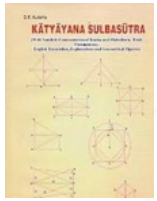

Indus Valley Weights= https://tamilandvedas.com/2019/09/

Ancient Indian Mathematics



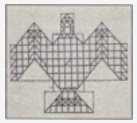




- :The Indus (or Harappan) civilisation c. 3300–1300 BCE.
- Vedic period (~800–200 BCE) Śulba Sūtras
- Aryabhata (~500 CE
  - Trigonometry - The Sine Table in the oral tradition.
- Zero
- Mathematics in Sanskrit poetry.

By Noyal francis - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=49835812

Indian Subcontinent - Rough Chronology


Vakrapaksa-syenacit. First layer of construction (after Baudhayana) - Mathematics in Ancient India by A. Dutta.



Vedic period
c. 15th–5th century BCE

Śulba Sūtras
c. 8th–2nd century BCE

Aryabhata
c. 5th century CE

Virahanka
c. 7th century CE

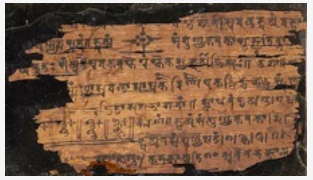
Indus (Harappan) civilization
c. 33rd–13th century BCE

Brahmagupta
c. 6th century CE

Bhāskara I
c. 6th century CE






One of the Bakhshali manuscripts. 799 –1102 CE

National Geographic, Public domain, via Wikimedia Commons

The Indus (or Harappan) civilisation.

3300 BCE to 1300 BCE.

The Indus (or Harappan) civilisation

- c. 3300–1300 BCE
- Urban civilization in the Indus region and neighboring areas
- Undeciphered script with several hundred signs
- Strong evidence of **standardized weights and some standardized measures**
- Known for urban planning, craft production, and long-distance trade
- The causes of its transformation and decline are still debated

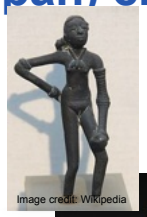


Image credit: Wikipedia




Indus Valley Weights= https://tamilandvedas.com/2019/09/

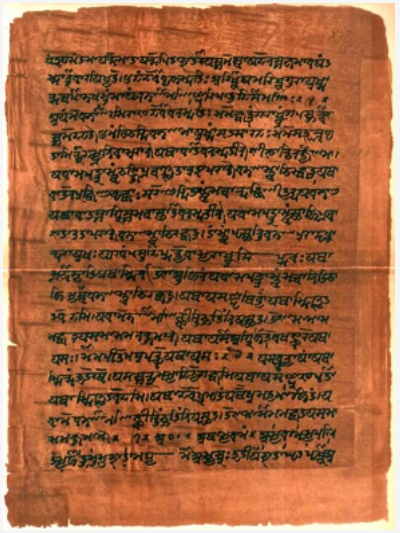

Terracotta Dice (Harapan Civilization) -2600/-1900. See also https://www.harappa.com/blog/ancient-indus-die

Vedas and the Śulba Sūtras 1500–500 BCE

Vedas and Sulba sutras

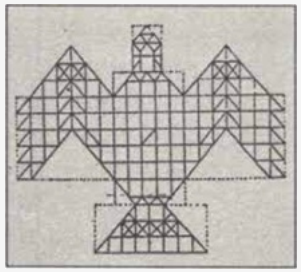
Vedas

- oldest scriptures of Hinduism, compiled **oral** wisdom starting around 1500 BCE.
- composed and transmitted orally in Vedic Sanskrit.
- gave instructions for religious purposes.
- veda means knowledge
- main source of our knowledge of early mathematics in India.
- most important for Math History:
  - calendars
  - astronomy,
  - Sulba-Sutras.



A page from the Atharvaveda.- Image of Codex Cashmiriensis folio 187a from Atharva-Veda Saṁhitā second half, by William Dwight Whitney and Charles Rockwell Lanman

Vedas and Sulba sutras



Vakrapaksa-syenacit. First layer of construction (after Baudhayana) - Mathematics in Ancient India by A. Dutta.

Śulba-sūtra: (**sutra = rule, śulba** = string or chord)

- Written mainly between **800 and 200 BCE**.
- Appendices to the **Vedas**
- Explained how to construct sacrificial fire altars, with
  - Precise measurements.
  - **Geometrical shapes** of given **areas**.
  - Using **stakes and marked cords** to make right-angled triangles

Vedas and Sulba sutras

The Śulba Sūtras are major surviving sources for Vedic geometry.

- No proofs or formal explanations—focus on construction methods
- Mathematics passed through families; texts copied across generations
- Tradition likely disrupted during periods of political uncertainty

https://artsandculture.google.com/asset/literature-from-the-pur%C4%81%E1%B9%87as/agH9hUtiTSKriw?hl=en



Rope Activity 1:

Sulba sutras Activity 1

https://www.geogebra.org/m/gc5setrq



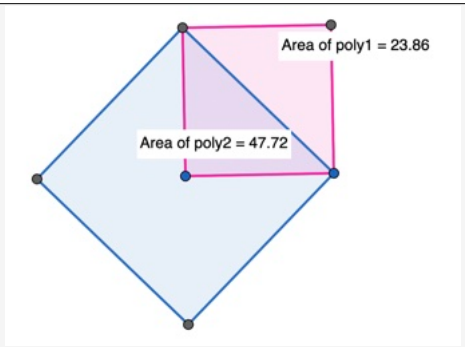
1. Construct a square (To do it, first click on the Regular Polygon tool, then click twice on the plane to create the endpoints of an edge of the square, finally, click "ok" if the number of sides offered is four, otherwise change it and then click OK)

2. Using the Regular Polygon tool again construct a new square such that one of its sides is one of the diagonals of the first square.

3. Using the area tool, compute the area of the two squares.

4.What is the relation between the areas of the two squares? If you have time, find an explanation for your answer. (answer in the platform)

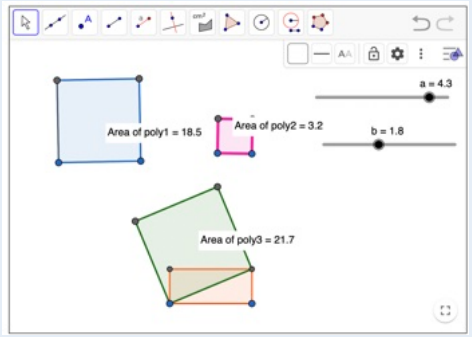
(Note: You can only move the first two points you constructed, which originally are blue. The rest of the construction depends on these two points).



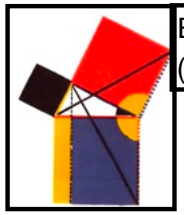
The rope which is stretched across the diagonal of a square produces an area double the size of the original square.

Activity 2: Explain why the area of the square on the diagonal of the rectangle you constructed is the sum of the areas of the given squares.

"The rope which is stretched along the length of the diagonal of a rectangle produces an area which the vertical and horizontal sides make together."

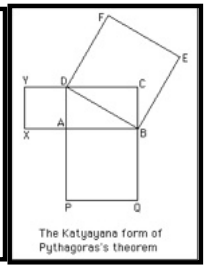


https://www.geogebra.org/m/ztcsqdum



Euclid's elements, ~300BCE
(Interpreted by Oliver Byrne ~ 1850)

"The rope which is stretched along the length of the diagonal of a rectangle produces an area which the vertical and horizontal sides make together"
Sulba Sutra-800BCE and 400BCE



The Katyayana form of Pythagoras's theorem

https://mathshistory.st-andrews.ac.uk/HistTopics/Indian_sulbasutras/

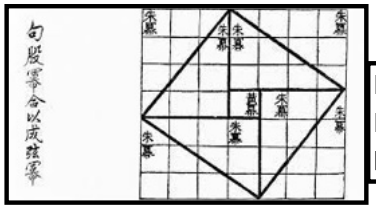


Diagram proof of the Gou-Gu rule - Pythagorean Theorem - text compiled between 100 BCE – 100 CE

Diagram added by Zhao Shuang to the Zhoubi Suanjing that can be used to prove the Pythagorean Theorem

Rope Activity 2: √2

Increase a unit length by its third and this third by its own fourth less the thirty-fourth part of that fourth

The Katyayana Śulbasūtra gives the following approximation to √2:

Increase a unit length by its third and this third by its own fourth less the thirty-fourth part of that fourth

How many digits after the decimal point of this approximation match the true value of √2?

The Katyayana Sulbasutra gives the following approximation to √2:

Increase a unit length by its third and this third by its own fourth less the thirty-fourth part of that fourth

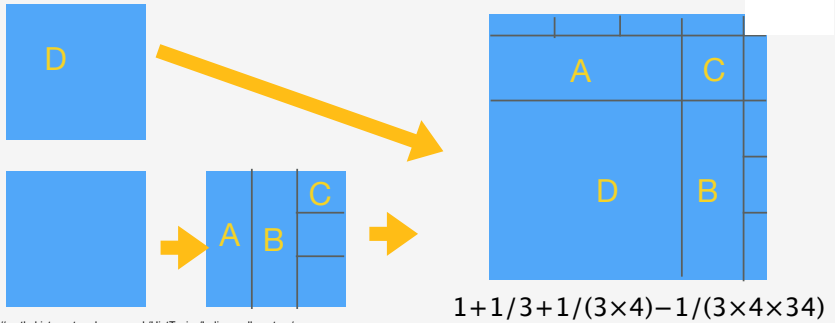
$=1+1/3+1/(3\times4)-1/(3\times4\times34)=$
$577/408 \sim 1.4142156863$

$\sqrt{2} = 1.414213562 \ldots$

How was the approximation of √2 found? A conjecture

Here is a **possible** explanation.

- Start with two squares, each of area 1.
- Goal: cut up and arrange one of the squares around the other, to make a new "almost square". The area of this "almost-square" is 2.



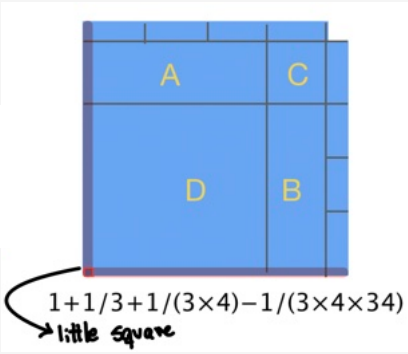
$1+1/3+1/(3\times4)-1/(3\times4\times34)$

https://mathshistory.st-andrews.ac.uk/HistTopics/Indian_sulbasutras/

How was the approximation of √2 found? A conjecture

- Area of missing square (top right): $(1/12)^2$
- Fill missing little square with strip of width x on the left hand side and bottom.
- Area of strip $2\cdot x\cdot(1+1/3+/12)= (1/12)^2$.
- $x = 1/(3\times4\times34)$
- But…there is a little square of side x overlapping, not accounted for.
- $2\cdot x\cdot(1+1/3+/12) - x^2 = (1/12)^2$, gives $x = 17/12-\sqrt{2}$

1/3x4x34~0.002451
17/12-√2~0.0024531



https://mathshistory.st-andrews.ac.uk/HistTopics/Indian_sulbasutras/

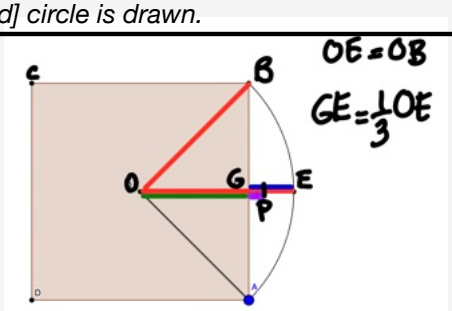
"Circling the square"

If it is desired to transform a square into a circle, [a cord of length] half the diagonal [of the square] is stretched from the centre to the east [a part of it lying outside the eastern side of the square]; with one-third [of the part lying outside] added to the remainder [of the half diagonal], the [required] circle is drawn.

What is the value of “π” in the construction "circling the square”?

If it is desired to transform a square into a circle,

- *[a cord of length] half the diagonal [of the square] is stretched from the centre to the east [a part of it lying outside the eastern side of the square];*
- *with one-third [of the part lying outside] added to the remainder [of the half diagonal], the [required] circle is drawn.*

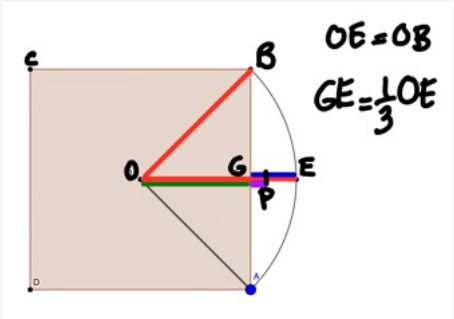


https://www.maa.org/press/periodicals/convergence/more-classroom-activities-based-on-ancient-indian-rope-geometry-transforming-a-square-into-a-circle

What is the value of “π” in the construction "circling the square”?

If $A(\square_s) = A(\bigcirc_\pi)$ then $\pi = \frac{1}{\sqrt{\pi}} s$

$\left(\frac{\sqrt{2}}{2}s - \frac{1}{2}s\right)\frac{1}{3} + \frac{1}{2}s = \frac{\sqrt{2}+2}{6}\cdot s$



$\frac{1}{\sqrt{\pi}} = 0.564189\ldots$

$\frac{\sqrt{2}+2}{6} = 0.569035\ldots$

$\left(\frac{6}{2+\sqrt{2}}\right)^2 = 3.088311\ldots$

$\text{“}\pi\text{”} = \left(\frac{6}{\sqrt{2}+2}\right)^2$

The Sine (Jya)

Trigonometry

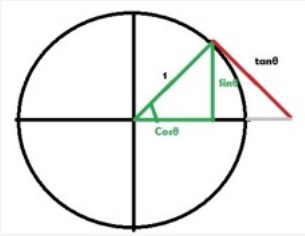
The study of the relationships between angles and lengths in triangles and their connection to circles.

Greek

trigōnon ⟶ triangle

metron ⟶ measure

Early trigonometry in India: The Sine-Jya



Unit circle and trigonometric functions.

Indian astronomers wanted angles and arc lengths to share the same numerical scale.

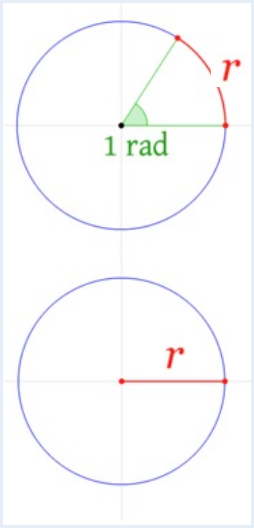
A full circle ↔ 360° ↔ 21 600′ (arc-minutes).

The circumference **was defined** as 21 600 linear units *(a numerical convention, not a physical measure)*.

2πR = 21 600 therefore **R ≈ 3438.**

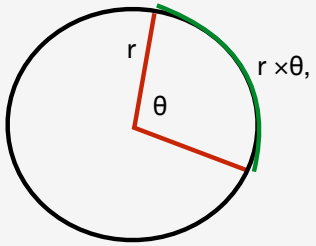
1 radian ≈ 3438′ → same as radius for Indian sine tables.

A radian is a unit for measuring angles.
Write a definition of one radian (see animation and figures below).



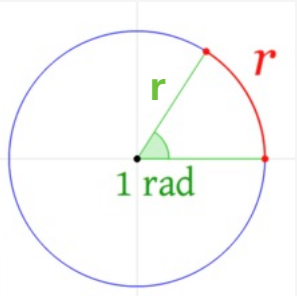
By Lucas Vieira - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=25112326

Definition: A *radian* is the measure of a central angle of a circle such that the corresponding arc length equals the radius.

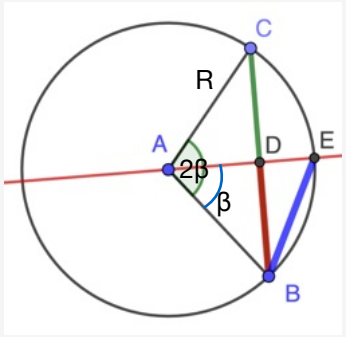


arc length = r × angle

(Angle in radians).



Early trigonometry in India: The Sine-Jya



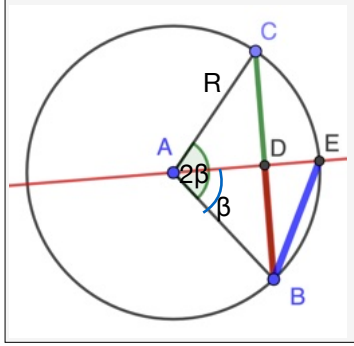
Chord(β) = BE

Chord(2β) = BC

BD (the half-chord) easier to compute because only radius and one perpendicular needed, not both chord endpoints.

Early trigonometry in India: The Sine-Jya

- Greek astronomers studied and tabulated geometric relations between chords.
- Indian mathematicians
  - simplified by working with half chords (called jya).
  - expressed these relations algebraically used recurrence → computation replaced construction.



BD = R sin β

Chord(2β) = BC = 2·BD = 2R sin β

Hence R sin β = ½ Chord(2β)

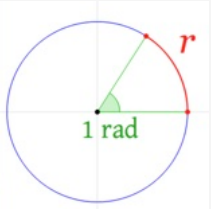
Jya(β) = ½ Chord(2β) (with R=3438)= half-chord of the double angle

Early trigonometry in India: The Sine-Jya

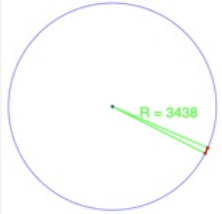
Why R=3438?

2π rad — 360x60 min = 21,600 min
1 rad — (360x60)/(2π) ~ 3438 min

- Unit of measure for angles and length: minutes.
- The **circumference of a circle** of radius R = **3438** was 21,600 minutes. (21,600=360 x 60.)



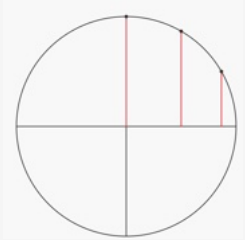
If r = 1, the arc of length 1 radian determines a central angle of 1 radian.



If R = 3438, the arc of length 1 minute determines a central angle of 1 minute.

Early trigonometry in India: The Sine-Jya

- A "table" of Sine was usually presented as a list ***in verse*** of
  - **twenty-four Sines** in the first quadrant
  - at **intervals of 3 3/4 ∘ (225 minutes)**,
  - a **rule** for linear interpolation to find Sines and arcs



There is usually no explanation of how the Sine-Jya was derived

Aryabhata

~500CE

Aryabhata ~500 CE

- Accurate value of π (as ratio 62,832/20,000)
- Described decimal place-value system
- Developed computational algorithms (square roots and cube roots, areas and volumes, equations, arithmetic and geometric progressions)
- Sine table construction methods
- Alphanumeric notation system for encoding large numbers
- Proposed earth's daily rotation
- Developed planetary motion calculations

A place should be 10 times the previous place.



मखि भखि फखि धखि णखि ञखि ङखि हस्झ स्ककि किष्ग श्घकि किघ्व ।
घ्लकि किग्र हक्य धकि किच स्ग झश ङ्व क्ल प्त फ छ कला-अर्ध-ज्यास् ॥

makhi bhakhi phakhi dhakhi ṇakhi ñakhi
ñakhi hasjha skaki kiṣga ghakhi kighva ।
ghlaki kigra hakya dhaki kica
sga śjha ṅva kla pta pha cha
kala-ardha- jyāḥ ॥

Aryabhata's poem-table of sines

Words from verse
makhi
bhakhi
phakhi
dhakhi
.nakhi
~nakhi
"nakhi
hasjha
skaki
ki.sga
"sghaki
kighva

Words from verse
ghlaki
kigra
hakya
dhaki
kica
sga
"sjha-jhasa
"nva
kla
pta
pha
cha

Complete the table -Teams of 3

https://docs.google.com/spreadsheets/d/1m_RaOfagWoNvBvlbHTtqDe66UCucl4lhmmyXSvWFVW8/

- Open the link with the QR code
- Make a copy of the spreadsheet and share it with me (my SB email)
- Fill the gaps (in yellow)

Classified consonants

ka	1
kha	2
ga	3
gha	4
"na	5
ca	6
cha	7
ja	8
jha	9
~na	10
.ta	11
.tha	12
.da	13
.dha	14
.na	15
ta	16
tha	17
da	18
dha	19
na	20
pa	21
pha	22
ba	23
bha	24
ma	25

Unclassified consonants

ya	30
ra	40
la	50
va	60
"sa	70
.sa	80
sa	90
ha	100

Vowels

a	1
i	100
u	100^2
.r	100^3
.l	100^4
e	100^5
ai	100^6
o	100^7
au	100^8

Word			Value
makhi	khi=kha x i=200	ma=25	225
bhakhi	khi=kha x i=200	bha=24	224
phakhi			
dhakhi	khi=kha x i=200	dha=19	219
.nakhi	khi=kha x i=200	.na=15	215
~nakhi			
"nakhi	khi=kha x i=200	"na=5	205
hasjha	ha=100	sa=90	jha=9	199
skaki	ki=100	sa=90	ka=1	191
ki.sga	ki=100	.sa=80	ga=3	183
"sghaki			
kighva	ki=100	va=60	gha=4	164
ghlaki	ki=100	gha=4	la=50	154
kigra			
hakya	ha=100	ya=30	ka=1	131
dhaki	dha=19	ki=ka x i=100		119
kica			
sga	sa=90 g	a=3		93
"sjha- jhasa	"sa=70	jha=9		79
"nva	"na=5	va=60		65
kla	ka=1	la=50		51
pta	pa=21	ta=16		37
pha				
cha	cha=7			7

Examples

	Decomposition	Computation	Value
ka	ka	1	1
ki	ka x i	1x100	100
gu	ga x u	3x10000	30000
gnu	(g+na) x u	(20+3)x10000	230000
khyughr	(gha x .r) + (ya + kha) x u	(4x100^3)+(30+2)x100^2	4,320,000

Aryabhata's Encoded Sine Table — Math Detective Work

Each Sanskrit word in the green table encodes a number using consonant and vowel values (see the purple tables on the right).
Your task: **decode the missing numbers in the yellow cells** using the same logic as in the completed examples.

Instructions

1. Work in teams of 2-3. List all team members in the "Team members:" section.
2. Open the file (you have *view-only* access).
3. Go to File → Make a copy.
4. Share one copy per group with me (as Editor).
5. Use the purple tables to find letter values and figure out how they combine. *(The pattern appears in the already filled rows.)*
6. When you finish, type "done" in Vevox.
7. Have math fun!



Aryabhata's explanation of his system for denoting numbers

Classified consonants

ka	1
kha	2
ga	3
gha	4
"na	5
ca	6
cha	7
ja	8
jha	9
~na	10
.ta	11
.tha	12
.da	13
.dha	14
.na	15
ta	16
tha	17
da	18
dha	19
na	20
pa	21
pha	22
ba	23
bha	24
ma	25

Unclassified consonants

ya	30
ra	40
la	50
va	60
"sa	70
.sa	80
sa	90
ha	100

Vowels

a	1
i	100
u	100^2
.r	100^3
.l	100^4
e	100^5
ai	100^6
o	100^7
au	100^8

- The classified consonants [starting] from k [are encoded] in the square [places],
- the non-classified consonants, [starting from] y which is [equal to] n˙m, to the non-square [places].
- Nine vowels [are assigned] to the square and non-square [places] in a double nine-tuple of zeros, and [beyond] the square [places] ending with nine.

Mathematics in India by Kim Plofker

Poem		Poem Number	Total from poem
		0	0
makhi	225	225	225
bhakhi	450	224	449
phakhi	675	222	671
dhakhi	900	219	890
.nakhi	1125	215	1105
~nakhi	1350	210	1315
"nakhi	1575	205	1520
hasjha	1800	199	1719
skaki	2025	191	1910
ki.sga	2250	183	2093
"sghaki	2475	174	2267
kighva	2700	164	2431
ghlaki	2925	154	2585
kigra	3150	143	2728
hakya	3375	131	2859

Why Aryabhata Listed Sine Differences

Differences reflect how the table was generated.

Suggests an **algorithmic mindset:** each value derived from the previous.

Later texts show similar finite-difference **methods** linking geometry and algebra.

Poem	Poem Number	Total from poem	Total for poem/ 3438	Actual sine	Actual Sine- Ariabhata Sine
	0	0	0.00000	0.00000	0.0000000
makhi	225	225	0.06545	0.06540	-0.0000467
bhakhi	224	449	0.13060	0.13053	-0.0000826
phakhi	222	671	0.19517	0.19509	-0.0000957
dhakhi	219	890	0.25887	0.25882	-0.0000715
.nakhi	215	1105	0.32141	0.32144	0.0000080
~nakhi	210	1315	0.38249	0.38268	0.0001654
"nakhi	205	1520	0.44212	0.44229	0.0001386
hasjha	199	1719	0.50000	0.50000	-0.0000368
skaki	191	1910	0.55556	0.55557	-0.0000262
ki.sga	183	2093	0.60878	0.60876	-0.0000676
"sghaki	174	2267	0.65939	0.65935	-0.0000978
kighva	164	2431	0.70710	0.70711	-0.0000425
ghlaki	154	2585	0.75189	0.75184	-0.0001062
kigra	143	2728	0.79348	0.79335	-0.0001897

"sghaki	-0.0000978
kighva	-0.0000425
ghlaki	-0.0001062
kigra	-0.0001897
hakya	-0.0001798
dhaki	-0.0002397
kica	-0.0002265
sga	-0.0002723
"sjha	-0.0002019
"nva	-0.0001139
kla	-0.0000898
pta	-0.0001930
pha	-0.0001785
cha	-0.0000737

Error

So accurate!

Word from verse	Number from verse	Actual sine	Difference
	0	0.00000	0.0000000
makhi	225	0.06540	-0.0000467
bhakhi	224	0.13053	-0.0000826
phakhi	222	0.19509	-0.0000957
dhakhi	219	0.25882	-0.0000715
.nakhi	215	0.32144	0.0000080
~nakhi	210	0.38268	0.0001654
"nakhi	205	0.44229	0.0001386
hasjha	199	0.50000	-0.0000368

Aryabhata “sine” table

Word from verse	Number from verse	Number from formula	Angle (in minutes)	Angle (in degrees)	Angle (in radians)	Total from verse	Total for verse/3438	Total/(180*60/Pi)	Actual sine	sin(real)-(Number/(180*60/Pi)-sin
	0		0	0	0.000	0	0.00000	0.00000	0.00000	0.0000000
makhi	225	225.00	225	4	0.065	225	0.06545	0.06545	0.06540	-0.0000467
bhakhi	224	224.00	450	8	0.131	449	0.13060	0.13061	0.13053	-0.0000826
phakhi	222	222.00	675	11	0.196	671	0.19517	0.19519	0.19509	-0.0000957
dhakhi	219	219.02	900	15	0.262	890	0.25887	0.25889	0.25882	-0.0000715
.nakhi	215	215.07	1125	19	0.327	1105	0.32141	0.32143	0.32144	0.0000080
~nakhi	210	210.16	1350	23	0.393	1315	0.38249	0.38252	0.38268	0.0001654
°nakhi	205	204.31	1575	26	0.458	1520	0.44212	0.44215	0.44229	0.0001386
hasjha	199	197.56	1800	30	0.524	1719	0.50000	0.50004	0.50000	-0.0000368
skaki	191	189.92	2025	34	0.589	1910	0.55556	0.55560	0.55557	-0.0000262
ki.sga	183	181.45	2250	38	0.654	2093	0.60878	0.60883	0.60876	-0.0000676
°sghaki	174	172.17	2475	41	0.720	2267	0.65939	0.65944	0.65935	-0.0000978
kighva	164	162.12	2700	45	0.785	2431	0.70710	0.70715	0.70711	-0.0000425
ghlaki	154	151.35	2925	49	0.851	2585	0.75189	0.75195	0.75184	-0.0001062
kigra	143	139.91	3150	53	0.916	2728	0.79348	0.79354	0.79335	-0.0001897
hakya	131	127.85	3375	56	0.982	2859	0.83159	0.83165	0.83147	-0.0001798
dhaki	119	115.22	3600	60	1.047	2978	0.86620	0.86627	0.86603	-0.0002397
kica	106	102.07	3825	64	1.113	3084	0.89703	0.89710	0.89687	-0.0002265
sga	93	88.48	4050	68	1.178	3177	0.92408	0.92415	0.92388	-0.0002723
°sjha	79	74.49	4275	71	1.244	3256	0.94706	0.94713	0.94693	-0.0002019
°nva	65	60.17	4500	75	1.309	3321	0.96597	0.96604	0.96593	-0.0001139
kla	51	45.58	4725	79	1.374	3372	0.98080	0.98088	0.98079	-0.0000898
pta	37	30.79	4950	83	1.440	3409	0.99156	0.99164	0.99144	-0.0001930
pha	22	15.86	5175	86	1.505	3431	0.99796	0.99804	0.99786	-0.0001785
cha	7	0.86	5400	90	1.571	3438	1.00000	1.00007	1.00000	-0.0000737
			5625	94	1.636	3438	1.00000	1.00007	0.99786	-0.0022147

Different recursive formulae

प्रथमाच्चापज्याधार्धद्यैरूनं खण्डितं द्वितीयार्धम् ।
तत्प्रथमज्यार्धशैस्तैस्तैरूनानि शेषाणि ॥ १२ ॥

Āryabhaṭīya, Gaṇitapāda 12
The rule for the sine differences
Text: Shukla & Sarma (1976)

$S_n = R \sin(n\alpha)$, $\alpha = 225'$, $R = 3438$, $n = 1, \ldots, 24$ $\quad d_n = S_n - S_{n-1}$, $S_1 = 225$

1 Shukla's translation

$d_2 = S_1 - S_1/S_1$

$d_{n+1} = S_1 - (S_1 + \ldots + S_n)/S_1$

2 Datta & Singh, after Parameśvara (c. 1430)

$d_2 = S_1 - S_1/S_1$

$d_{n+1} = d_n - S_n/S_1$

3 Nīlakaṇṭha (c. 1500)

$d_2 = S_1 - S_1/S_1$

$d_{n+1} = d_n - (S_n/S_1)(d_1 - d_2)$

Three readings, one table

- Subtract consecutive lines of ①: you get ②.
- With $d_2 = 224$, $d_1 - d_2 = 1$, so ③ is ②.
- Exact value: $d_1 - d_2 = 2R \sin \alpha (1 - \cos \alpha) \approx 0.96$. With it, ③ is the exact second-difference rule.

But there is geometry (Gaṇitapāda 11)

- Half-chords are obtained “from trilaterals and quadrilaterals.”
- Bhāskara I (c. 630) computes 1719, 2978, 890 from right triangles.
- Plofker: the 24 values were probably computed this way.

Apparently, Aryabhata did not use his own rule! (7th difference: verse 205, rule 204.31)

Copied from a Greek table (Hipparchus, not Ptolemy)? Debated.

Shukla & Sarma, Āryabhaṭīya (INSA 1976) · Katz sourcebook, Plofker, pp. 407–409 · Toomer, Centaurus 18 (1974)

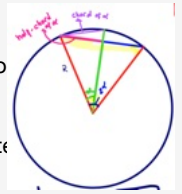
Large numbers and verses

- Indian mathematicians transmitted knowledge **orally**, often in Sanskrit verse.
- As in Ptolemaic astronomy, early Indian astronomy began by finding the mean position of celestial bodies at a given time.
- These mean position were expressed as **ratios of large integers**—numbers of cycles completed over fixed intervals.
- **Hence, they needed a method to encode very large numbers (often 10+ digits) within verse.**

Mathematics in India by Kim Plofker

Origin of the word “sine”

- **jya** **ज्या** The mathematical technical term **jya**¯ (bowstring) appears originally to have meant “**chord**.”
- **ardhajya** Later the term **ardhajya** “half-chord,” denoted the Sine.
- **jya** People dropped the “**ardha**” and kept “**jya**.”
- **jiba** When Arabic writers translated his works from Sanskrit into Arabic, they referred it as **jiba**.
- **jb** Since in Arabic writings, vowels are omitted, it was abbreviate as **jb**.
- **jaib** Later writers substituted **jb** with “**jaib**”, meaning "pocket" “bay” or "fold (in a garment)”.
- **sinus** Later in the 12th century, these writings were translated from Arabic into Latin, the Arabic **jaib** was replaced with its Latin counterpart, **sinus**, which means "**cove**" or “**bay**” which is the word that arrived to us.



Bakhshali Manuscript
799 –1102 CE
Earliest extant Indian mathematical manuscript.

Bakhshali manuscript -possibly 799–1102 CE

One of the Bakhshali manuscripts.



National Geographic, Public domain, via Wikimedia Commons

- Mathematical manuscript
- Written on birch bark
- Found in present-day Pakistan
- Written mainly in Sanskrit.
- Sometimes described as containing one of the earliest known written uses of a zero symbol as a placeholder
- Oldest extant manuscript of Indian mathematics.
- Contains rules and worked examples in arithmetic, algebra, geometry, and mensuration
- Composed by different folios that may date from different centuries.
- The mathematical content itself may be older than the manuscript.
- Radiocarbon dating places the sampled folios in 799–1102 CE, but the manuscript's date remains debated.

https://ora.ox.ac.uk/objects/uuid:5a6d1dd7-f20c-4209-adb6-33849f5b08f4

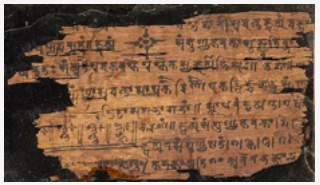
The positional, base 10 number system

Did the concept of zero appear first as a number or as a numeral? (Recall 1922 is a number, 0, 1,2...9 are the numerals in our decimal number system) Give reasons for your answer.

Development of Place-Value (positional) Number System and Zero in India

- **Bakhshali manuscript surviving birch bark dated 799–1102 CE**; early use of a dot as a zero placeholder.

One of the Bakhshali manuscripts.



National Geographic, Public domain, via Wikimedia Commons

- **Zero as a number (7th c. CE)** — Brahmagupta defines arithmetic rules; zero becomes operational, not just notation

Zero in Gwalior

- Gwalior, India: Medieval fort located south of Delhi, near Agra
- Inside the fort, a temple tablet dates to the 9th century CE
- This tablet records the oldest *definitively* dated zero in India: the inscription reads 876 CE





Decimal number 270 (top left) inscribed in a temple in Gwalior. (Plofker, Mathematics in India)

https://www.ams.org/publicoutreach/feature-column/fcarc-india-zero

Zero as a numeral around the world

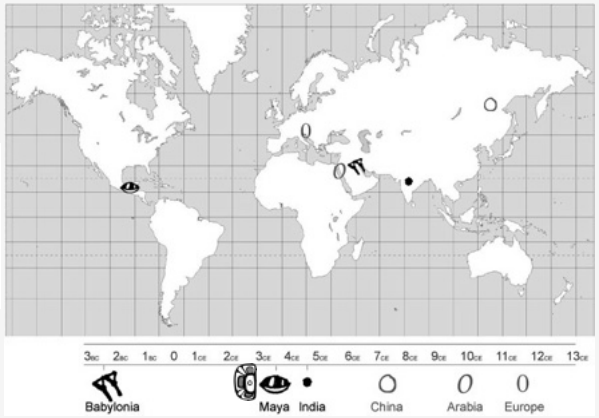


0 in Mesopotamia

The number 605 in Khmer numerals, (a date that corresponds to AD 683) one of the earliest known uses of zero as a numeral



https://commons.wikimedia.org/wiki/File:Khmer_Numerals_-_605_from_the_Sambor_inscriptions.jpg

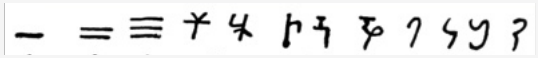


From A BRIEF HISTORY OF ZERO by Kristen McQuillin, July 1997 (revised January 2004)

https://mt.mediatinker.com/blog/archives/008821.html

Development of Place-Value (positional) Number System and Zero in India

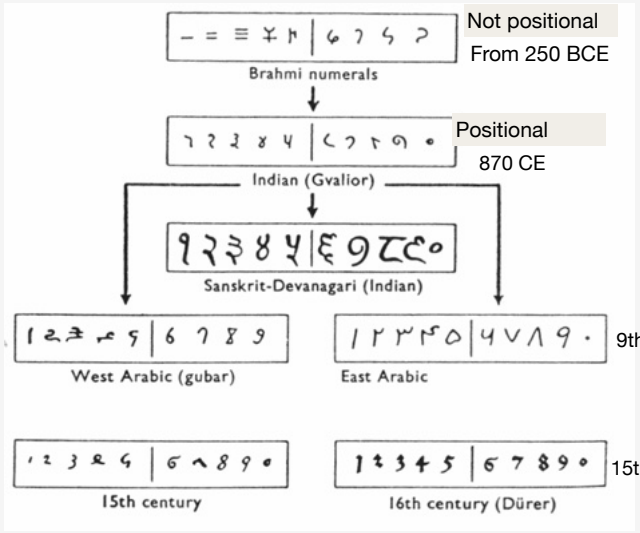
- **Brāhmī numerals (3rd c. BCE)** — Written additive/ multiplicative system; separate symbols for 1–9, tens, hundreds; non-positional; no zero



- **Bhūta-saṅkhyā system (possibly 3rd–5th c. CE)** — Verbal place-value system using object-words; positional concept; no zero

- **Inscriptions (mid-1st millennium CE)** — Written positional decimal notation appears; spreads to Southeast Asia; no found zero initially.

The evolution of decimal numerals
From Brahmi to Arabic to modern numerals



Higgins, Peter. Number Story: From Counting to Cryptography, Gottingen: Copernicus, 2008. P.189.

How Indian number system (positional, base 10) reached Europe
India → Islamic world → Europe

- **India:** Decimal place-value notation developed.
- **9th century:** Al-Khwārizmī wrote a treatise explaining Indian arithmetic.
- **12th century:** Latin translations helped spread these methods in Europe.
- **1202**: Fibonacci's Liber Abaci helped popularize decimal arithmetic in Europe.

Sanskrit Poetry Throughout 500–1200 CE

Between 600 and 800 CE, in India, a poet named Virahanka gave a rule for counting certain variations in a given rhythm of Sanskrit poetry.

The basic units in Sanskrit poetry are ***mora*** or "syllabic instant" which can be either

- **Short**, lasting 1 beat, denoted by S.
- **Long**, lasting 2 beats, denoted by L.

Thus, we have

- exactly one verse type lasting one beat - S
- exactly two verses types lasting two beats - SS and L
- exactly three verses types lasting three beats: SSS, SL and LS

Let's write down verses (in English!) reinterpreting the rules as follows:
Each

- **mora** (or "syllabic instant") is a **word**,
- **beat** is a **syllable** of the word

Each of the words we use is:

- **S**hort, lasting 1 syllable (one beat), denoted by **S**. (for instance: "green")
- **L**ong, lasting 2 syllables (2 beats), denoted by **L**. (for instance,"purple")

Math is great

Math (1) — S

is (1) — S

great (1) — S

Total: 1 + 1 + 1 = 3 syllables

Type: **SSS**

Find verses of 3 syllables state their type.
Example: Math is great- SSS

SSS (1 + 1 + 1)

- Cats eat cake.
- Fish love jazz.
- Dogs chase cars.
- Bees make tea.
- Pigs play chess.

SL (1 + 2)

- Cats follow.
- Dogs quibble.
- Kids gallop.
- Fish ripple.
- Bees narrow.

LS (2 + 1)

- Purple cat.
- Orange dog.
- Silly pig.
- Happy frog.
- Yellow bee.

.

ChatGPT

Let's write down verses (in English!) reinterpreting the rules as follows:
Each

- **mora** (or "syllabic instant") is a **word**,
- **beat** is a **syllable** of the word

Each word is either:

- Short, lasting 1 syllable (one beat), denoted by S. (for instance: "green")
- Long, lasting 2 syllables (2 beats), denoted by L. (for instance,"purple")

The kitten sleeps

The (1) — S

kit-ten (2) — L

sleeps (1) — S

Total: 1 + 2 + 1 = 4 syllables

Type: **SLS**

Find verses of 4 syllables and state their type.
Example: The kitten sleeps - SLS

SSSS (1 + 1 + 1 + 1)

- Cats eat cold pie.
- Dogs jump high cliffs.
- Bees hum near ponds.
- Fish swim past rocks.
- Pigs bake soft bread.

SSL (1 + 1 + 2)

- Cats eat butter.
- Dogs chase teacher.
- Pigs love drama.
- Bees make honey.
- Fish hug water.

SLS (1 + 2 + 1)

- Cats follow rules.
- Dogs gallop fast.
- Bees quibble less.
- Kids ripple songs.
- Pigs narrow minds.

LSS (2 + 1 + 1)

- Purple cats run.
- Orange pigs eat.
- Silly bees hum.
- Happy dogs nap.
- Yellow frogs jump.

LL (2 + 2)

- Purple teacher.
- Orange kitten.
- Silly water.
- Happy maple.
- Narrow drama.

ChatGPT

Let's write down verses (in English!) reinterpreting the rules as follows:
Each

- **mora** (or "syllabic instant") is a **word**,
- **beat** is a **syllable** of the word

Each word is either:

- Short, lasting 1 syllable (one beat), denoted by S. (for instance: "green")
- Long, lasting 2 syllables (2 beats), denoted by L. (for instance,"purple")

My students are smart

My (1) — S

stu-dents (2) — L

are (1) — S

smart (1) — S

Total: 1 + 2 + 1 + 1 = 5 syllables

Type: **SLSS**

Find verses of 5 syllables and state their type. Example: My students are smart - SLSS

SSSSS (1 + 1 + 1 + 1 + 1)

- Cats eat cake at noon.
- Dogs run past the lake.
- Pigs dig holes for fun.
- Bees buzz near the barn.
- Fish jump over logs.

SSSL (1 + 1 + 1 + 2)

- Cats eat cake slowly.
- Dogs chase pigs loudly.
- Bees hum songs softly.
- Fish swim fast always.
- Kids draw clouds purple.

SLL (1 + 2 + 2)

- Cats follow teacher.
- Dogs quibble drama.
- Bees ripple water.
- Kids gallop orange.
- Pigs narrow maple.

LSSS (2 + 1 + 1 + 1)

- Purple cats eat pie.
- Orange dogs run far.
- Silly frogs jump high.
- Happy bees make tea.
- Yellow pigs wear hats.

LLS (2 + 2 + 1)

- Purple butter cat.
- Orange drama pig.
- Silly water bee.
- Happy maple dog.
- Yellow teacher ant.

ChatGPT

How many *types* of verses of one syllable are there? How many of two? three? four? five?

Let's write down verses (in English!) reinterpreting the rules as follows: Each

- **mora** (or "syllabic instant") is a **word**,
- **beat** is a **syllable** of the word

Each word is either:

- Short, lasting 1 syllable (one beat), denoted by S. (for instance: "green")
- Long, lasting 2 syllables (2 beats), denoted by L. (for instance,"purple")

Between 600 and 800 CE, in India, a poet named Virahanka gave a rule for counting certain variations in a given rhythm of Sanskrit poetry.

The basic units in Sanskrit poetry are ***mora*** or "syllabic instant" which can be either

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Thus, we have

- exactly one verse type lasting one beat - S
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- exactly three verses types lasting three beats: SSS, SL and LS

of Rythms of N syllables composed by short and long words.

Types of words we can use for verses

- Short, lasting 1 syllable, denoted by S (for instance: “green”)
- Long, lasting 2 syllables, denoted by L. (for instance,” purple”)

Table 1-2

Number of syllables	How many	List of types
1	1	
2	2	
3	3	
4	5	
5	8	
6	13	

of Rythms of N syllables composed by short and long words.

Types of words we can use for verses

- Short, lasting 1 syllable, denoted by S. (for instance: “green”)
- Long, lasting 2 syllables, denoted by L. (for instance,” purple”)

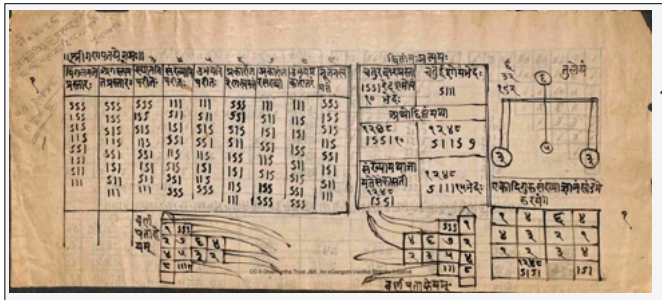
Table 1-2

Number of syllables	How many	List of types
1	1	S
2	2	SS; L
3	3	SSS; SL; LS
4	5	SSSS; LL; SLS; SSL; LSS
5	8	SSSSS; LLS; LSL; SLL; LSSS; SLSS; SSLS; SSSL
6	13	SSSSSS, SSSSL, SSSLS, SSLSS, SLSSS, LSSSS, SSLL, SLSL, SLLS, LSSL, LSLS, LLSS, LLL

Why the Fibonacci numbers count the number of rhythms of n syllables?

All types of length n+1

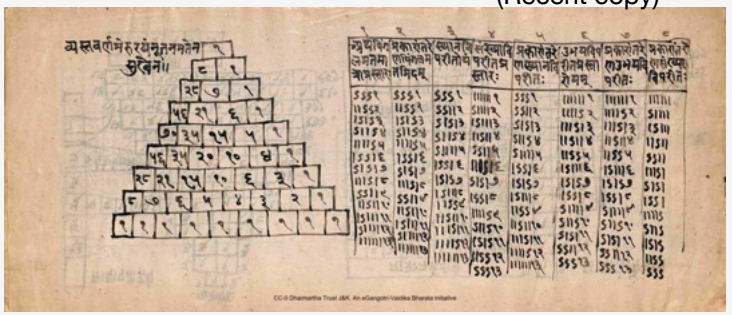
Length n+1 , ends with L,, Counts all types of length n-1	Length n+1 , ends with S, Counts all types of length n



Prakrit Pingala Prastara; Raghunath Temple, Jammu,J&K; https://archive.org/details/PrakritPingalaPrastaraVarnaMatraPatakadiYantrani775GhaAlm4Shlf3DevanagariAlankarShastra/page/n3/mode/2up

Pingala’s Chandaḥśāstra tradition (ancient Indian treatise on prosody and combinatorics).

Content origin: ~3rd BCE – 2nd BCE (Recent copy)



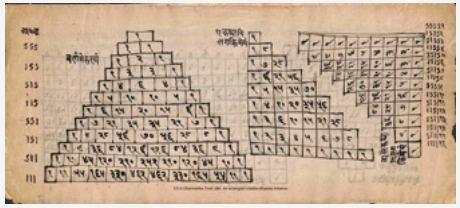
Fibonacci numbers also appear in the Pascal Triangle and were found by Sanskrit scholars.



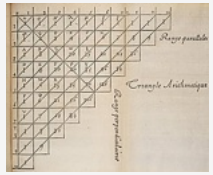
By RDBury - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=15045063



Yang Hui's triangle in Jade Mirror of the Four Unknowns, a mathematical work by Zhu Shijie, dated 1303.



Pingala binomial coefficients triangle. https://archive.org/details/PrakritPingalaPrastaraVarnaMatraPatakadiYantrani775GhaAlm4Shlf3DevanagariAlankarShastra/page/n3/mode/2up



Pascals Pascal Triangle Blaise Pascal - Cambridge University Library, https://cudl.lib.cam.ac.uk/view/PR-CCB-00013-00024/5

Pingala (~250 BCE) classifying poetic meters of long and short syllables, presents the Mount Meru (known now as Pascal Triangle)

Fibonacci numbers in the Liber Abaci (1202)

A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

These are very special rabbits:

- They never die,
- They “make” a new pair of rabbits exactly once every month
- This pair is always composed of male and female.

Fibonacci numbers in the Liber Abaci (1202)

A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

1. In the first month, how many pairs do we have? Are they able to produce offspring in this first month?

2. In second month, can our original pair produce offspring? How many pairs total?

3.In the third month how many pairs can produce offspring? Why can't the pair born in month 2 reproduce yet?

4.The fourth month how many pairs can produce offspring? What is the total number of rabbits?

These are very special rabbits:

- They never die,
- They “make” a new pair of rabbits exactly once every month
- This pair is always composed of male and female.

Fibonacci numbers in the Liber Abaci (1202)

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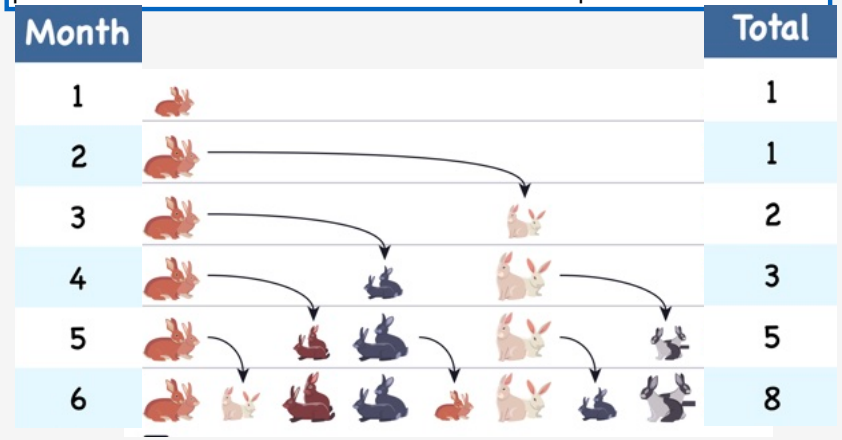


https://mathigon.org/course/sequences/fibonacci

Fibonacci numbers in the Liber Abaci (1202)

A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

- Start with **1 newborn pair**.
- **Month 1:** that pair **matures** (no birth yet).
- **Month 2**: the mature pair produces 1 baby pair.
- Pairs we have 1 mature + 1 baby.
- Each later month:
  - Every mature pair produces 1 baby pair.
  - Every baby pair becomes mature.

In **month n**

- M_n =Mature pairs
- B_n =Baby pairs
- **Total: $T_n = M_n + B_n$**

In **month n+1** we have

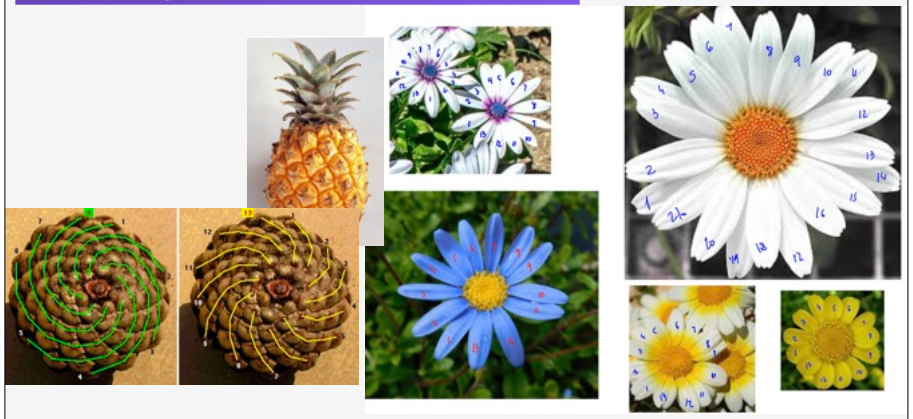
- $B_{n+1} = M_n$.
- $M_{n+1} = M_n + B_n$
- **$T_{n+1} := M_{n+1} + B_{n+1}$**

In **month n+2** we have

- $B_{n+2} = M_{n+1}$.
- $M_{n+2} = M_{n+1} + B_{n+1}$
- **$T_{n+2} = M_{n+2} + B_{n+2} = T_{n+1} + T_n$**

Fibonacci numbers appear in some natural patterns, such as the spirals of sunflower heads and pinecones, where growth and efficient packing help shape their arrangement.



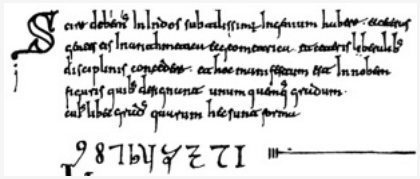


Liber Abaci (1202)

"A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?"

The Liber Abaci, authored by Fibonacci, helping popularize Indian **decimal positional number system**, which utilizes Hindu-Arabic numerals, to Europe. This is the same number system that we use today.





The first Arabic extant numerals in a Western manuscript, AD 976

The numerals appeared in Europe by the 900s, but entered daily mathematical life only after Liber Abaci in the 1200s.

Liber Abaci (1202)

The Liber Abaci, authored by Fibonacci, helping popularize Indian **decimal positional number system**, which utilizes Hindu-Arabic numerals, to Europe. This is the same number system that we use today.



The ingenious method of expressing every possible number using a set of ten symbols (each symbol having a place value and an absolute value) emerged in India. The idea seems so simple nowadays that its significance and profound importance is no longer appreciated. Its simplicity lies in the way it facilitated calculation and placed arithmetic foremost amongst useful inventions. the importance of this invention is more readily appreciated when one considers that it was beyond the two greatest men of Antiquity, Archimedes and Apollonius

Laplace

Brahmagupta — 598–668 CE

Rules for zero and negative numbers; quadratic equations; geometry of cyclic quadrilaterals.

India: Bhaskara → negative roots for quadratic equations but…

*The second value is in this case not to be taken, for it is **inadequate**; people do not approve of negative roots"*

0 — 600 — 800 — 1100

600 C.E., India: Brahmagupta → rules for operations with negative numbers.

Negative numbers in intermediate computations

- A debt minus zero is a debt; an asset minus zero is an asset.
- Zero minus zero is zero.
- Zero minus a debt is an asset. Zero minus an asset is a debt.
- The product (quotient) of a debt and an asset is a debt, of two debts or of two assets is an asset.
- The product of zero with an asset, a debt or with zero is zero.

Bhāskara I c. 600–680 CE

— Approximation formulas for sine; commentary on Āryabhaṭa.

Bhāskara I
c. 600–680 CE

- Indian mathematician and astronomer.
- Wrote commentaries explaining and extending Aryabhata's mathematics.
- Developed an accurate rational approximation to the sine function.
- His formula required only arithmetic operations, avoiding trigonometric tables.

Mahāvīra (9th century CE)

Mahāvīra (9th century CE)

A Jain mathematician in South India, probably at the court of a king

"*Epitome of the Essence of Calculation*"

- Earliest Sanskrit book devoted only to mathematics that survives complete.
- Has more than 1,100 verses of rules and word problems: arithmetic, fractions, proportion, series, areas and volumes.
- Includes rules for zero and negative numbers..
- Negative numbers have no square root, "Because a negative quantity, on account of its own form, is nonsquare, therefore [there is] no square root of that.."
- States mathematics is useful everywhere: in love, commerce, music, cooking, medicine and poetry.

A mathematician ["calculator"] is to be known by eight qualities:

1. Swift work
2. Deliberation
3. Refutation
4. Non-idleness
5. Comprehension
6. Concentration (or memory)
7. Inventiveness
8. Having the answers

Problem from Mahāvīra's book

A king took a sixth of a quantity of mangoes, the queen a fifth of the remainder, three princes a fourth, a third [and] a half [of successive remainders]. The youngest son took the three remaining mangoes. [You], skilled in the miscellaneous [methods], tell [me] the amount of them.

Step	What happened	Mangoes
Youngest son	Took the 3 remaining mangoes.	3
Third prince	Took half, leaving 3.	3 × 2 = 6
Second prince	Took 1/3, leaving 2/3.	6 × 3/2 = 9
First prince	Took 1/4, leaving 3/4.	9 × 4/3 = 12
Queen	Took 1/5, leaving 4/5.	12 × 5/4 = 15
King	Took 1/6, leaving 5/6.	15 × 6/5 = 18

Bhāskara II (Bhāskarācārya) — 1114–1185 CE

Algebra, calculus-like reasoning, indeterminate equations (chakravāla method).

Bhāskara II (Bhāskarācārya) — 1114–1185 CE

- Indian mathematician and astronomer.
- Author of the Līlāvatī (arithmetic) and Bījagaṇita (algebra).
- Studied indeterminate equations, including equations of the form $x^2 - Dy^2 = 1$.
- Developed methods involving zero, negative numbers, and quadratic equations.
- Investigated instantaneous motion and other ideas resembling aspects of differential calculus.
- He wrote that
  - a mathematician without demonstrations will not shine "in the assemblies of the eminent." In the sphere, the proof is seen "like a fruit in the hand."
  - Number divided by zero is like the god Viṣṇu: creatures go in and out and he is unchanged. (This is one of the earliest links between division by zero and infinity.)

Kerala School of Astronomy and Mathematics — c. 1300–1600 CE

Infinite series for sine, cosine, arctangent; proto-calculus

Kerala School of Mathematics (c. 1300–1600 CE)

- Kerala, southwestern India, closely connected to astronomy.
- **Mādhava** (c. 1340–1425): Developed infinite series for π and trigonometric functions.
- **Nīlakaṇṭha Somayājī** (15th–16th century): Extended mathematical and astronomical methods.
- **Yuktibhāṣā** (16th century): Presented mathematical reasoning and derivations, including infinite series.

These developments preceded Newton and Leibniz by centuries.