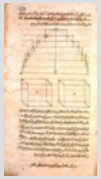
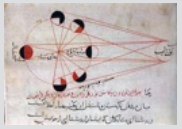
MAT 336: Mathematics in the Islamic Golden Age

Wikipedia: Engraving of Abū Sahl al-Qūhī's perfect compass to draw conic sections.





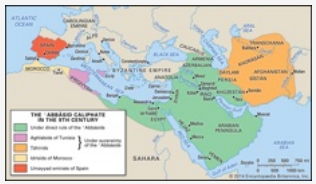




- Islamic Golden Age
- al-Khwarizm
- Omar Khayyam
- Perfect compass

A (too) brief history of the Islamic Empire

Rough Chronology

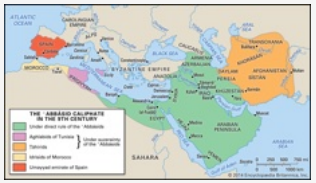


https://www.britannica.com/place/Caliphate/The-Abbasid-caliphate

- **c. 600**: Rise of Islam in the Arabian Peninsula.
- **c. 650**: Islamic rule expands and unites diverse peoples across a large region.
- **c. 750**: Abbasid dynasty comes to power.
- **c. 800**: Baghdad becomes a major center of learning and scholarship.
- **c. 800–1300**: Period often called the Islamic Golden Age.
- **c. 800**: The House of Wisdom is associated with Baghdad and the translation movement.
- **c. 800**: Arabic becomes a major language of science and scholarship.

About the denomination "Islamic"

"Islamic mathematics" refers to mathematics developed within societies shaped by Islamic civilization, not exclusively mathematics done by Muslims. Scholars included Muslims, Christians, Jews, and others. Arabic was a major shared language of scholarship.



https://www.britannica.com/place/Caliphate/The-Abbasid-caliphate

Why Did Islamic Scholarship Flourish?

- Abbasid dynasty encouraged scholarship.
- House of Wisdom.
- Translation movement
- Confluence of many cultures.
- Freedom of scholarship
- Religious tolerance and strong interest in knowledge.
- Paper (from China) although no printing press. (woodblock printing)

One Thousand and One Nights

- **The story:** Scheherazade marries a sultan who kills each wife after one night. She survives by telling a story each night, always stopping at dawn at a cliffhanger. 1001 nights later, she has her life and his respect.
- **The book**: Compiled in Arabic from Persian and Indian sources, 8th-14th centuries.

- **Sinbad the Sailor**
- **The Fisherman and the Genie**
- **Aladdin and the Magic Lamp** (Aladdin entered the collection on the 18th-century)

Legacy and impact: from Borges to Prince of Persia, from Magic: The Gathering to Disney, stories we know but don't realize where they came from.

Scheherazade and the sultan by Iranian painter Sani al Mulk (1849–1856)



By Sani ol-Molk (1814-1866) - http://www.ir-tmca.com/Exhibition/negargari/work1.htm [1], Public Domain, https://commons.wikimedia.org/w/index.php?curid=2631435

The House of Wisdom

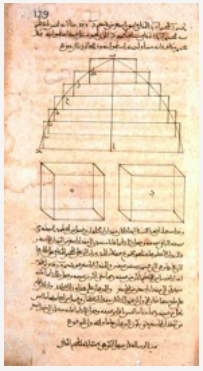
Baghdad's **center of translation and scholarship** — where manuscripts from Greece, Persia, and India were collected, translated, and built upon. (The exact nature of the institution is still debated.)



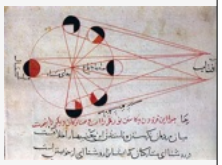
library, painted by Yaḥyā al-Wāsiṭī (Baghdad, 1237) for al-Ḥarīrī's Maqāmāt: the trickster Abu Zayd in a library in Basra. BnF Arabe 5847, fol. 5v. Book "Maqamat al-Hariri". Painter: Yahya al-Wasiti. Baghdad 1237 AD. At French Nat'l Library, Codex Parisinus Arabus 5847, fol. 5v. Online at: https://gallica.bnf.fr/ark:/12148/btv1b8422965p/f20

Mathematical Achievements

- **Translation movement**: Mathematical manuscripts from all around the world were collected and **translated**. Islamic scholars **amalgamated knowledge** from Egypt, Babylonia, Greece and India.
- Close relations between theory and practice.
- Systematization of decimal number system, including fractions.
- Started to understand and systematize relations between algebra and geometry.
- Advances in number theory and in geometry (plane, 3D and spherical)
- Combinatorics were brought from India.



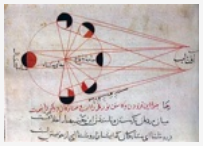
Geometrical shapes from Suhayl al-Quhi's book Fī istihraci mesaha al-muhassama al-maqafī or Risala-i abu Sahl [Suleymaniya library, Ayasofya-4832]



Wikipedia: al-Biruni's explanation of the phases of the moon

Mathematicians of the Islamic Golden Age

- **al-Khwārizmī** (~800): algebra; Indian numerals
- **Ibn al-Haytham** (~1000): optics; sums of powers; parallel postulate
- **al-Bīrūnī** (~1000): astronomy; trigonometry; Earth's radius
- **Omar Khayyam** (~1100): cubic equations solved with conic section.
- **Naṣīr al-Dīn al-Ṭūsī** (~1250): systematic treatment of plane and spherical trigonometry



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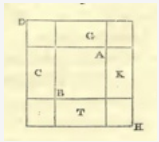
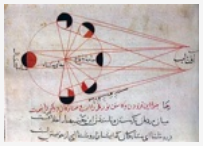


More Mathematicians of the Islamic Golden Age

- **Thābit ibn Qurra** (~850): translations of Greek mathematics; amicable numbers
- **al-Battānī** (~900): sines instead of chords; astronomical tables used by Copernicus
- **Abū Kāmil** (~900): algebra with irrational numbers; read by Fibonacci
- **al-Karajī** (~1000): algebra of polynomials; binomial coefficients



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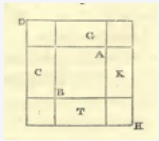


Educated guess: Name authors of books which were translated into Arabic during the Islamic Golden age. (Just write one, if you know more than one, tell the name to a classmate)
Hint: we studied most of them in class

Mathematical works translated or adapted into Arabic (c. 750–900 CE)

GREECE (c. 300 BCE–250 CE)

- **Euclid**: Elements, Data, Optics
- **Archimedes**: On the Sphere and Cylinder
- **Apollonius**: Conics
- **Diophantus**: Arithmetica
- **Ptolemy**: Almagest



Euclid's Element Translated into Arabic Origin: 1237 CE https://www.fihrist.org.uk/catalog/manuscript_1798

PERSIA (c. 600 CE)

- **Sasanian astronomical tables**: Used to predict planetary positions and other astronomical phenomena.

INDIA (c. 650 CE)

- **Brahmagupta**: Mathematical and astronomical works, including methods of calculation and astronomical tables.

These works were **translated, adapted, studied, and developed further,** not simply preserved.

Places and languages of knowledge

Main routes



Before Alexandria

- Babylon → Greece: base-60 angles (degrees, minutes) and astronomical records.

Into Baghdad (~750–900)

- From Greece, often through Syriac
- From India: numerals and astronomy (Sindhind ~770)
- From Persia: astronomical tables

Transmission to new centers

- West: al-Andalus → Toledo (~1100s): Euclid, al-Khwārizmī and Ptolemy translated into Latin
- Sicily: Greek → Latin directly (~1160)
- North Africa → Pisa: Fibonacci learns the Indian numerals (~1200)
- East: Marāgha (~1260) → Samarkand (~1420)

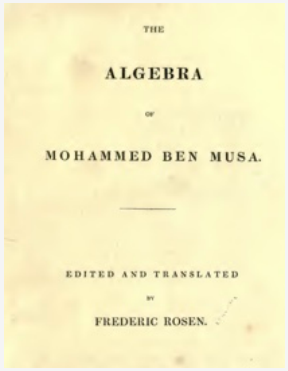
Socrates and his students" As imagined by a 13th century Turkish manuscript. Greek knowledge didn't just pass through Islamic scholarship, it was absorbed, debated, and reimagined.



Socrates and his Students, illustration from 'Kitab Mukhtar al-Hikam wa-Mahasin al-Kilam' by Al-Mubashir, Turkish School, (13th c) Photo by Bridgeman

https://aeon.co/ideas/arabic-translators-did-far-more-than-just-preserve-greek-philosophy

Al-Khwārizmī



THE

ALGEBRA

OF

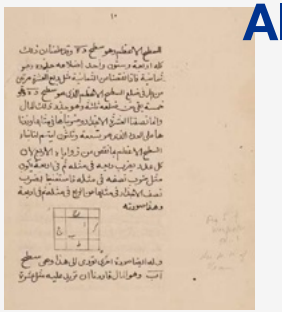
MOHAMMED BEN MUSA.

EDITED AND TRANSLATED

BY

FREDERIC ROSEN.

https://www.wilbourhall.org/pdfs/The_Algebra_of_Mohammed_Ben_Musa2.pdf



This is a page from al-Khwārizmī's algebra text, Kitāb al-jabr wa'l-muqābala, written in about 825, one of the earliest survivingt algebra text, by al-Khwārizmī. https://www.maa.org/press/periodicals/convergence/mathematical-treasures-al-Khwārizmī s-algebra

Al-Khwārizmī ~800 CE Bagdag



Statue of al-Khwārizmī in front of the Faculty of Mathematics of Amirkabir University of Technology in Tehran, IRAN

By M. Tomczak - http://www.es.flinders.edu.au/~mattom/science+society/lectures/illustrations/lecture16/alkhwarizmi.html, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=27754448

Sometimes called "the father of algebra" (same as Diophantus). (My opinion is that Algebra has many, many parents)

(e.g. his astronomical tables were based on the Sindhind, an Arabic version of Indian astronomy going back to Brahmagupta.)

al-Khwārizmī ~ 800CE

- **CLASSIFICATION:** six standard forms for linear or quadratic equations
- **ABSTRACTION:** solutions using **algebraic methods and geometrical diagrams -**
- **NEGATIVE NUMBERS**
  - thinking linked to geometry, thus he rejected negative results.
  - in laws inheritance, used negatives as **debts**.



al-Khwārizmī ~ 800

Wrote The Book of Restoration and Balancing
(Kitāb al-jabr wa'l-muqābala)



- **al-jabr:** "restoration," "completion," or "rejoining" (the operation of moving a term from one side of = to the other to eliminate a **minus** sign)
- **al-muqābala:** "balancing," "comparison," or "setting opposite" (canceling equal terms on both sides of =)

The word algorithm derives from his name

https://en.wikisource.org/wiki/The_Compendious_Book_on_Calculation_by_Completion_and_Balancing/Compendium_on_calculating_by_completion_and_reduction

1. What reasons does al-Khwārizmī give for writing this book? 2.What stands out to you most about this reason?

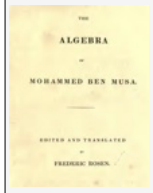


That fondness for science, by which God has distinguished the Imam al-Mamun, the Commander of the Faithful, … that affability and condescension which he shows to the learned, that promptitude with which he protects and supports them in the elucidation of obscurities and in the removal of difficulties, has encouraged me to compose a short work on calculating by al-jabr and al-muqabala, confining it to what is easiest and most useful in arithmetic.



When I considered what people generally want in calculating, I found that it always is a number.

Muhammad ibn Musa al-Khwārizmī



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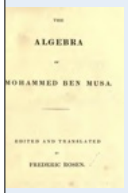
I observed that the numbers which are required in calculating by completion and reduction are of three kinds, namely, roots, squares, and simple numbers relative to neither root nor square.

A root is any quantity which is to be multiplied by itself, consisting of units, or numbers ascending, or fractions descending.

A square is the whole amount of the root multiplied by itself.

A simple number is any number which may be pronounced without reference to root or square.

A number belonging to one of these three classes may be equal to a number of another class; you may say, for instance, " squares are equal to roots," or " squares are equal to numbers," or " roots are equal to numbers.



1. According to al-Khwārizmī, what are the three kinds of numbers? Use his terms. 2. In modern notation, write an example of an equation where 'squares are equal to roots'.



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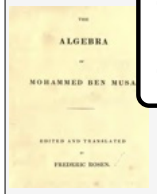


root → unknown, x
square → its square, x^2
simple number → constant term

Al-Khwārizmī classified terms of a linear and quadratic equations as follows:

A. Square (of the unknown, x^2 for us)
B. Root (the unknown, x)
C. Number (the constant term)

Classification

Al-Khwārizmī used

- Only positive coefficients
- Equations with at least one positive solution.
- Zero as numeral but not as a number.

Write down as many different types of equations as you can find with the above rules. (For instance: a x^2 =c is one of these types)

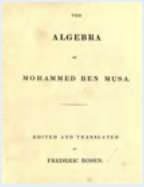
Types of Equations According to al-Khwārizmī (Linear and Quadratic)

1. Squares equal to roots: $ax^2=bx$
2. Squares equal to numbers: $ax^2=c$
3. Squares equal to roots and numbers: $ax^2=bx+c$
4. Squares and roots equal to numbers: $ax^2+bx=c$
5. Squares and numbers equal to roots: $ax^2+c=bx$
6. Roots equal to numbers: $bx=c$



Of the case in which squares are equal to roots, this is an example. "A square is equal to five roots of the same;" the root of the square is five, and the square is twenty-five, which is equal to five times its root.

Express "A square is equal to five roots of the same" as an equation" in modern notation.



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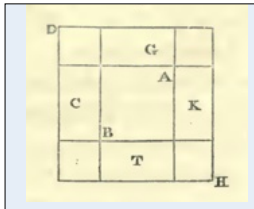

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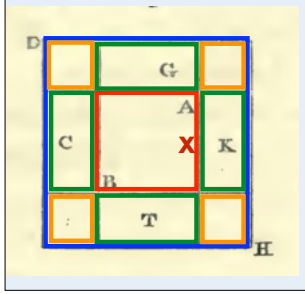
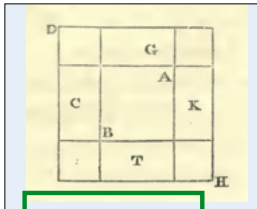


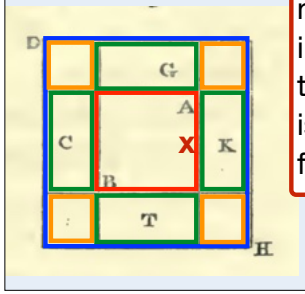
Questionnaires



Demonstration of the Case: "a Square and ten Roots are equal to thirty-nine Dirhems."

The figure that explains this is a square whose sides are unknown. It represents the square whose root you wish to find. This is the figure AB, where each side represents a root. If you multiply one side by any number, that product represents the number of roots added to the square. Since in this problem the roots are combined with the square, we take one-fourth of ten—that is, two and a half—and add it to each of the four sides of the figure.





Questionnaires

al-Khwārizmī- Case a Square and ten Roots are equal to thirty-nine dirhams.

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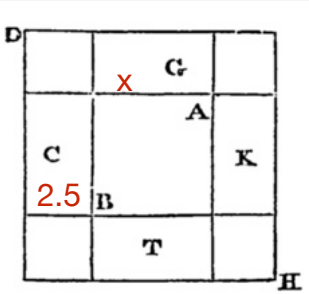


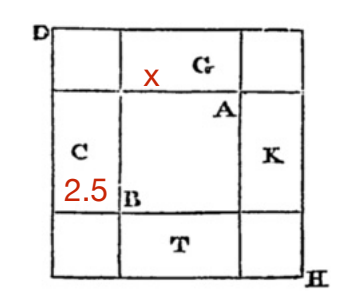
Find the area of **one** of the four attached rectangles (C, G, T, or K) in terms of x.

al-Khwārizmī cont.

With the original square AB, we attach four rectangles, each having a side of the square as its length and two and a half as its width. These are rectangles C, G, T, and K. We now have a figure with unknown but equal sides, but at each of the four corners a small square of two and a half by two and a half is missing. To fill this gap and complete the large square, we must add four times the square of two and a half—that is, twenty-five.

Find the area of one of the four corner little squares.

Add the areas of the four corner squares to the areas of AB, C, G, T and K to get the area of the large square DH.

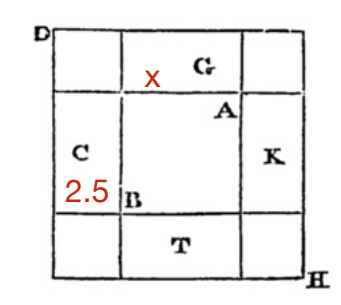


al-Khwārizmī cont.

We know from the problem that the original square, together with the four rectangles around it (representing the ten roots), equals thirty-nine. If we add twenty-five (the four corner squares that complete the large figure DH), the total is sixty-four. One side of this large square is its root: eight.

Note that the original square AB plus the four rectangles equals 39. What is the total area of the completed large square DH?

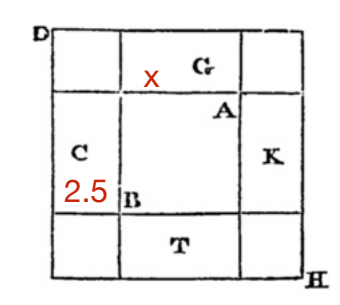
What is the side length of the large square DH?



al-Khwārizmī cont.

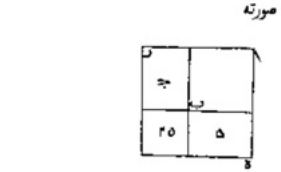
If we subtract twice two and a half (that is, five) from eight—removing the added lengths from both ends of the side of the large square DH—the remainder is three. This is the root of the original square—that is, the side of figure AB. Note that we halved the number of roots and added the square of this half to thirty-nine in order to complete the four corners of the large figure. This is because one-fourth of any number, squared and then multiplied by four, equals the square of half that number. Therefore, we squared only half of the roots rather than squaring one-fourth and then multiplying by four.

What is x, the side length of the original square AB?

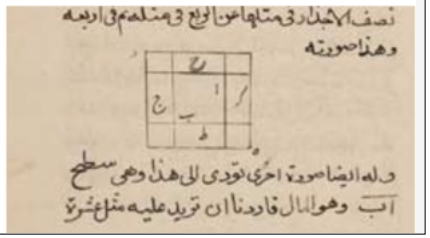


Each of the solutions below, al-Khwārizmī completed the square.

Solution 1



Solution 2



Why negative numbers did not appear as solutions?

About algebra and meaning

One way to put the dichotomy in a more philosophical or literary framework is to say that algebra is to the geometer what you might call the Faustian offer. As you know, Faust in Goethe's story was offered whatever he wanted (in his case the love of a beautiful woman), by the devil, in return for selling his soul. Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine. (Nowadays you can think of it as a computer!)

Sir Michael Atiyah

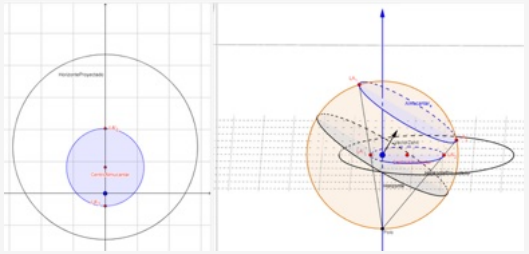
What do you think Atiyah means by " algebra is to the geometer what you might call the Faustian offer"

Of course we like to have things both ways; we would probably cheat on the devil, pretend we are selling our soul, and not give it away. Nevertheless, the danger to our soul is there, because when you pass over into algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking about the meaning.

I am a bit hard on the algebraists here, but fundamentally the purpose of algebra always was to produce a formula which one could put into a machine, turn a handle and get the answer. You took something that had a meaning; you converted it into a formula, and you got out the answer. In that process you do not need to think any more about what the different stages in the algebra correspond to in the geometry. You lose the insights, and this can be important at different stages. You must not give up the insight altogether! You might want to come back to it later on. That is what I mean by the Faustian offer. I am sure it is provocative.

Do you agree? Why or why not? **Sir Michael Atiyah**

Astrolabe



Vgonzalez630 - Sitio web: http://www.geogebra.org

Astrolabe

The moving parts simulate the sky's rotation for any date, time, and latitude.

Astrolabe made by Abd al-Karim al-Misri/al-Madrabi, brass with silver and copper inlay, AH633 (1236AD), probably Mayyafariqin (Turkey)



https://www.britishmuseum.org/collection/object/W_1855-0709-1

- Handheld astronomical instrument.
- Stereographic projection of the celestial sphere on a plane.

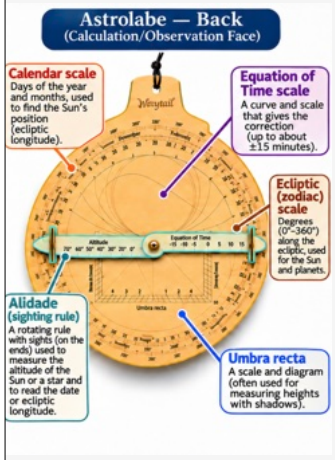
Uses

- Solve problems geometrically (no calculation)
- From Sun/star position one can find time, latitude, direction of Mecca (qibla).

Parts

- Plate (depends on latitude)
- Rotating star map (rete)
- Rule

Astrolabe Uses

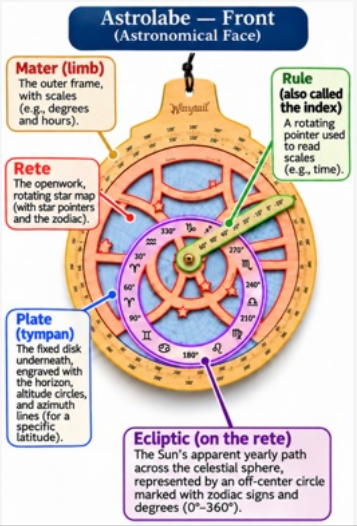


- Time: Finding local time from the Sun or stars.
- Latitude: Estimate or verify latitude from celestial observations
- Prayer times: Determine times of prayer
- Qibla: Find the direction of Mecca
- Astrology: Often used to cast horoscopes
- Surveying: Measure heights and distances
- Teaching: Teaching and calculating problems in spherical astronomy.

https://archive.org/details/ilmetaugeet_gmail_659/page/n168/mode/thumb The Astrolabe by James E. Morrison Rehoboth Beach, DE: Janus, 2007

How does the astrolabe work?

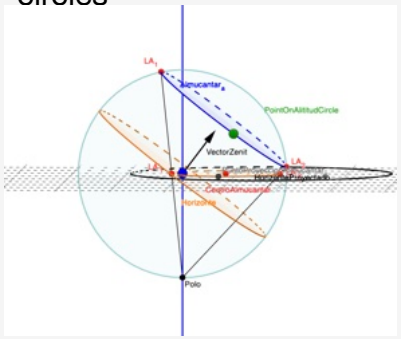
- **Back**: measure the altitude of the Sun or a star with the sighting rule.
- **Front**: stereographic projection (from the south celestial pole onto the plane of the equator)
  - turns the sky's rotation around the pole into the rotation of the star map (rete) around the center;
  - sends circles to circles (or lines), so they can be drawn with a compass;
  - preserves angles between smooth curves; for example, circles of equal height and circles of equal direction still meet at right angles.



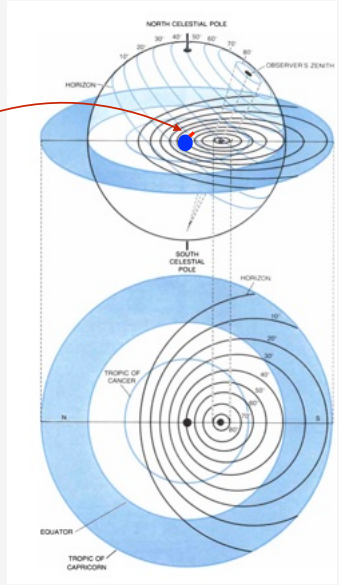
Stereographic projection

- Explained by Ptolemy (Alexandria ~200)
- Preserves angles and circles





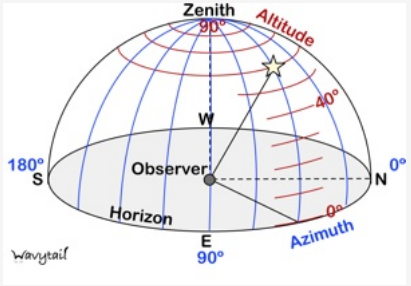
https://www.geogebra.org/classic/kya3qthv



https://www.jstor.org/stable/10.2307/24949987

Astrolabe Latitud ~ 41

Basic diagram to show how the lines on the plate relate to the sky above you:
This star has an Azimuth of 45° and an altitude of 60°

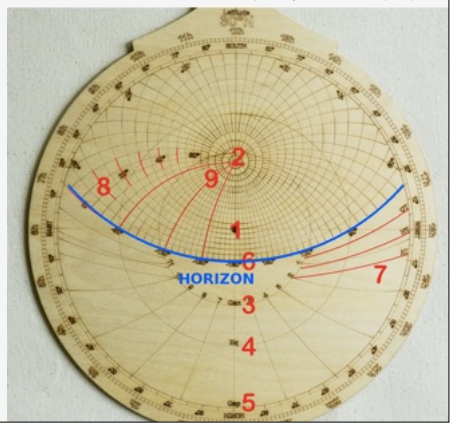


Words of Arabic origin

- **Azimuth** → from *al-sumūt*, "the directions"
- **Almucantar** → from *al-muqanṭarāt*, "circles of altitude"
- **Nadir** → from *naẓīr*, "opposite"
- **Zenith** → from *samt al-ra*'s, via Latin corruption

1. North Pole
2. Zenith (the point in the sky directly above your head)
3. Tropic of Cancer
4. Equator
5. Tropic of Capricorn
6. Horizon
7. Civil, Nautical and Astronomical Twilight Arcs
8. Altitude circles go around the zenith, from 0° at the horizon to 90° at the Zenith
9. Azimuth circles converge at the zenith, from 180° facing South to 0° facing North

https://wavytail.com/astrolabe-parts-plates/

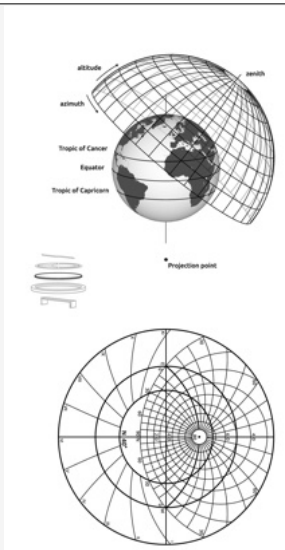


~150 BCE	**Hipparchus** associated with early stereographic projection methods
~2nd c. CE	**Ptolemy** formalizes the mathematics (*Planisphaerium*)
~4th–5th c.	**Theon of Alexandria** writes an early astrolabe treatise
~8th c.	Arabic translations; adoption in the Islamic world
~9th–10th c.	Abbasid development; earliest surviving instruments
~10th c.	**al-Sufi** and others refine use and design
~10th–11th c.	**al-Biruni** systematizes theory and applications
~11th–12th c.	Transmission to Europe via al-Andalus
~13th–15th c.	European flourishing; **Chaucer's** treatise (~1391)
~16th–17th c.	Gradual decline; replaced by specialized instruments and mechanical clocks

Astrolabe

Animation showing how celestial and geographic coordinates are mapped onto an astrolabe's base plate (tympan) through a stereographic projection.

Hypothetical tympan (40 degrees North Latitude) of a 16th century European planispheric astrolabe.



By This image has been created during "DensityDesign Integrated Course Final Synthesis Studio" at Politecnico di Milano, organized by DensityDesign Research Lab in 2016. Credits goes to Michele Invernizzi - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=45222752

Hebrew Astrolabe

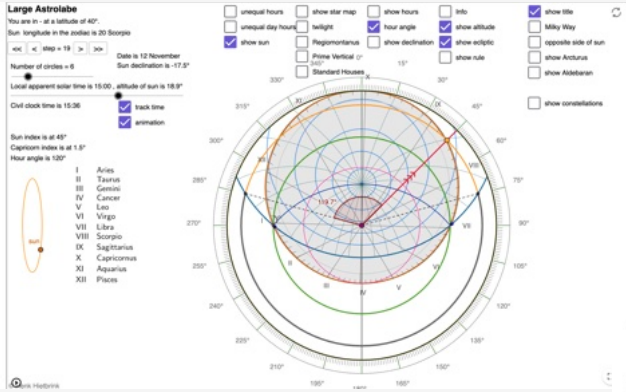
Astrolabe inscribed with words in Hebrew and Arabic



https://www.bbc.co.uk/sounds/play/b00st9z8

From A History of the World in 100 Objects

Astrolabe in Geogebra



https://www.geogebra.org/m/HNFgEkzf

https://www.fransvanschooten.nl/astrolabe.htm

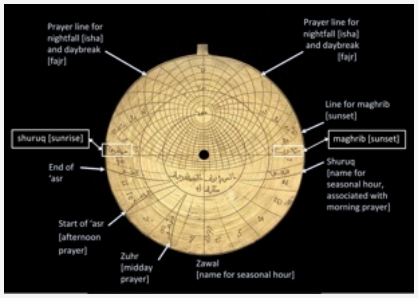
Astrolabe used for measuring



16th-century woodcut of the measurement of a building's height with an astrolabe



By Johannes Stöffler - http://christies.com/img/LotImages/2017/CKS/2017_CKS_14298_0349_000(stoffler_johannes_elucidatio_fabricae_usuque_astrolabii_ex_secunda_aut051338).jpgfromhttp://christies.com/lot/lot-stoffler-johannes-1452-1531-elucidatio-fabricae-usuque-astrolabii-6069643, Public Domain, https://commons.wikimedia.org/w/index.php?curid=108928273



Lines of prayer on Astrolabe, North African, 1800s, Inv. 47714 https://www.hsm.ox.ac.uk/prayer



Gazetteer for Astrolabe Inv. 35313 showing locations in the Islamic world

Astrolabe in Istanbul

Painting of the Istanbul Observatory. It shows workers at the observatory of Taqi al-Din at Istanbul in 985 H (1577). Two observers are working with an astrolabe. A universal astrolabe of the saphea form is on the table in front of the man with the dividers and paper. The painting is from Shahinshah-nama (History of the King of Kings), an epic poem by 'Ala ad-Din Mansur-Shirazi, written in honour of Sultan Murad III (reigned 1574-95 [AH 982-1003]).



https://muslimheritage.com/using-an-astrolabe/

Omar Khayyam ~1100 CE

How swiftly does this caravan of life pass;
Seek thou the moment that with joy does lapse.
Saghi, why lament tomorrow's misfortunes today?
Bring forth the chalice, for the night shall pass.

From the Rubáiyát of Omar Khayyam
Translation by Joobin Bekhrad



Omar Khayyam
May 18, 2019 Google Doodle celebrating his 971st Birthday
~1100 CE

Khayyam was Persian, but wrote scientific works in Arabic, the scholarly language of his era. He wrote poetry in Persian.

Symbols (according to chatGPT)

- **Sun:** his calendar reform was one of history's most accurate solar calendars
- **Glowing Earth:** Astronomical observations. Yellow lines = stylized Arabic script.
- **Sphere with crosshairs:** Spherical geometry and astronomical instruments.
- **Golden curve:** Mathematical and astronomical contributions
- **Book with Arabic script:** Treatises on algebra, geometry, and Euclid
- **Grapes:** Wine imagery from the Rubā'iyāt
- **Persian landscape:** Nishapur, his birthplace

You must realize that this treatise will be understood only by those who have mastered Euclid's Elements and Data as well as the two [first] books of Apollonius's Conics. He who lacks knowledge of one of these treatises will not reach an understanding of this book.

Omar Khayyam - Treatise on Algebra

1.What does this paragraph reveal about how mathematical knowledge was organized and transmitted at the time?

2. How does this compare to modern expectations for reading a math text? (Prerequisites, background, structure.)

What does this paragraph reveal about how mathematical knowledge was organized and transmitted at the time?

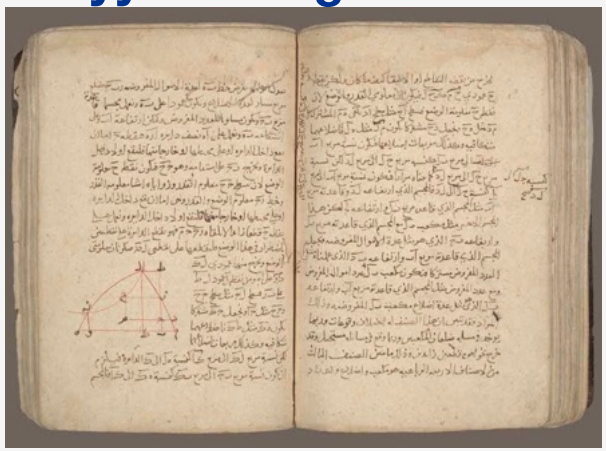
- Knowledge transmission was text-based and hierarchical: you couldn't skip foundational steps or learn in isolation.
- Scholars were expected to have direct familiarity with specific foundational texts—not just general concepts.
- This suggests a relatively small, well-defined set of essential mathematical works that mathematically trained scholars were expected to know.
- Omar Khayyam naming Euclid's Elements and Data, and the first two books of Apollonius's Conics shows a **common set of reference texts.**

How does this compare to modern expectations for reading a math text? (Prerequisites, background, structure.)

Today we still rely on prerequisites, but in practice:

- Modern textbooks are more self-contained.
- Authors usually review necessary background instead of expecting total mastery.
- Students learn through lectures, problem sets, and multiple sources, not a single canonical text.
- The structure is modular, not strictly linear: you can enter a subject at different points.

Omar Khayyam's Algebra

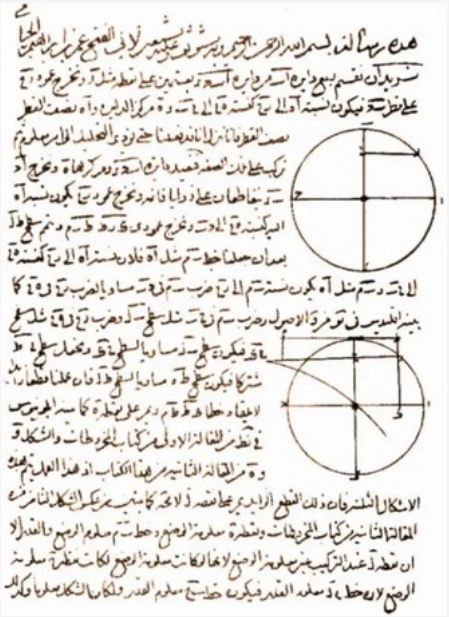


Maqalah fi al-jabr wa-al muqabalah - Omar Khayyam Algebra
https://dlc.library.columbia.edu/catalog/ldpd:113656

Solution of the various cases of the cubic equation by finding the intersections of appropriately chosen conic sections.

On this page, the case "a cube, sides and numbers are equal to squares," or, in modern notation, $\mathbf{x^3 + cx + d = bx^2}$.

Khayyam's cubic equation and intersection of conics manuscript



https://digitaleditions.sheridan.com/publication/?i=672591&article_id=3759503&view=articleBrowser
manuscript kept in Tehran University.

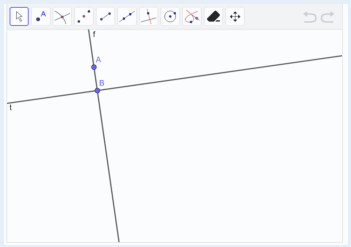
For Khayyam, numbers could only be positive, which meant there were 19 different types of cubic equations with positive coefficients and solutions containing absolute numbers, sides, squares, and cubes. (Al-Khwārizmī similarly considered only positive coefficients for quadratics.)

List some of these types, ensuring the coefficient of the cubic term is 1.

Example: $x^3 + c\,x^2 = ax + b$

There is some evidence suggesting that Khayyam recognized that certain cubic equations would have more than one positive real solution, while other cubic equations would have no positive real solutions

Equation form	Translation of equation	Method of solution
$x^3 = c$	Cube equals number	Cube root c
$x^3 = bx$	Cube equals sides	Reduce to $x^2 = b$
$x^3 = ax^2$	Cube equals square	Reduce to $x = a$
$x^3 + ax^2 = bx$	Cube and square equal sides	Reduce to $x^2 + ax = b$
$x^3 = ax^2 + bx$	Cube equals square and sides	Reduce to $x^2 = ax + b$
$x^3 + bx = ax^2$	Cube and sides equal square	Reduce to $x^2 + b = ax$
$x^3 + bx = c$	Cube and sides equal number	Parabola and semicircle
$x^3 + c = bx$	Cube and number equal sides	Parabola and hyperbola
$x^3 + c = ax^2$	Cube and number equal square	Parabola and hyperbola
$x^3 + ax^2 = c$	Cube and square equal number	Parabola and hyperbola
$x^3 = ax^2 + c$	Cube equals square and number	Parabola and hyperbola
$x^3 = bx + c$	Cube equals sides and number	Parabola and hyperbola
$x^3 + ax^2 = bx + c$	Cube and square equal sides and number	Hyperbola and hyperbola
$x^3 + c = ax^2 + bx$	Cube and number equal square and sides	Hyperbola and hyperbola
$x^3 = ax^2 + bx + c$	Cube equals square, sides and number	Hyperbola and hyperbola
$x^3 + ax^2 + c = bx$	Cube, square and number equal sides	Hyperbola and hyperbola
$x^3 + ax^2 + bx = c$	Cube, square, and sides equal number	Hyperbola and semicircle
$x^3 + bx = ax^2 + c$	Cube and sides equal square and number	Hyperbola and circle
$x^3 + bx + c = ax^2$	Cube, sides, and number equal square	Hyperbola and circle

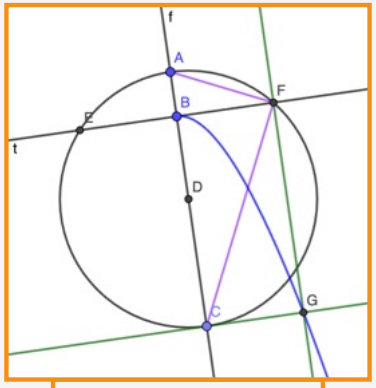


https://www.geogebra.org/m/ua9j88ty

https://www.geogebra.org/m/ua9j88ty

1. We have line **f** through **A** and **B**. and line **t** through **B**, perpendicular to **f** and point **C** on line **f**, below **B**.
2. Draw a circle with diameter **AC** (find midpoint with the midpoint tool).
3. Let **E** and **F** be the intersection points of this circle with line **t**.
4. Draw the line through **E** perpendicular to **t**.
5. Draw the line through **C** perpendicular to **f**.
6. Let **G** be their intersection.
7. Use the **locus tool**: select **G**, then **C**, to find the locus of G as C moves.
8. Repeat steps 4–7 using F instead of E.
9. Answer the Wooclap question: what type of curve is the locus?

Constructing the parabola $x^2 = a \cdot y$ with straightedge and compass, as Khayyam might have done (following Apollonius)



Similar triangles yield
AB/BF = BF/BC

Set

- B = (0, 0), A = (0, –a), with a > 0 fixed
- C = (0, y), with y > 0 varying as C moves
- F = (x, 0), where the circle with diameter AC meets t
- G = (x, y): same x as F, same y as C

By similar triangles

- $x^2 = BF^2 = AB \cdot BC = a \cdot y$
- Hence $x^2 = a \cdot y$ and $G = (x, (1/a)\, x^2)$

The locus of **G** (that is all points G we get varying y) is a **parabola** with vertex at **B** and axis the line **f**.

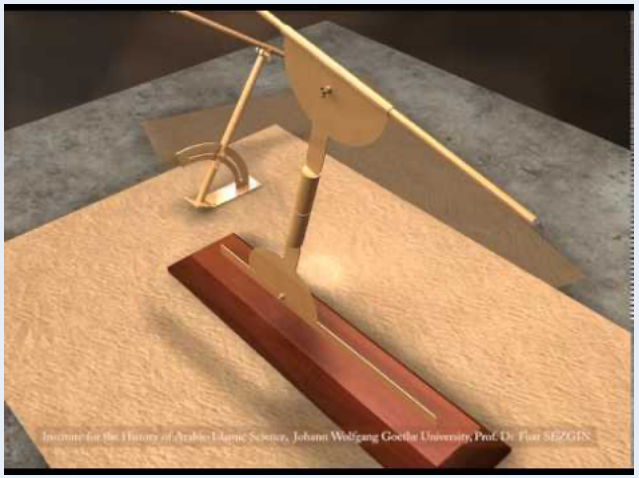
Perfect compass

The perfect compass

- **6th c.**: Eutocius credits Isidore of Miletus with an instrument for drawing a parabola. What it was is unknown.
- **c. 984**: Ibn Sahl draws the conics mechanically.
- **Late 10th c.:** al-Qūhī's perfect compass draws lines, circles and all the conics. al-Sijzī writes a treatise on it too.
- **12th–13th c**.: new treatises in Mosul (Ibn al-Ḥusayn, al-Abharī).
- **c. 1500**: it reappears in Florence (della Volpaia, Leonardo).

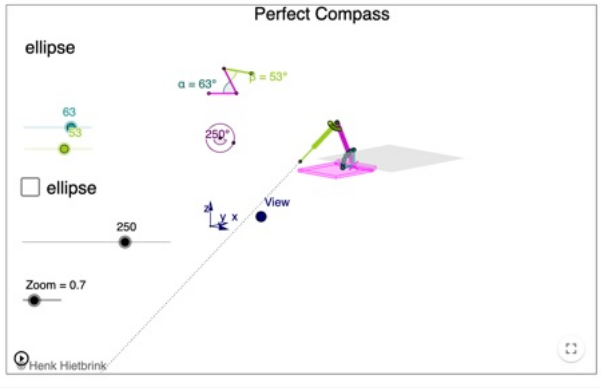
References: Martines, Nuncius 29 (2014); Rashed, Isis 81 (1990); Rashed, Arabic Sci. & Phil. 13 (2003); Raynaud, Arabic Sci. & Phil. 17 (2007).

Why was this compass capable of drawing conic sections?



https://www.youtube.com/watch?v=lQp-W_Lt_gE

Perfect Compass in Geogebra



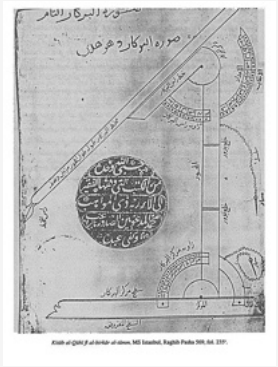
https://www.geogebra.org/m/x2ge2bwv

https://www.geogebra.org/m/x2ge2bwv

Activity and geogebra app by Henk Hietbrink https://fransvanschooten.nl/perfectcompass.htm

Al-Qūhī, ~1000 and the perfect compass

- Born in Persia (now Iran)
- Directed the Baghdad observatory.
- Wrote a treatise on his perfect compass:
  - Proved that the compass can draw straight lines, circles, and conic sections.
  - Explained how to draw a given curve.
- Claimed he knew of no earlier description of such an instrument.



[Wit this compass it is possible] "*to construct astrolabes on plane surfaces or surfaces of revolution, as well as to construct sundials on any surface, and likewise to construct all the instruments on which the lines that are the intersections of a conic surface and of any surface are situated*". Abü Sahl al-Qühi

How Does the Perfect Compass Work?

- One leg (the axis) is fixed at an angle to the drawing plane.
- The other leg makes a fixed angle with the axis but can change length, keeping its tip on the plane.
- As the compass rotates, the moving leg sweeps out a cone.
- The tip traces the intersection of the cone with the plane.

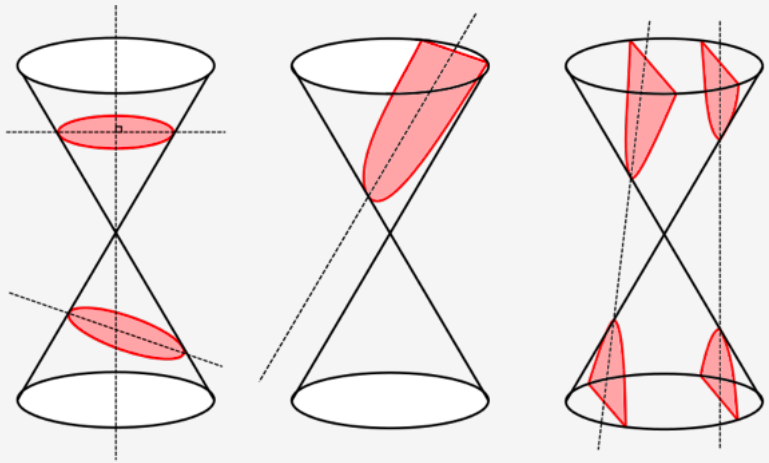


Rashed, Geometry and Dioptrics in Classical Islam (2005), pp. 726–796 (translation);

Let ϕ = angle between the axis and the plane; θ = angle between the legs.

- $\phi = 90°$: Circle
- $\phi > \theta$: Ellipse
- $\phi = \theta$: Parabola
- $\phi < \theta$: Hyperbola
- $\theta = 90°$: Straight line (degenerate case)

Conic Sections



underground mathematics Rich resources for teaching A level mathematics
Copyright © and Database Right 2013-2022 University of Cambridge
https://undergroundmathematics.org/circles/conic-sections-in-real-life

Galileo Compass

