Al-Battani & the Law of Cosines

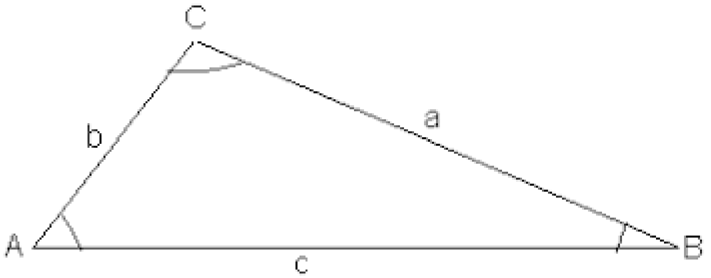
BY {INSERT NAME}

What is the Law of Cosines?

For any **oblique** triangle ABC, with sides a,b,c and angles A,B,C

$a^2 + b^2 - 2ab \times cosC = c^2$ or $\frac{(a^2+b^2-c^2)}{2ab} = \cos(C)$

Oblique triangles have NO right angles!



Given any two sides, (take **a & b)** and angle **adjacent to a & b, (angle C),** you can find the value of the third side, **c.**

This geometric property is referred to SAS, or “Side Angle Side”, and utilizes the law of Cosines to find missing side lengths and angles of any oblique triangle

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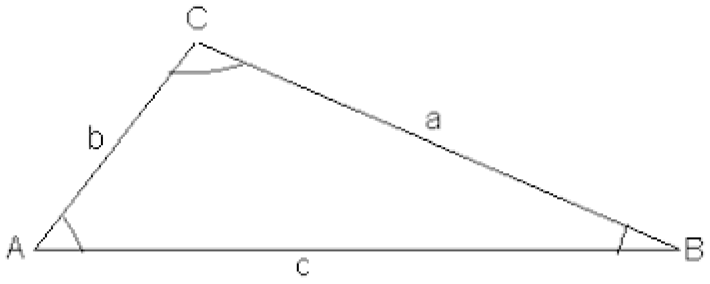
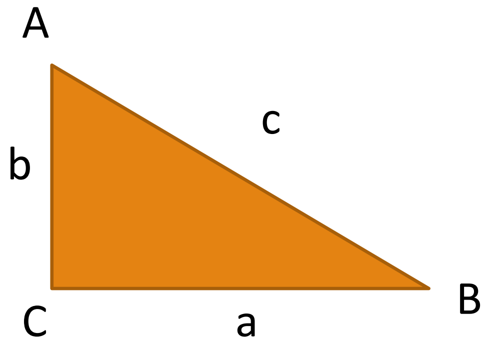




Given any right triangle ABC, we can denote:

$\cos(A) = \frac{b}{c}, \cos(B) = \frac{a}{c}$, $\cos(C) = \cos(90) = 0$

NOTE: the Law of Cosines might approximate something familiar…..

Contributions of not just one, but many!

Euclid

Ptolemy

Regiomontanus

al-Battani

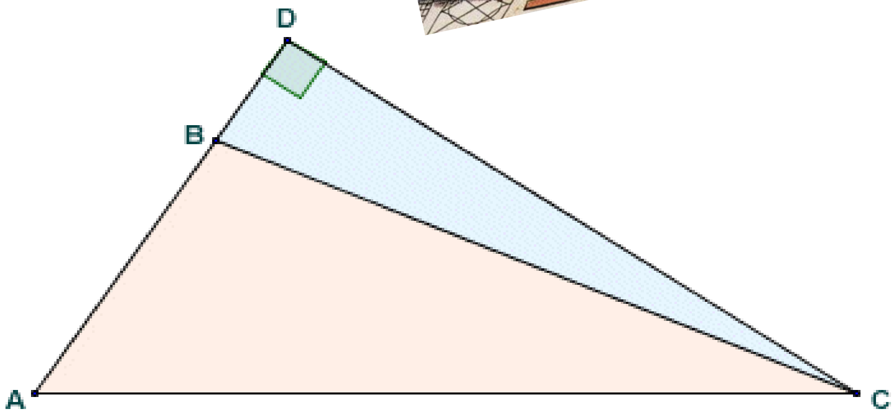
al – Kashi

Euclid’s *Elements* - (~ 300 B.C.E)

“In obtuse-angled triangles the square on the side opposite the obtuse angle is greater than the sum of the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle.” - Book II, Proposition 12

- We have Square AC greater than squares AB + BC, which is also added to “twice the rectangle contained” by
  - ***“which the perpendicular falls”*** – AB
  - ***“line cut off outside by the perpendicular” – BD***

Thus, $(AC)^2 \geq (AB)^2 + (BC)^2 + 2 \times (AB \times BD)$



Euclid's *Elements*

We want to show that given any obtuse $\Delta\, ABC$, that:

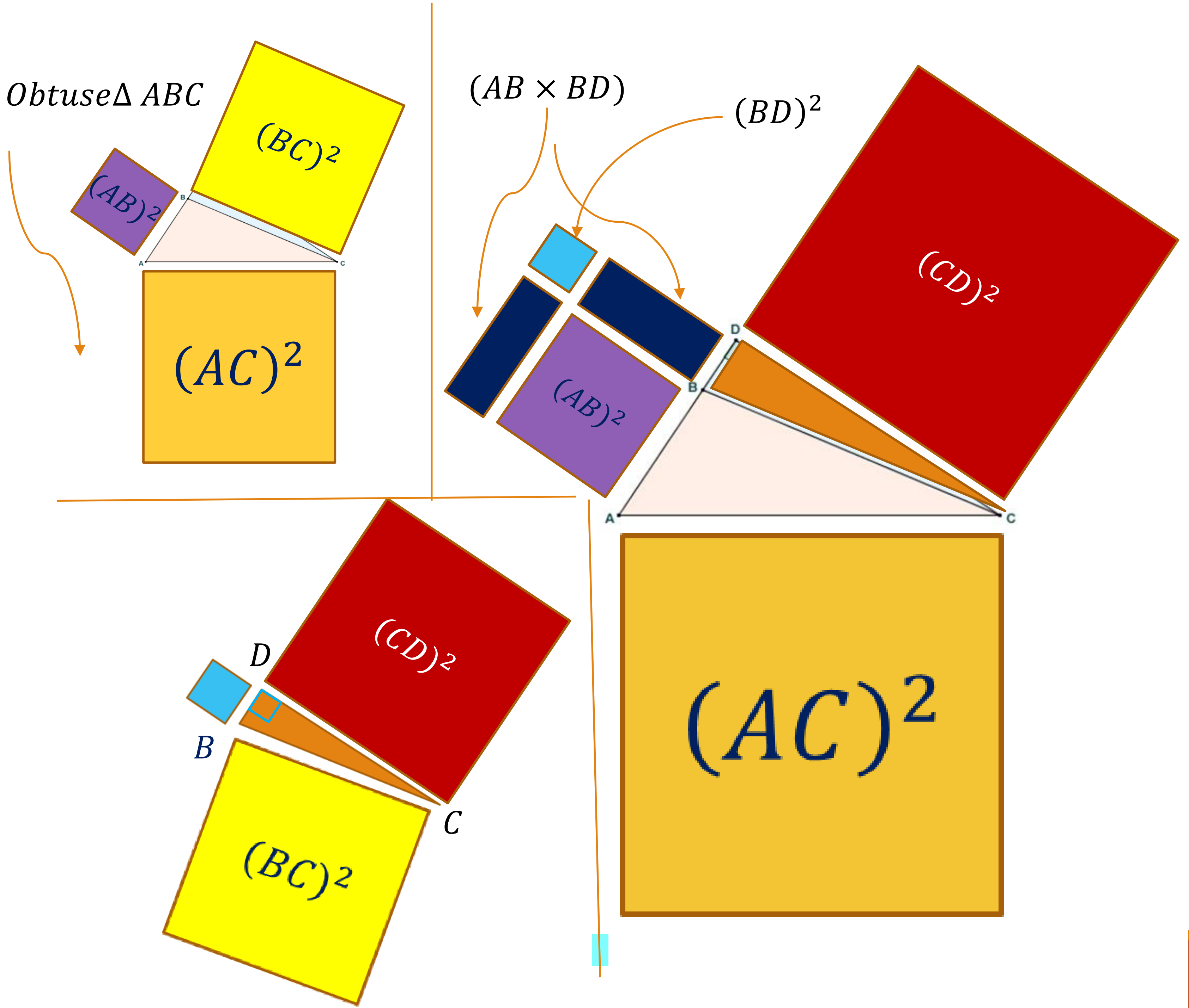
$(AC)^2 \geq (AB)^2 + (BC)^2 + 2 \times (\mathrm{AB} \times BD)$

$(BC)^2 = (BD)^2 + (CD)^2$ *by Pyth. Thm*

$(AC)^2 \geq (AB)^2 + (BD)^2 + (CD)^2 + 2 \times (\mathrm{AB} \times BD)$

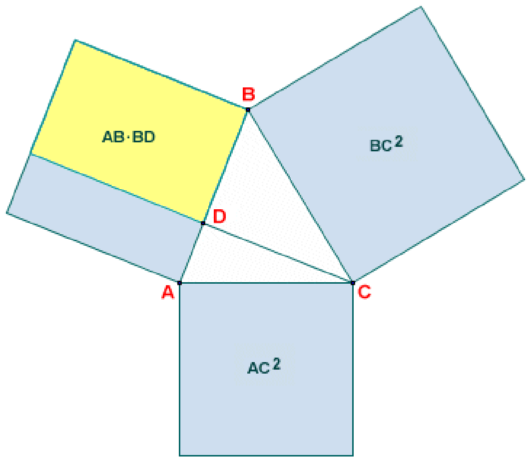
The area of the right triangle is larger than that of the obtuse one

■



Euclid’s *Elements*

“In acute-angled triangles the square on the side opposite the acute angle is less than the sum of the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle.” - Book II, Proposition 13

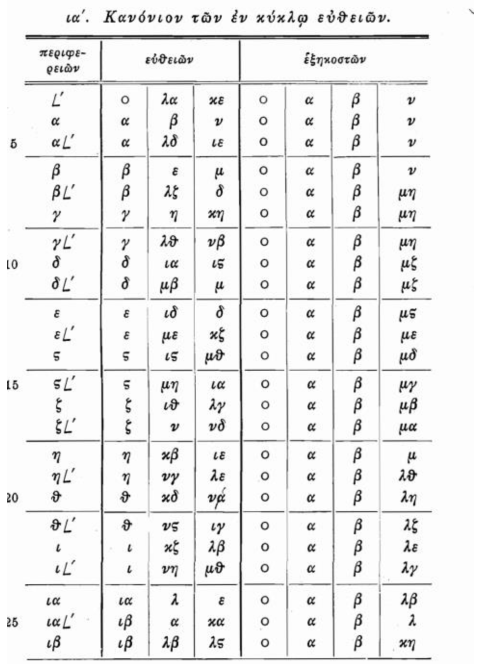


Similar, but with acute angles!

$$(AC)^2 \leq (BC)^2 + (AB)^2 - 2 \times (AB \times BD)$$

$$c^2 = a^2 + b^2 - 2 \times \cos(C)$$

NO TRIGONOMETRY!



ια΄. Κανόνιον τῶν ἐν κύκλῳ εὐθειῶν.

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β∟΄	β	λζ	δ	ο	α	β	μη
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Trigonometry through Ptolemy's eyes – (~100 C.E.)

- Greek/Egyptian mathematician/astronomer

- Trigonometry - the branch of mathematics that analyzes the relationships between triangles' side lengths and angles

- Ptolemy used his charts of chord lengths and angles to find computational differences between chords, arc lengths, and other geometric figures with regards to plane triangles or spherical triangles

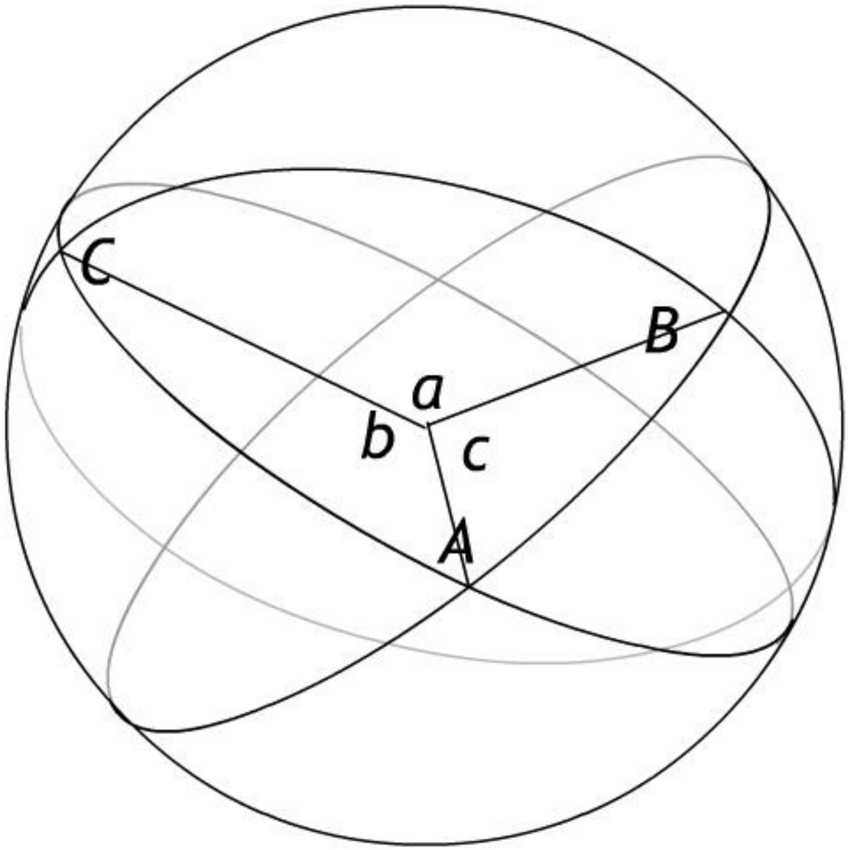
Ptolemy's *Almagest* – predicted motion of the Sun, moon, stars and other planets

al–Battani – (858-929 C.E.)
“Ptolemy of the Arabs”

- Well known for his contributions to astronomical science
- Was known for his discovery of secant and cosecant (as reciprocals of sine and cosine)
- The tangent defined as an “extended shadow”

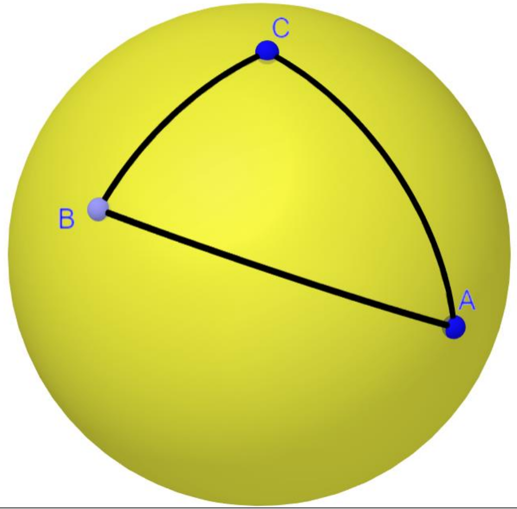
“… after having lengthily applied myself in the study of this science, I have noticed that the works on the movements of the planets differed consistently with each other, and that many authors made errors in the manner of undertaking their observation, and establishing their rules. I also noticed that with time, the position of the planets changed according to recent and older observations; changes caused by the obliquity of the ecliptic, affecting the calculation of the years and that of eclipses. Continuous focus on these things drove me to perfect and confirm such a science.”

A quick note on “Spherical” Geometry



<u>Non-Euclidean</u>, or “Spherical” Geometry:

- Used in astronomy, navigation
- three-dimensional shapes
- “Great circles” – circular intersections into two dimensional planes
- Form spherical triangles



“In a nutshell, Al-Battani had the equipment, but did not formulate the law. He works in astronomical, not geometric terms.” – Tony Phillips, SBU

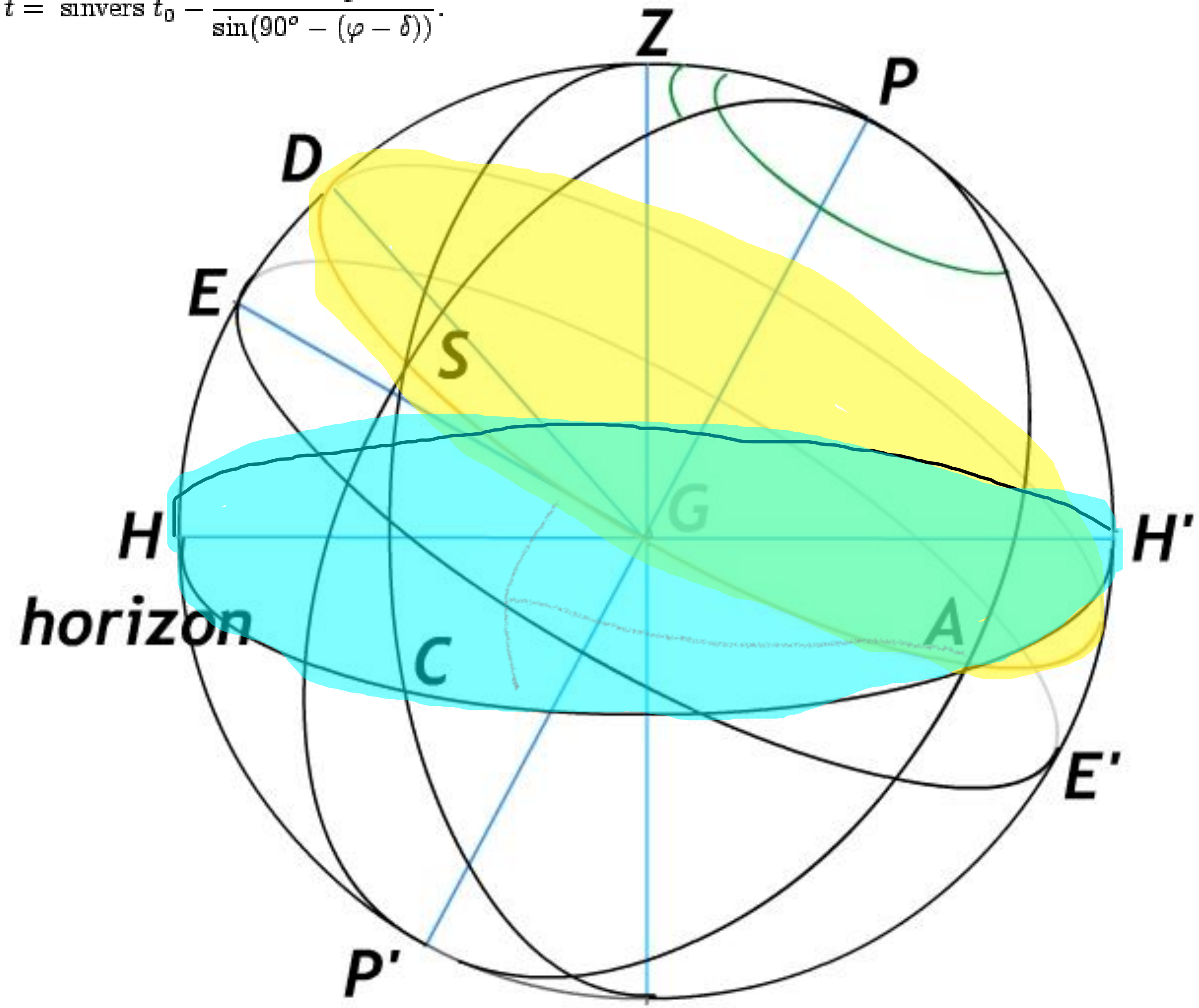
- He calculated the length of the day given of the “**latitude**” (at any given point of the observer) and the “**ecliptic**”

Take the horizonal plane (labelled H,H’) as a **latitude**

Take a circular plane (for instance, (D,A)) as an **ecliptic**

Al-Battani used variables relating the two planes to form “spherical triangles” ΔAHE of “angles” t, t_0, and then his $sinvers\ (t)$ equation to calculate t.

$$\text{sinvers } t = \text{sinvers } t_0 - \frac{\text{sinvers } t_0 \sin h}{\sin(90^\circ - (\varphi - \delta))}.$$



"In a nutshell, Al-Battani had the equipment, but did not formulate the law. He works in astronomical, not geometric terms." – Tony Phillips, SBU

Consider al-Battani's given

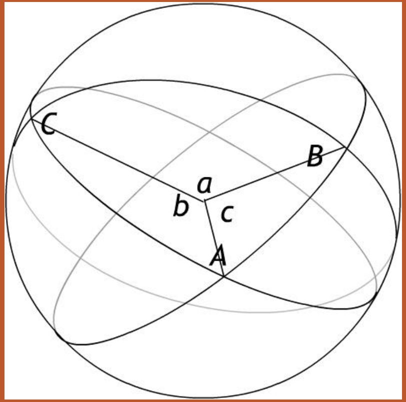
$$sinvers\ (t) = 1\ - \cos(t)$$

function with:

sides of **spherical** triangles (in terms of $a, b\ \&\ c$)

instead of **times** (in terms of t, t_0)

$\angle C$ representing the angle between side "a" and side "b"

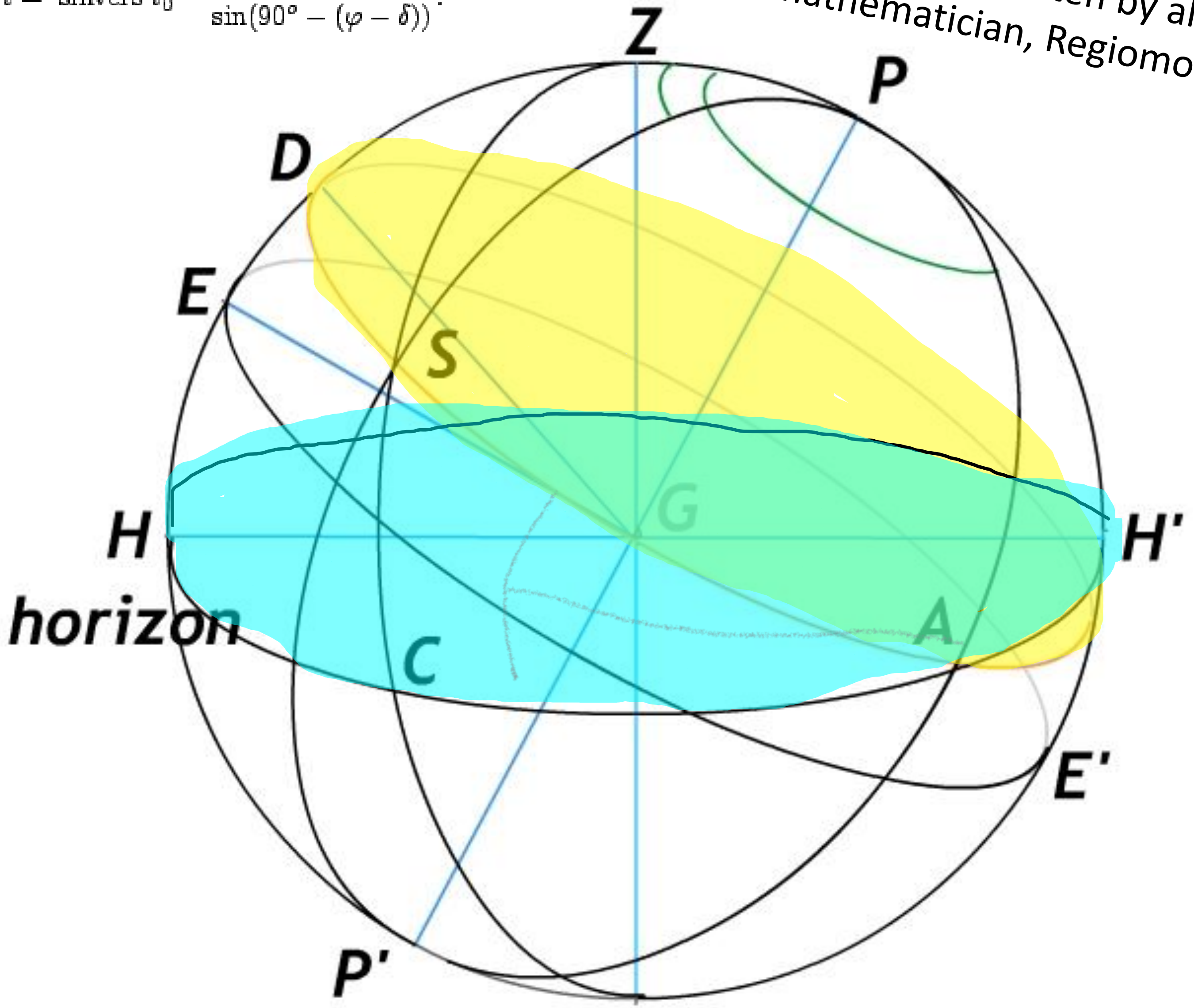


$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(C)$$

$$\frac{\cos(c) - \cos(a)\cos(b)}{\sin(a)\sin(b)} = \cos(C)$$

$$\text{sinvers } t = \text{ sinvers } t_0 - \frac{\text{sinvers } t_0 \sin h}{\sin(90^\circ - (\varphi - \delta))}.$$

This generalization was not written by al-Battani, but by German mathematician, Regiomontanus!

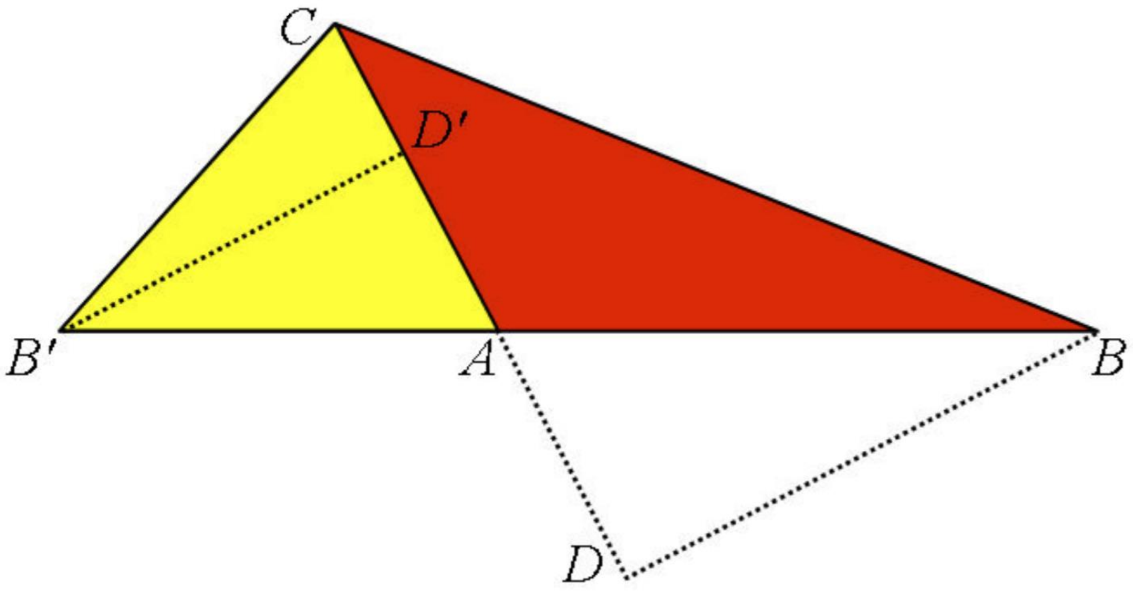


al-Kashi (1380-1449 C.E.)

- Persian mathematician/astronomer

- Known for being the first to officially state the law of cosines as a combination of the two former:

- using Euclidean geometry and trigonometric tables, he was able to state the theorem simple enough for general oblique triangles in a two - dimensional setting



Law of Cosines (*Théorème d’Al-Kashi*):

If A is the angle at one vertex of a triangle, a is the opposite side length, and b and c are the adjacent side lengths, then

$$a^2 = b^2 + c^2 - 2bc \times cos(A)$$

$$(AC)^2 \geq (AB)^2 + (BC)^2 + 2 \times (AB \times BD)$$
$$(AC)^2 \leq (AB)^2 + (BC)^2 - 2 \times (AB \times BD)$$

Planar vs Spherical Law of Cosines

$$\cos(C) = \frac{\cos(c) - \cos(a)\cos(b)}{\sin(a)\sin(b)}$$

We've found two distinct and early forms of the Law of Cosines:

Planar:

- Based in two-dimensional triangles, more theoretical and basic
- No trigonometry!

$$a^2 = b^2 + c^2 - 2bc \times cos(A)$$

Spherical:

- Based in a three-dimensional space, theoretical and mathematical definition with applications in finding astronomical discoveries
- Heavy implication of trigonometric conversions and table use

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