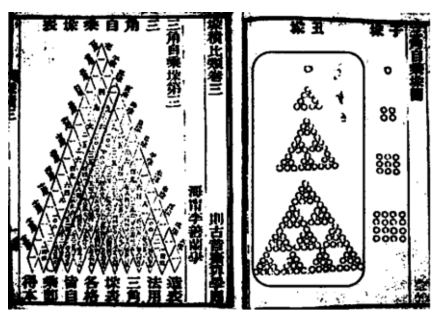


Figure 7. Li Shanlan's representation of the square binomial coefficients using the arithmetical triangle.[40]

Even though it is known in the West as Pascal's triangle, the arithmetical triangle had even been studied by other Westerners before Pascal. The first known western European

[36] A. W. F. Edwards, "Arithmetical Triangle," 170.
[37] Andrea Bréard, "China," in *Combinatorics: Ancient and Modern*, ed. Robin J. Wilson and John J. Watkins (Oxford: Oxford University Press, 2015), 73.
[38] Ibid.
[39] Ibid, 78-79.
[40] Ibid, 79.

mathematician to study the triangle was Niccolò Tartaglia of Italy. In his 1556 *General Trattato di Numeri et Misure* ("General Treatise on Numbers and Measures"), he illustrates the arithmetical triangle (see fig. 8) and uses the first six columns of it to describe the number of possible combinations when rolling six-sided dice.[41] Additionally, Michael Stifel of Germany presented his version of the arithmetical triangle (see fig. 9) in 1544, as did Gerolamo Cardano (see fig. 10) in his 1570 *Novum de Proportionibus Numerorum* ("New Work on the Proportions of Numbers").[42]

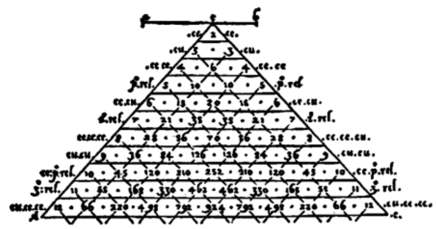
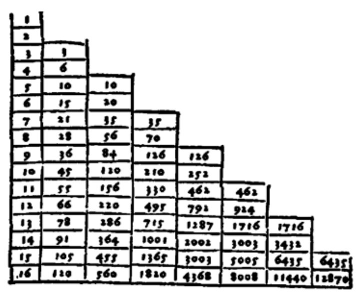


Figure 8. Niccolò Tartaglia's version of the arithmetical triangle.[43]



1							
2							
3	3						
4	6						
5	10	10					
6	15	20					
7	21	35	35				
8	28	56	70				
9	36	84	126	126			
10	45	120	210	252			
11	55	156	330	462	462		
12	66	220	495	792	924		
13	78	286	715	1287	1716	1716	
14	91	364	1001	2002	3003	3432	
15	105	455	1365	3003	5005	6435	6435
16	120	560	1820	4368	8008	11440	12870

Figure 9. Michael Stifel's version of the arithmetical triangle.[44]

41 A. W. F. Edwards, "Arithmetical Triangle," 171.
42 Ibid, 172-173.
43 Ibid, 172.
44 Ibid.

1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1
2	3	4	5	6	7	8	9	10	11	
3	6	10	15	21	28	36	45	55		
4	10	20	35	56	84	120	165			
5	15	35	70	126	210	330				
6	21	56	126	252	462					
7	28	84	210	462						
8	36	120	330							
9	45	165								
10	55									
11										

Figure 10. Gerolamo Cardano's version of the arithmetical triangle.[45]

Clearly, Pascal was not exactly breaking new ground in his discovery of the arithmetic triangle. Plenty of other mathematicians from all around the world studied this mathematically significant triangular figure throughout history. Nevertheless, Pascal's role in the analysis of the arithmetical triangle does not go unnoticed in the West, as he continues to be credited with its discovery.

Conclusion

Pascal's triangle, a triangular figure named for French mathematician Blaise Pascal, is an exceedingly interesting diagram in mathematics. It holds much mathematical significance in the many numerical patterns it presents, such as the powers of two and the Fibonacci sequence. It also has several applications, like determining combinations and binomial coefficients.

Considering the mathematical complexity of the arithmetical triangle, it is no surprise that numerous other mathematicians studied the figure prior to Pascal. Although Blaise Pascal is credited with its discovery in the West, other mathematicians in places like India and China discussed the arithmetical triangle in their own works throughout history. They all sketched similar diagrams of Pascal's triangle and recognized its patterns and applications.

[45] A. W. F. Edwards, "Arithmetical Triangle," 173.

Regardless of who originally discovered the arithmetical triangle, its mathematical significance cannot go unnoticed. Whether one refers to this famous triangular figure as Pascal's triangle, Yang Hui's triangle, or Tartaglia's triangle, the many interesting patterns and applications remain the same. Ultimately, throughout the world, Pascal's triangle is best known for its importance to mathematics—not who discovered it first.

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