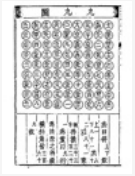
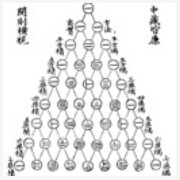
Ancient and Medieval Chinese Mathematics



1. China, India, and the Islamic World
2. Rod numerals and counting boards
3. Nine Chapters / Book of Numbers and Computations
4. Liu Hui
5. Liu Hui on the volume of the pyramid
6. Liu Hui on the volume of the truncated pyramid
7. Liu Hui on the volume of the sphere
8. Zu Geng on the volume of a sphere
9. Computation of π by Liu Hui and Zu Chongzhi
10. Gougu theorem
11. Magic squares
12. Square root computations
13. Medieval China
14. Timeline, summary

China, India, and the Islamic World

China, India, and Islamic World

Difficulties Studying Math History

- Large (in space and time).
- Not homogeneous (for instance, different languages)
- Western centered approach
  - tendency to see them as alien or exotic
  - accounts from historians or writers with a variety of agendas.
- Lack of documents (destroyed by climate, wars, fires, and people)

China, India, and Islamic World

Some reasons for doing mathematics

- Astronomy - including computations of the calendar, astrology and cosmology.)
- Religious: Calculating the direction of Mecca for the Islamic world, constructing altars in India.
- Measuring time.
- Land surveying
- Estimating areas and volumes
- Taxation and division of states.
- Teaching numeracy to an elite.
- Math for the sake of it.

Who did math professionally in Antiquity? Priests? Scribes? Wealthy people? Bureaucrats? Somebody else? Who? (Answer the question for one or more of the following societies)

- In Egypt?
- In Babylonia?
- In the Hellenic world?
- In China?

Who did math professionally in Antiquity? Priests? Scribes? Wealthy people? Bureaucrats? Somebody else? Who? (Answer the question for one or more of the following societies)

- Egypt
  - Scribes and priest-scribes (bureaucratic administrators)
  - **Greek point of view:** "Thus the mathematical sciences originated in the neighborhood of Egypt, because there the priestly class was allowed leisure." Aristotle (Metaphysics):
- Babylonia
  - Scribes and scholar-scribes in temple, palace, and scribal-school contexts
- Hellenic world
  - Scholars, teachers, philosophers, astronomers, engineers, surveyors, and other technical specialists
- China
  - State officials and technical specialists in the imperial bureaucracy, especially in astronomical and calendrical offices.

Oracle bone (~1200 BCE)
Numbers between 1 to 30,000.
Oldest extant Chinese numerals



From A history of Chinese mathematics Jean-Claude Martzloff copy





Inscribed tortoise carapace ("oracle bone"), Anyang period, late Shang dynasty, c. 1300–1050 B.C.E., tortoise shell, China, 6.5 high x 10.8 x 2.3 cm - Smithsonian Institution, Washington, D.C.

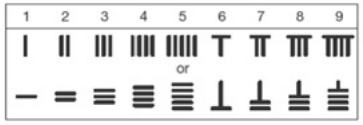
Rod numerals and counting boards

Complete the table using the hints (row a = 72, row b = 26). What are the numbers in rows c, d, and e?

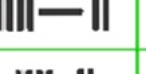
Rod numerals were used (approximately) between 500 BCE and 1500 CE.

	72	a
	26	b
		c
		d
		e

Counting rods and counting boards

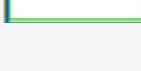


Rod numerals were used in China for over 2000 years. This number system was:
a) Additive b) Positional
c) Multiplicative d) Ciphered

	72	a
	26	b
	42	c
	412	d
	402	e

Rod number system

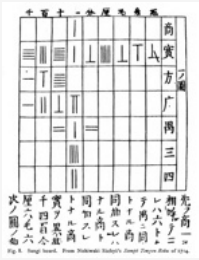
- 475-221 B.C. A **base ten positional number system** was in common use in China
- Calculations were done using rods made from bone or bamboo, on a **counting board**.
- The numerals from 1 to 9 were represented by rods.
- Rods representing odd powers of ten were rotated 900 for clarity in distinguishing the powers.
- A zero was represented by a blank square.

	72	a
	26	b
	42	c
	412	d
	402	e

Counting boards/ Calculating Surfaces

Calculations were performed by arranging counting rods on a surface, commonly called a **counting board**. The precise physical form of these surfaces is uncertain.

- **Positional** number system: Columns = units, tens, hundreds (blank = zero)
- **Advanced set of algorithms** for
  - multiplication, division,
  - computation of square and cubic roots
  - solving linear and polynomial equations
- Use of **negative numbers**



Japanese sangi board (Nishiwaki, 1714), in Smith & Mikami 1914.
https://kartsci.org/kocomu/computer-history/history-abacus-ancient-computing/

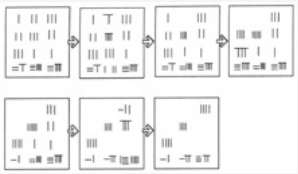
Counting rods and counting boards

The Chinese counting board consisted on a grid of square cells for storing and manipulating rows and columns of numbers. This same grid concept later appeared in the West under different names. Which of these terms describe such grids? (choose all the appropriate): lattice, matrix, magic square, spreadsheet?

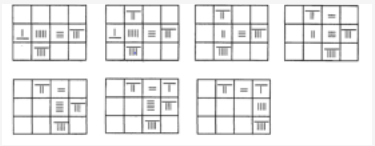


Examples of Counting Board Uses

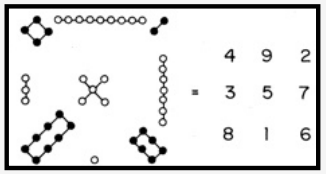
Gauss-Jordan Algorithm



Division Algorithm



Magic Squares



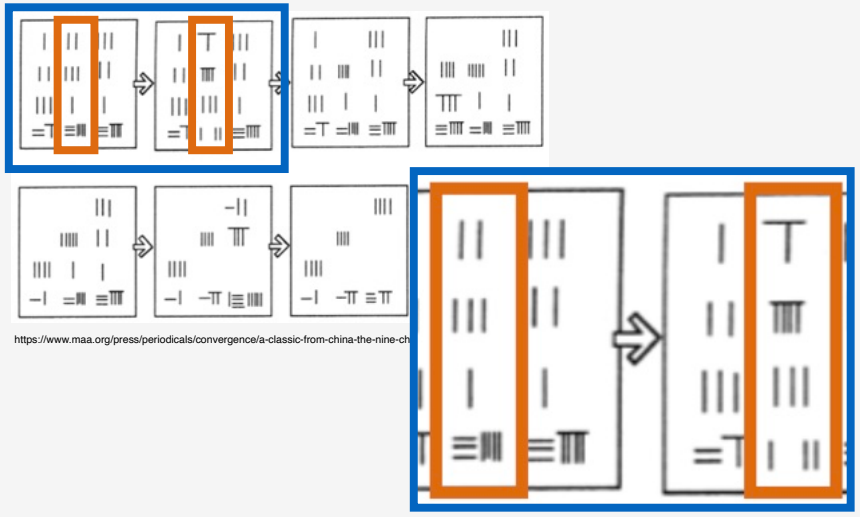
Square Root Algorithm



The Chinese counting board is a good example of how a technological invention can influence how science develops, and even how people think.

Randy K. Schwartz- *A Classic from China: The Nine Chapters* -
https://www.maa.org/press/periodicals/convergence/a-classic-from-china-the-nine-chapters-numbers-and-units

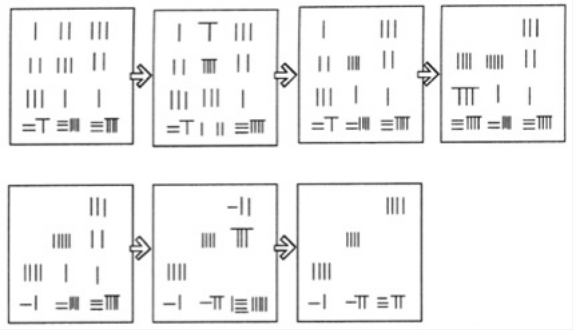
What is the relation between the two highlighted columns?



https://www.maa.org/press/periodicals/convergence/a-classic-from-china-the-nine-ch

"Matrices" in the Nine Chapters

The illustration shows how the above problem **would** be solved on a traditional Chinese counting board.

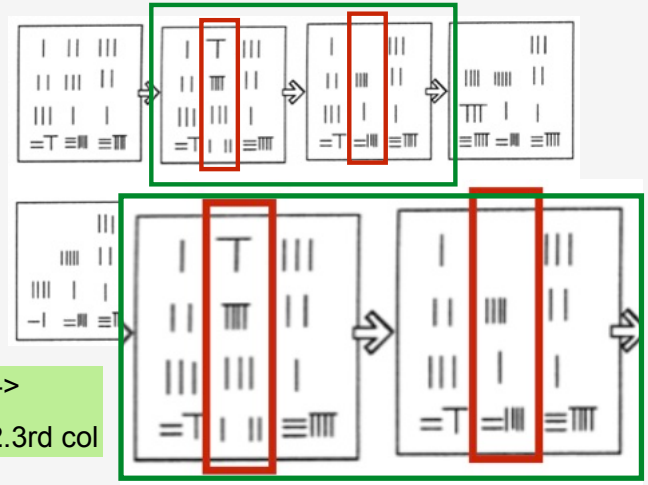


https://www.maa.org/press/periodicals/convergence/a-classic-from-china-the-nine-chapters-matrices

“Matrices” in the Nine Chapters

“Matrices” already appeared in the Nine chapters

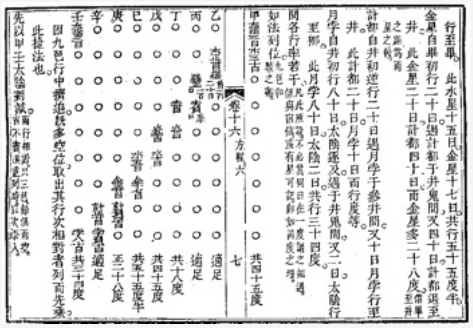
The following illustration shows how the above problem **would** be solved on a traditional Chinese counting board.



2nd col. ->

2nd col-2.3rd col

A 9x9 linear system ~1700

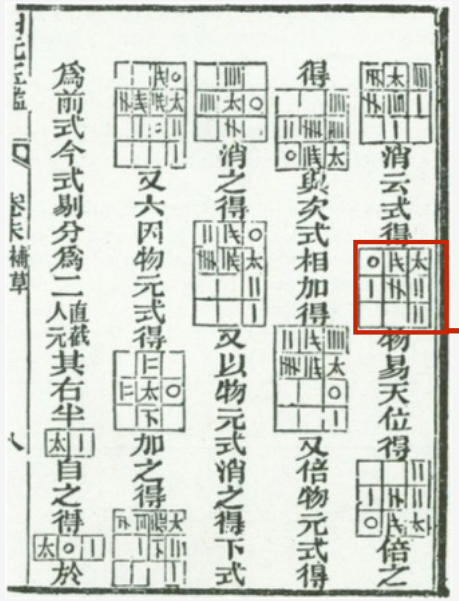


Source: http://rhart.org/algebra/

$$\begin{bmatrix} 3 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 \\ 0 & 30 & -120 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & -50 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 30 & 30 & 0 & 0 & 0 & 0 & 18 \\ 0 & 0 & 0 & 0 & 50 & 10 & 0 & 0 & 0 & 45 \\ 0 & 0 & 0 & 0 & 0 & 15 & 17 & 0 & 0 & 111/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 20 & -40 & 0 & 28 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 20 & -10 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 80 & 34 \end{bmatrix}$$

A *fangcheng* problem with 9 conditions in 9 unknowns of the form of the generalized “well problem,” from Mei Wending’s 梅文鼎 (1633–1721) *On* Fangcheng (*Fangcheng lun* 方程論, c. 1674)

Polynomials Equations - 1303



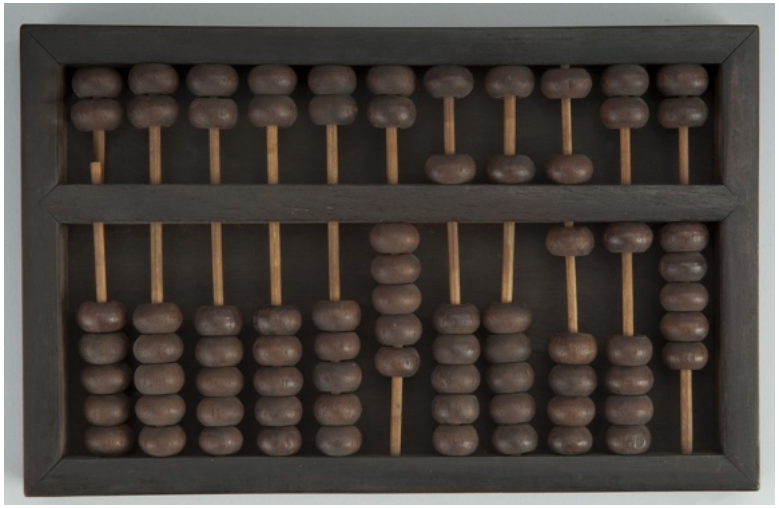
A page from a 19th century edition of the *Ssu Yuan Yii Chien of* Chu Shih-Chieh (**1303 a.D.**)

0	−120	·
1	−2	2
		2

xy^2-l20y $-2xy+2x^2+2x$

Source: On ancient Chinese mathematics by D. Struik, The Mathematics Teacher , Vol. 56, No. 6 (OCTOBER 1963), pp. 424-432

Modern Chinese Abacus
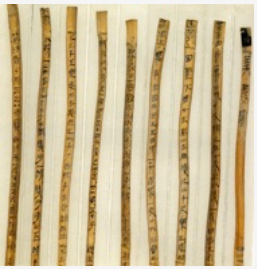


https://old.maa.org/press/periodicals/convergence/mathematical-treasure-modern-chinese-abacus

Nine Chapters of the Mathematical Art

Book of Numbers and Computations

The Book of Numbers and Computations - Suan Shu Shu

1. **235 BCE** (discovered in **1983** in a 186 BCE tomb).
2. Anonymous
3. Earliest known Chinese mathematical text.
4. **original source** (not a copy)
5. Possibly from a variety of sources.
6. 68 problems, order unknown
7. About 200 bamboo strips.
8. **practical mathematics** for administrators: fractions, proportions, wastage, and solid volumes
   - Calculations with fractions and proportions,
   - Rule of False Position
   - Volumes of various solid shapes.
   - tax calculations, grain conversions.



算數書

The Book of Numbers and Computations - Suan Shu Shu

A fox, raccoon, and hound go through customs, and (together) pay tax of 111 qian. The hound says to the raccoon, and the raccoon says to the fox: since your fur is worth twice as much as mine, then the tax you pay should be twice as much! How much should each one pay?

Find an equation to solve this problem.



Bamboo strips - The Book of Numbers and Computations
Suan Shu Shu

The Book of Numbers and Computations - Suan Shu Shu

Statement of the problem

A fox, raccoon, and hound go through customs, and (together) pay tax of 111 qian. The hound says to the raccoon, and the raccoon says to the fox: since your fur is worth twice as much as mine, then the tax you pay should be twice as much! How much should each one pay?

The result says: the hound pays 15 6/7 qian, the raccoon pays 31 5/7 qian, and the fox pays 63 and 3/7 qian.

The method says: let each one double the other; adding them together (1 + 2 + 4), 7 is the divisor; taking the tax, multiplying by each (share) is the dividend; dividing the dividend by the divisor gives each one's (share).

The Book of Numbers and Computations - Suan Shu Shu

(The tax on) 3 (square) bu of millet is 1 dou; (on) 4 (square) bu of wheat is 1 dou; (and on) 5 (square) bu of small beans is 1 dou. If the combined tax (on all of them together) is 1 shi (capacity), then how much is the tax (on each one)?

The result says: the tax on millet is 4/12 dou; the tax on wheat is 3 9/47 dou; (and) the tax on beans is 2 26/47 dou.

The meaning of "Put down" here is "Put down on the counting board

The method says: put down (**on the counting board** the amount of) millet 3 bu, wheat 4 bu, and beans 5 bu; let the product of the millet and wheat be the dividend for the beans; the product of the beans and the millet be the dividend for the wheat; (and the product of the wheat and the beans be the dividend for the millet); for each of the different (amounts) put down (on the counting board) one shi multiplied by each (of the amounts for beans, wheat, and millet) as the dividends; (taking) 47 as the divisor gives the result in dou.

Fundamental source. Compilation

The Nine Chapters of the Mathematical Arts

1. **~200 BCE**
2. Earliest copy made in 1213 CE.
3. From a variety of sources (including Book of Numbers)
4. Practical handbook of mathematics to provide methods for everyday problems of engineering, surveying, trade, and taxation.
5. 246 problems
   a. statement
   b. numerical answer
   c. method to calculate the solution.
6. Organized in 9 chapters by topics (often by nature of the problem).
7. Often sophisticated and difficult in its treatment of algorithms.
8. Later commentators made **important contributions**

Method for approximating π from a sixteenth-century version of **Liu Hui**'s third-century edition and commentary on the Nine Chapters.



Source: Mathematical Treasures - *Jiuzhang suanshu*, MAA *Convergence*)

The Nine Chapters on the Mathematical Art, around 200BC.

There are three types of grain (millet): best quality, medium quality, and worst quality.

- Three bundles of the best quality, two bundles of medium quality, and one bundle of the worst quality produce 39 units of grain.
- Two bundles of the best quality, three bundles of medium quality, and one bundle of the worst quality produce 34 units of grain.
- One bundle of the best quality, two bundles of medium quality, and three bundles of the worst quality produce 26 units of grain.

Question: How much grain does one bundle of each type produce?

Modern formulation

$$3x + 2y + 1z = 39$$
$$2x + 3y + 1z = 34$$
$$2 + 2y + 3z = 26$$

Table 1

	Suan Shu Shu	Nine Chapters on the Mathematical Art
Date (approx.)	3rd entury BCE	1st century BCE
Number	68 problems	246 problems
Medium	190-200 bamboo strips	Text on silk/book form (later)
Organization / content	No chapters, loosely grouped by topic, order uncertain. Practical problems: fractions, proportions, taxation, volumes, grain conversion	Divided into 9 organized chapters by topic: arithmetic, algebra, geometry, proportions, root extraction, volumes, linear systems, right triangles
Level	Basic/proto-algebraic; lacks advanced matrix or geometric algorithms	More advanced: includes matrix algebra and Pythagorean theorem applications
Influence	Practical handbook for officials, possibly source for later texts	**Enormous influence**, basis for Chinese mathematical tradition
Known Authors	Anonymous (scribal names appear in places)	Traditionally attributed to Zhang Cang, Geng Shouchang; major commentary by Liu Hui

The Nine Chapters on the Mathematical Art, around 200BC.

Liu Hui Commentary (~200 CE): Things from different groups are put together in wholes that are mixed up. For each type, in a row, there are quantities, and one expresses their products/dividends [that is, their constant term] globally. One does it in such a way that each column is made of lu. If there are two things, one measures [each] twice, if there are three things; one measures [each] thrice; all types of things] are measured [as many times as] the number of things. One brings rows together to make columns, this is why one calls this [procedure] 'measures in square. To the right and to the left of a column, no [other column] exists that is the same. Moreover, one ensures that they are formulated only if there is something on which to ground them. **This is a universal procedure. However, with abstract formulations it would be difficult to understand them. This is why the [authors] linked it on purpose to millet to eliminate the obstacle.** Further, one lays down the middle and the left columns like the right one.

The special terms which are used to denote mathematics […] indicate that it is concerned with numbers shu and with computation suan and that it is an art (shu, fa, dao). […] In addition to its general meaning of "art," shu also means process, method, device, trick, stratagem, artifice or prescription. Thus, the various shu are essentially collections of prescriptions which, in the case of mathematics, are more or less similar to our algorithms. In addition, etymologically, suan denotes a set of specific objects used for calculation: counting-rods. Thus, at least originally, it relates to the faculty of knowing how to manipulate certain rods with dexterity in order to carry out computations in accordance with fixed prescriptions. A History of Chinese Mathematics Jean-Claude Martzloff

Chinese mathematicians were clearly concerned about justifying their methods and establishing the validity of their results. Their proofs were not axiomatic proofs, but they were proofs nevertheless, and they were clearly able to establish the truth of correctness of the solutions they proffered. Joseph Dauben

Liu Hui (~250)
劉徽

Liu Hui (~250 CE)

- Outstanding and original mathematician with a deep understanding of difficult concepts.
- Familiar with the literary and historical classics of China.
- Never claimed results of which he was not fully confident. He wrote:-

  Let us leave the problem to whoever can tell the truth.

- Cared about the conditions of people and about the economy of the country

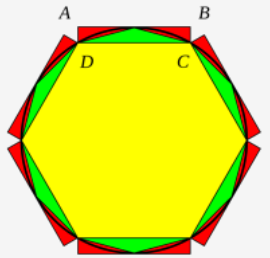


Diagram of Liu Hui's π inequality
Source: derivative work: Pbroks13 (talk) Gisling - Wikimedia Commons

Worked on:

- Volumes of plane and solid figures.
- Solution of linear equation with two unknowns.
- Algorithm to compute what we now call π.

https://mathshistory.st-andrews.ac.uk/Biographies/Liu_Hui/

Liu Hui Introduction to his commentary of the Nine Chapters

I read the Nine Chapters as a boy, and studied it in full detail when I was older. [I] observed the division between the dual natures of Yin and Yang [the positive and negative aspects] which sum up the fundamentals of mathematics. **Thorough investigation shows the truth therein, which allows me to collect my ideas and take the liberty of commenting on it.** Things are known to belong to various classifications. Just as the branches of a tree are to its trunk, so are a multitude of things to an archetype. Therefore I have tried to explain the whole theory as concisely as possible, with spatial forms shown in diagrams, so that the reader should have a reasonably good all-around understanding of it.

Liu Hui Introduction to his commentary of the Nine Chapters

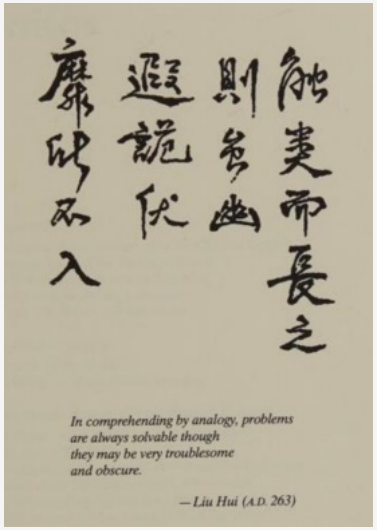
Some of the material in the Nine Chapters predates the great book-burning and burial-alive of scholars of 213 B.C., ordered by emperor ShihHuang-ti of the Qin dynasty. Indeed, Liu Hui writes in the preface of his commentary:

In the past, the tyrant Qin burnt written documents, which led to the destruction of classical knowledge ... Because of the state of deterioration of the ancient texts, Zhang Cang and his team produced a new version ...filling in what was missing

Liu Hui and the First Golden Age of Chinese Mathematics, Philip D. Straffin, Jr.,Mathematics Magazine, Jun., 1998, Vol. 71, No. 3 (Jun., 1998)

Page 5 from The sea island mathematical manual : surveying and mathematics in ancient China by Liu Hui (translated by Frank Swetz)

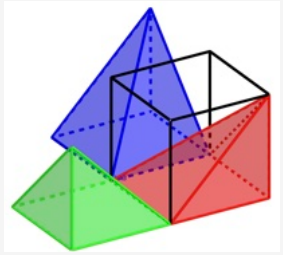
In comprehending by analogy, problems are always solvable though they may be very troublesome and obscure.

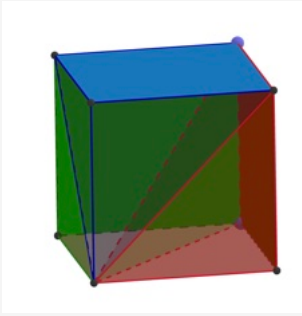


https://archive.org/details/isbn_2083776009956/page/n5/mode/2up

Liu Hui on the volume of the pyramid

Cube dissected into Yangmas - Square faces marked





https://www.geogebra.org/m/cqydu4an

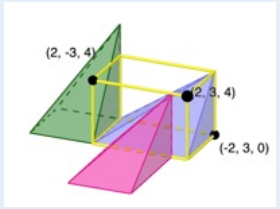
https://www.geogebra.org/m/v6CavHd2

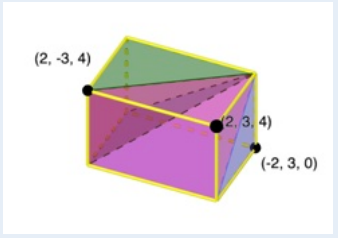
Liu Hui on the volume of yangma (rectangular pyramid)

A ***yangma*** is a
- pyramid with rectangular base and
- one of its lateral edges perpendicular to the base

Liu Hui (among many other topics) studied the volume of the Yangma.

1.Are these three yangmas congruent? 2.Educated guess: Do they have the same volume?

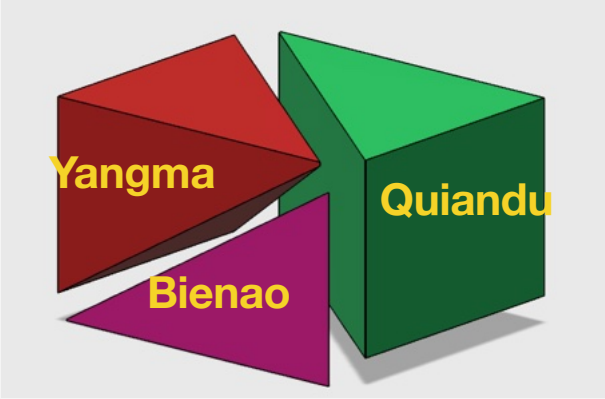




https://www.geogebra.org/m/weg6d6hx

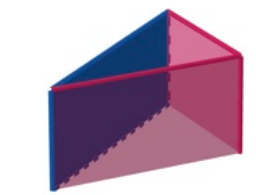
In a cube the three pieces are congruent; in a box they aren't, so symmetry can't give 1/3. Liu Hui gives us another argument.

Liu Hui dissection of the rectangular prism



Liu Hui Components for the dissection of the rectangular prism

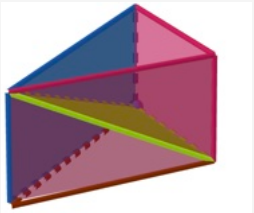
Quiandu



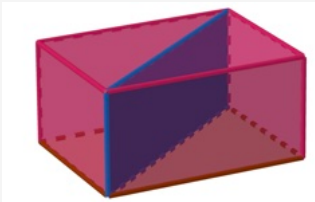


=Quiandu

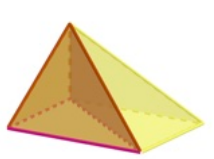
Quiandu = Yangma+ Bienao



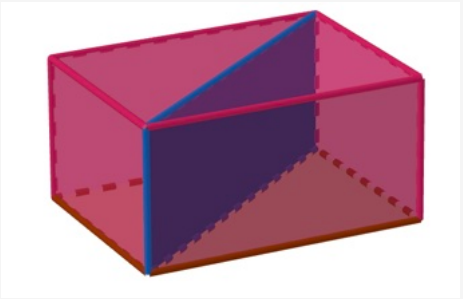
Rectangular Prism = 2Quiandu



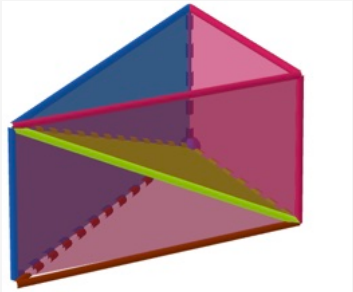
Yangma=2Bienao



Rectangular Prism = 2Quiandu

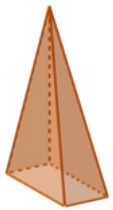


Quiandu = Yangma+ Bienao



Liu Hui on the volume of yangma (rectangular pyramid in ancient Chinese)

A **yangma (阳马)** is a pyramid with rectangular base and one of its lateral edges perpendicular to the base

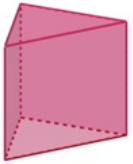


Rectangular base

A **bienao** is a pyramid with base a right triangle, one of its lateral edges perpendicular to the base and the apex over a one of the not right angles vertices.

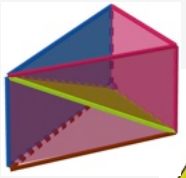


A **qiandu** is a a right triangular prism.

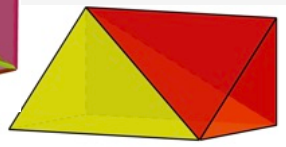


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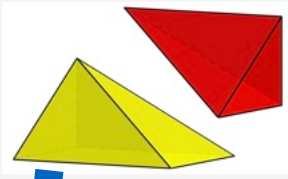
Quiandu = Yangma+ Bienao



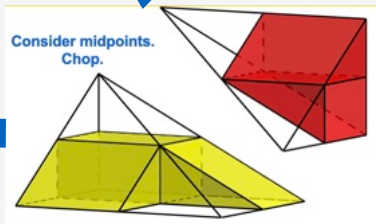
Quiandu = Yangma+ Bienao



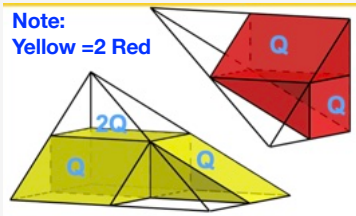
Yangma+ Bienao



Consider midpoints. Chop.



Note:
Yellow =2 Red

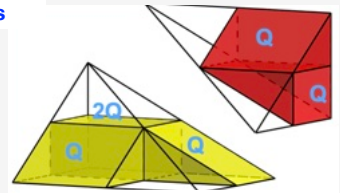


Remaining:
Yellow -> Two smaller yagnmas
Red-> Two smaller bienaos



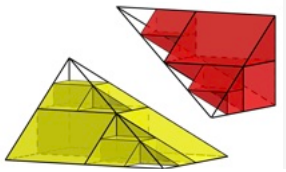
Remaining:
Yellow -> smaller yagnmas
Red-> smaller bienaos

Yellow Vol =
2 Red Vol

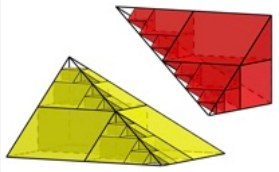


In color:6 quiandus halved.
No color: 2 quiandus halved.

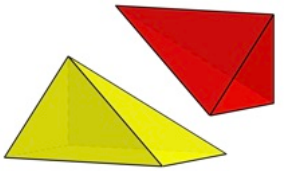
Repeat

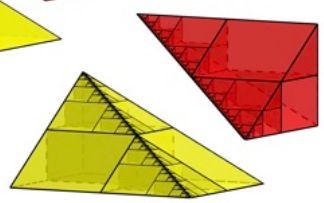


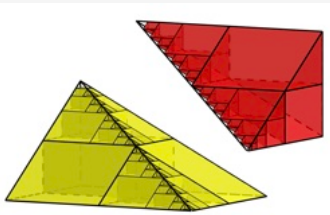
In color: 12 Q halved+6 Q quartered
No color: 4 Q quartered.



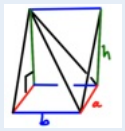
Yangma+ Bienao







Liu Hui on the volume of yangma (rectangular pyramid)



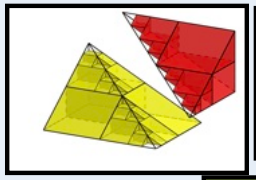
Liu Hui showed that the volume of a yangma is **(1/3)a.b.h** , where **a** and **b** are the sides of the rectangular base and **h** is the height.

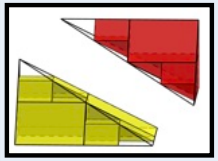
This proof is from ~250 CE.What does this proof suggest to you about early Chinese mathematical thinking?

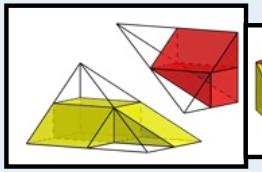
https://www.geogebra.org/m/vwvuznur

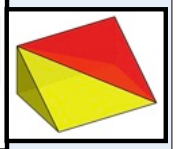


Start: Red Volume =(1/2)Yellow volume. The "leftover" (with no color) gets smaller and smaller, so red and yellow fill more and more.
Limit: Red Volume =(1/2)Yellow volume.
Therefore, Red=(1/3) of qiandu and Yellow=(2/3) of qiandu





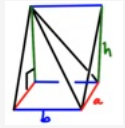




Vol(Quiandu)=(1/2)Vol(Rect. prism)=(1/2)a.b.h=(3/2)Vol(Yangma)
Then **Vol(Yangma)=(1/3)a.b.h**

http://donwagner.dk/Pyramid/Pyramid-4.html

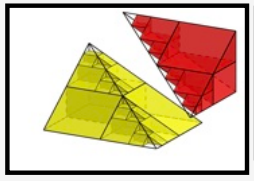
Liu Hui on the volume of yangma (rectangular pyramid)

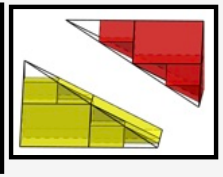


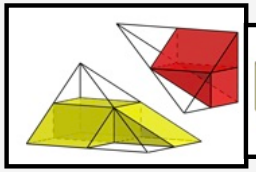


Note: Liu Hui has to consider "infinitely many" pieces to prove his result. Could he had done it with finitely many?

Start: Red Volume =(1/2)Yellow volume. The "leftover" (with no color) gets smaller and smaller, so red and yellow fill more and more.
Limit: Red Volume =(1/2)Yellow volume.
Therefore, Red=(1/3) of qiandu and Yellow=(2/3) of qiandu









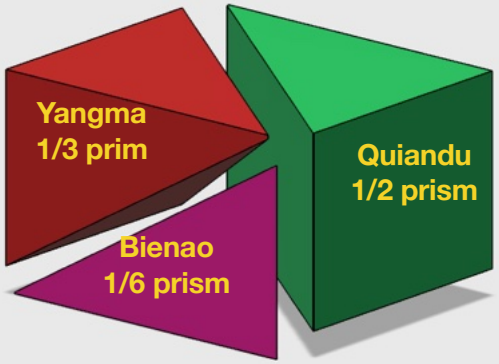
http://donwagner.dk/Pyramid/Pyramid-4.html

Liu Hui on the volume of yangma (rectangular pyramid)

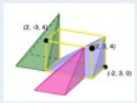


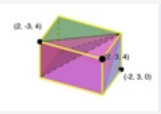
http://donwagner.dk/Pyramid/Pyramid-4.html

Red=(1/3) of qiandu and
Yangma=(2/3) of qiandu



These three pyramids have the same volume





Scissor CongruenceTheorem (Wallace–Bolyai–Gerwien): If two polygons have the same area, then it is possible to cut one of them into polygonal pieces and rearrange these pieces to form the other.

(Note: The number of pieces is finite here)



Two shapes of a given kind are ***scissor congruent*** if one can be cut into finitely many pieces of that kind, which can then be rearranged to form the other.



The equality of two volumes of two tetrahedra of equal bases and equal altitudes

- In two letters to Gerling, Gauss expresses his regret that certain theorems of solid geometry depend upon the method of exhaustion, i. e., in modern phraseology, upon the axiom of continuity (or upon the axiom of Archimedes).
- **Gauss** mentions in particular the theorem of **Euclid**, that **triangular pyramids of equal altitudes are to each other as their bases**.
- Now the analogous problem in the plane has been solved.
- Gerling also succeeded in proving the equality of volume of symmetrical polyhedra by dividing them into congruent parts.
- Nevertheless, **it seems to me probable that a general proof of this kind for the theorem of Euclid just mentioned is impossible, and it should be our task to give a rigorous proof of its impossibility.**
- **This would be obtained, as soon as we succeeded in specifying two tetrahedra of equal bases and equal altitudes which can in no way be split up into congruent tetrahedra, and which cannot be combined with congruent tetrahedra to form two polyhedra which themselves could be split up into congruent tetrahedra.**



- **Max Dehn (1900):** He found an invariant built from the edge lengths and dihedral angles, now called the **Dehn invariant**, that cutting and rearranging never changes.
  - Cube: 0. Regular tetrahedron: not 0.
  - So a regular tetrahedron cannot be cut up and rearranged into a cube of the same volume.
- **Consequence:** the volume of a pyramid cannot be obtained by finite cut-and-paste alone. An infinite process (Euclid XII, Liu Hui) or calculus is needed.

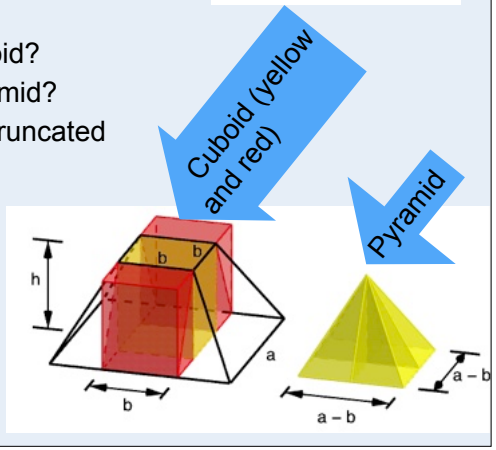
Liu Hui on the volume of the truncated pyramid

Volume of a truncated pyramid

- Consider a truncated pyramid of square base as in the figure.
- The Geogebra app shows how to cut this solid and rearranged into two pieces, a pyramid and a cuboid.
- What is volume of the cuboid?
- What is volume of the pyramid?
- What is the volume of the truncated pyramid?



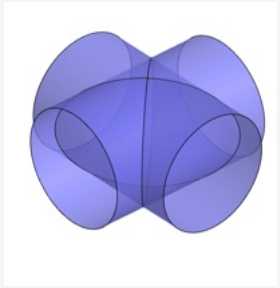
https://www.geogebra.org/m/azanhyje



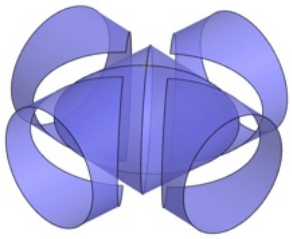
Liu Hui on the volume of the sphere

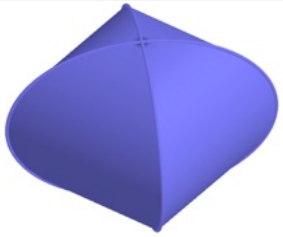
Liu Hui on the Volume of a Sphere

Box Lid - Bicylinder - Mouhefanggai - 牟合方蓋



https://www.geogebra.org/m/dYQHpXQY





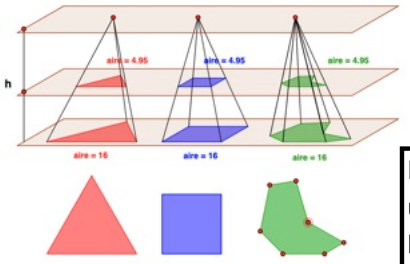
Liu -200 CE- discovered the Mouhefanggai (牟合方蓋面面觀) -the solid formed by two square umbrellas

Zu Geng's (~450 CE) statement

疊棋成立積
緣冪勢既同
則積不容異

If blocks are piled up to form volumes,
And corresponding areas are equal,
Then the volumes cannot be unequal

Source: http://donwagner.dk/SPHERE/SPHERE.html



This is called now the **"Cavalieri principle"** ~1600

Is it fair?

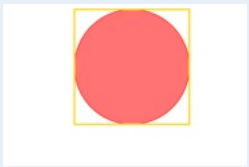


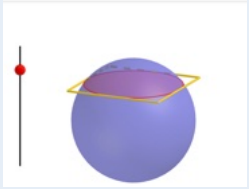
Trattato della sfera e prattiche per uso di essa, Roma, 1682

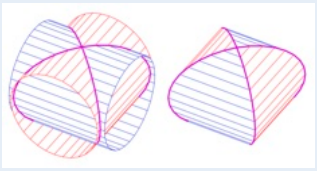
Liu Hui (~250 CE) used this principle before Zu Geng's statement

Recalll Archimedes!

M = volume of the Mouhefanggai 牟合方蓋
S =volume of the sphere (S)
Find M/S (Hint: Cavalieri Principle)







By Ag2gaeh - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=63549042

Yellow -> Mouhefanggai (box lid)

https://www.geogebra.org/m/pfj8sd75

M = volume of the Mouhefanggai 牟合方蓋
S = volume of the sphere
Find M/S

By "sections" principle (or Cavalieri)





Since the ratio of all slices is 4/π, the ratio of the volume is 4/π. That is,

M/S = 4/π

https://www.geogebra.org/m/pfj8sd75

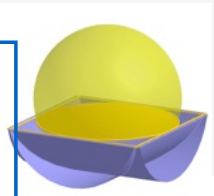
Yellow -> Mouhefanggai (box lid)

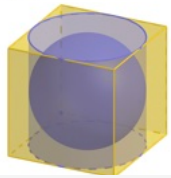
Summary: Liu Hui and the volume of the sphere

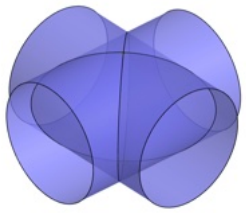
Liu -200 CE- showed that the relation

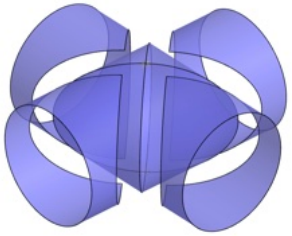
Vol(cylinder)/Vol(sphere)= 4/π

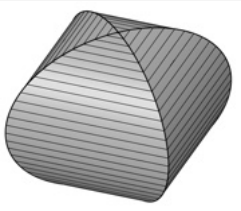
does not hold. The Mouhefanggai "squeezes" the sphere more than the cylinder.











Math Poem

Liu Hui concludes with the following
The geometer's frustration by Liu Hui

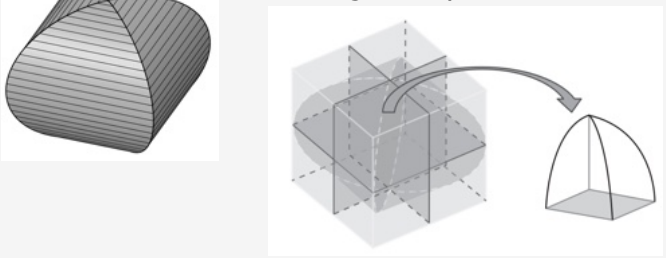
Look inside the cube
And outside the box-lid;
Though the diminution increases,
It doesn't quite fit.
The marriage preparations are complete;
But square and circle wrangle,
Thick and thin make treacherous plots,
They are incompatible.
I wish to give my humble reflections,
But fear that I will miss the correct principle;
I dare to let the doubtful points stand,
Waiting for one who can expound them.

Source: http://donwagner.dk/SPHERE/SPHERE.html

Zu Gengzhi (Zu Geng, 5th c.).on the Volume of a Sphere

Zu Gengzhi on the Volume of a Sphere

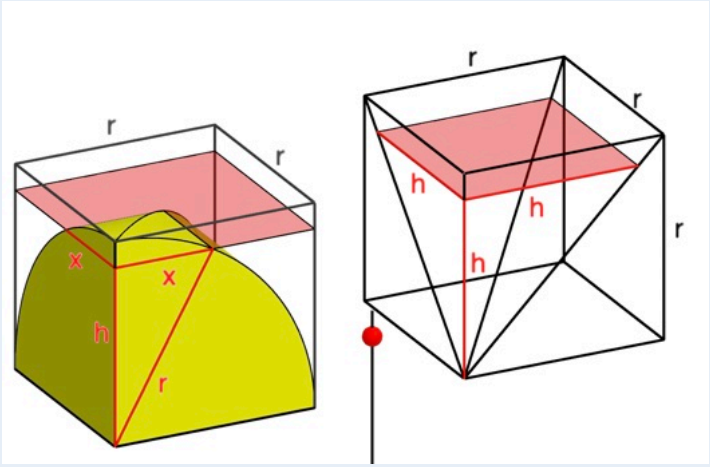
Divide M(r) into eight congruent pieces.



Each of the eight blocks is “like” a rounded *yangma* and “fits” in C(r).

https://www.geogebra.org/m/bprtgwh8#material/gevzbyfd
https://www.geogebra.org/m/bprtgwh8

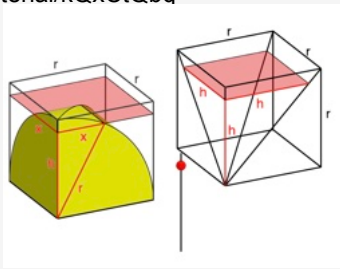
Compute the areas of the two red shapes in terms of r and h.



https://www.geogebra.org/m/ygrrfnsw

Zu Geng on the Volume of a Sphere

https://www.geogebra.org/m/nkdqansv#material/kQxCtQbq



Cube(r) denotes cube of side r

Pyr(r) denotes slanted pyramid in cube of side r.

Sph(r) sphere of radius r

M(r) Mouhefangai around S(r)

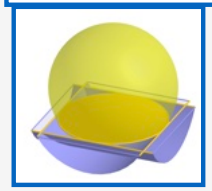
Volume(M(r))/8 = Vol(Cube(r)) - Vol(Pyr(r)) = (2/3)Vol(Cube(r))

Volume(M(r)) = = (16/3)Vol(C(r))

Zu Geng on the Volume of a Sphere

C(r) = cube of side r
P(r) = slanted pyramid in cube of side r.
S(r) = sphere of radius r
M(r)= Mouhefangai enclosing S(r)

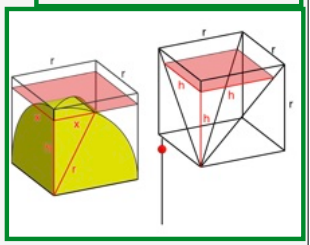
$$\frac{\text{Volume}(S(r))}{\text{Volume}(M(r))} = \pi/4$$



Liu Hui

Volume(S(r)) =
(π/4) Volume(M(r))=
= (π/4) (16/3)Vol(C(r)) =
(4/3)π r^3

Volume(M(r)) =
(16/3)Vol(C(r))



Math Poems

Liu Hui concludes with the following
The geometer's frustration
by Liu Hui

Look inside the cube
And outside the box-lid;
Though the diminution increases,
It doesn't quite fit.
The marriage preparations are complete;
But square and circle wrangle,
Thick and thin make treacherous plots,
They are incompatible.
I wish to give my humble reflections,
But fear that I will miss the correct principle;
I dare to let the doubtful points stand,
Waiting for one who can expound them.

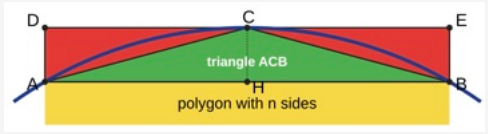
The geometers triumph
by Zu Gengzhi

The proportions are extremely precise
And my heart shines.
Zhang Heng copied the ancient,
Smiling on posterity;
Liu Hui followed the ancient,
Having no time to revise it.
Now what is so difficult about it?
One need only think.

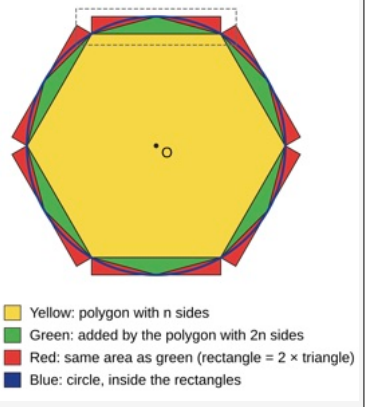
Source: http://donwagner.dk/SPHERE/SPHERE.html

Computation of π by Liu Hui and Zu Chongzhi

Liu Hui approximation of π (3rd century CE)



yellow + green < circle < yellow + green + red
2n-gon < circle < 2n-gon + (2n-gon − n-gon)



- Inscribed polygons
- **Doubling the sides** from 6 to 192.
- **Gougu** (Pythagorean) gives new sides.
- Considered 96-gon, 2x96-gon=192-gon
- **Areas**: 192-gon < circle < 192-gon + (192-gon − 96-gon)
- By the Nine Chapters rule, area = ½ circumference × ½ diameter.
- Result: **diameter : circumference = 50 : 157 (≈ 3.14).**

Zu Chongzhi approximation of π (5th century CE)

- Bounds: 3.1415926 < π < 3.1415927.
- Method unknown (his book is lost).
- Gave remarkably accurate approximation: 355/113.
- With Liu Hui's method, his bounds would need polygons of 12,288 and 24,576 sides ($3 \cdot 2^{12}$ and $3 \cdot 2^{13}$).

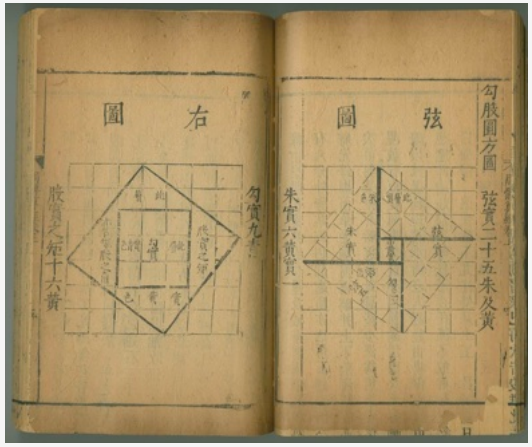


Later edition of the Nine Chapters, with a figure added by an editor ("corrections and supplementary figures," in the left margin). The note at right names "Mr. Zu's approximate ratio: diameter 7, circumference 22." Image: MAA Convergence, Mathematical Treasures: Jiuzhang suanshu.

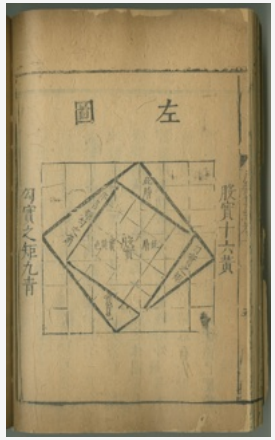
Gougu theorem

Pythagorean or Gou-Gu Theorem in

Zhoubi suanjing - Arithmetical Classic of the Gnomon and the Circular Paths of Heaven - Book on astronomy and mathematics ~ 100 BCE.



https://maa.org/press/periodicals/convergence/mathematical-treasures-zhoubi-suanjing



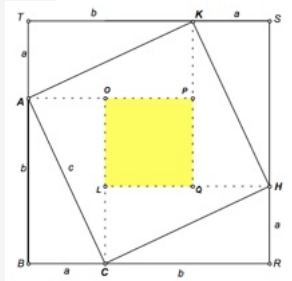
https://maa.org/press/periodicals/convergence/mathematical-treasures-zhoubi-suanjing

Pythagorean or *Gou-Gu* Theorem in the Zhoubi suanjing (The arithmetic classic of the gnomon and the circular paths of heaven

Let us cut a rectangle (diagonally), and make the width 3 (units) wide, and the length 4 (units) long. The diagonal between the (two) corners will then be 5 (units) long. Now, after drawing a square on this diagonal, circumscribe it by half-rectangles like that which has been left outside, so as to form a (square) plate. Thus the (four) outer half-rectangles, of width 3, length 4 and diagonal 5, together make two rectangles (of area 24); then (when this is subtracted from the square plate of area 49) the remainder is of area 25. This (process) is called "piling up the rectangles." Ji ju

1st century BC

(translation by Needham,1959)



https://cd1.edb.hkedcity.net/cd/maths/en/ref_res/material/MSS_e/Exemp21.pdf

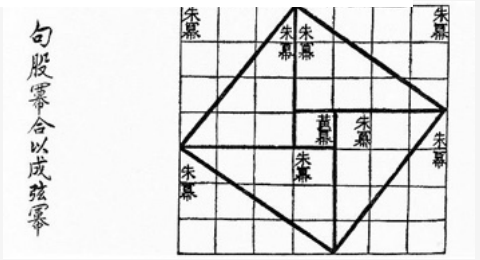


Diagram added by Zhao Shuang to the Zhoubi Suanjing that can be used to prove the Pythagorean Theorem

https://en.wikipedia.org/wiki/Zhoubi_Suanjing#Background_behind_Pythagorean_derivation

Chapter 9 of the Nine Chapters: Gou-gu 勾股

Let the square on the gou be red in color, the square on the gu be blue. Let the deficit and excess parts be mutually substituted into corresponding positions, the other parts remain unchanged. They are combined to form the square on the hypotenuse. Extract the square root to obtain the hypotenuse."

The Mathematics of Egypt, Mesopotamia, China, India, and Islam, p. 286.

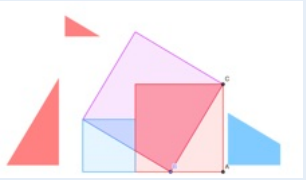
勾自乘為朱方，股自乘為青方，令出入相補，各從其類，因就其餘不移動也，合成弦方之冪。

- States that on a right triangle, the square on the hypotenuse equals the sum of the squares on the other two sides (the Pythagorean theorem).
- Gives many practical surveying problems using that rule.
- Liu Hui gives a **verbal dissection** argument, explaining how rearranging areas demonstrates the equality of the squares.



Modern reconstruction of Liu Hui's dissection argument.

Note: The diagram for Liu Hui's proof is lost.



https://www.geogebra.org/m/zzkst6ep

https://www.geogebra.org/m/zzkst6ep

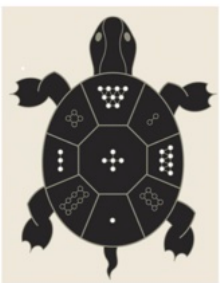
The pieces of the two small squares that stick out of the big square fill exactly the parts of the big square they don't cover.

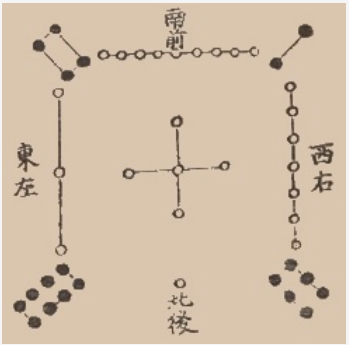
1. Why does rearranging these pieces show that the big square has the same area as the two small squares together?

2. What must we know to conclude that this is a proof for all right triangles?

Magic Squares

The Mythological Emperor Yu, (~2000BCE) received a divine gift from a Lo river tortoise. The gift, a diagram called the Lo shu, is believed to contain the principles of Chinese Mathematics.





4	9	2
3	5	7
8	1	6

Lo Shu from "The Astronomical Phenomena" (*Tien Yuan Fa Wei*). Compiled by Bao Yunlong in 13th century, published during the Ming dynasty, 1457–1463.

Can you find a pattern?

These arrangements of numbers are magic squares.

Write down the the definition of magic square

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

4	14	15	1
9	7	6	12
5	11	10	8
16	2	3	13

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

2	9	4
7	5	3
6	1	8

A ***magic square*** is an nxn matrix, such that
1. The entries are the numbers 1,2, …, n^2.
2. The sum of columns, rows and diagonals is constant.

Write down the magic constant of an n x n magic square in terms of n.

2	9	4	
7	5	3	
6	1	8	

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

4	14	15	1
9	7	6	12
5	11	10	8
16	2	3	13

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

The sum of the rows (or columns, or diagonals) of a magic square is called the ***magic constant***.

8	1	6	
3	5	7	
4	9	2	

4	3	8	
9	5	1	
2	7	6	

2	9	4	
7	5	3	
6	1	8	

6	7	2	
1	5	9	
8	3	4	

6	1	8	
7	5	3	
2	9	4	

2	7	6	
9	5	1	
4	3	8	

4	9	2	
3	5	7	
8	1	6	

8	3	4	
1	5	9	
6	7	2	

(2 minutes) These eight magic squares are *equivalent* in some sense. Can you explain how are they equivalent?

4	14	15	1
9	7	6	12
5	11	10	8
16	2	3	13

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Note: These two magic squares are *not* equivalent

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

not **equivalent**

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

not **equivalent**

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

2	9	4
7	5	3
6	1	8

3x3 all **equivalent**

Order	Number of magic squares
1	1
2	0
3	1
4	880
5	275305224
6	?

It is an unsolved problem to determine the number of magic squares of an arbitrary order, but the number of distinct magic squares (excluding those obtained by successive rotations and reflections)

Order	Number of magic squares
1	1
2	0
3	1
4	880
5	275305224
6	?

~1.775399 ·10^19

If you want to keep reading about magic squares

http://www.mathematische-basteleien.de/magsquare.htm



A page displaying 9×9 magic square from Cheng Dawei's Suanfa tongzong (1593)



Melencolia I

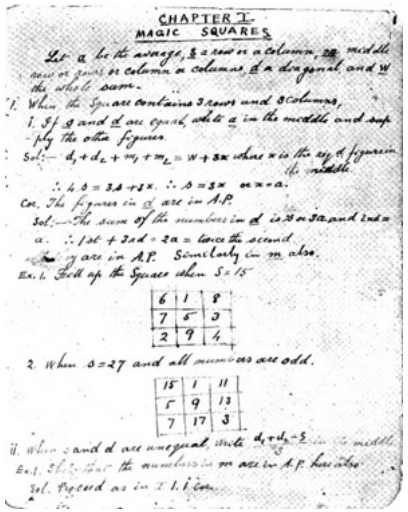
1514

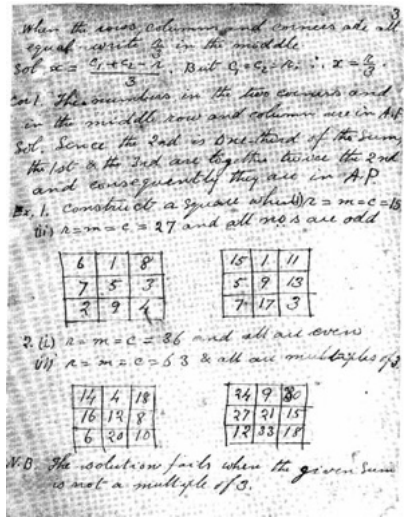
Albrecht Dürer German



https://www.metmuseum.org/en/art/collection/search/336228

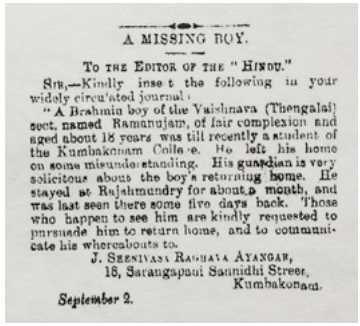
Two Pages from Ramanujan's Notebooks





https://www.imsc.res.in/~rao/ramanujan/introindex.html

By the time Ramanujan got to college, all he wanted to do was mathematics—and he failed his other classes, and at one point ran away, causing his mother to send a missing-person letter to the newspaper:



A MISSING BOY.

To the Editor of the "Hindu."

Sir,—Kindly insert the following in your widely circulated journal:

"A Brahmin boy of the Vaishnava (Thengalai) sect, named Ramanujam, of fair complexion and aged about 18 years was till recently a student of the Kumbakonam College. He left his home on some misunderstanding. His guardian is very solicitous about the boy's returning home. He stayed at Rajahmundry for about a month, and was last seen there some five days back. Those who happen to see him are kindly requested to persuade him to return home, and to communicate his whereabouts to.

J. Srinivasa Raghava Ayangar,
15, Sarangapani Sannidhi Street,
Kumbakonam.

September 2.

https://writings.stephenwolfram.com/2016/04/who-was-ramanujan/



What is the license plate of Hardy's cab? Why does Ramanujan think this ins an interesting number?

https://youtu.be/YykgE_z4-38?feature=shared&t=137

A recipe for generating magic squares of odd size:

Step 1: Write down 1 in the middle of the top row.
Step k+1: Write down the number k+1 on the closest entry to the right and above the entry k, "pac-man" maner. If that entry is already filled, write down k+1 directly below k.

			2	

		1	8	
	5	7		
4	6			
				3
			2	

A recipe for generating magic squares of odd size:

Step 1: Write down 1 in the middle of the top row.
Step k+1: Write down the number k+1 on the closest entry to the right and above the entry k, "pac-man" manner. If that entry is already filled, write down k+1 directly below k.

Homework problem 2: Construct a magic square of order 7. Find its magic number.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Extra Credit Homework problem 2: Find a "your own" magic square of size 4x4 or larger (different from the ones we saw in class, and not constructed with the above algorithm

Explain why there all 3x3 magic squares are equivalent

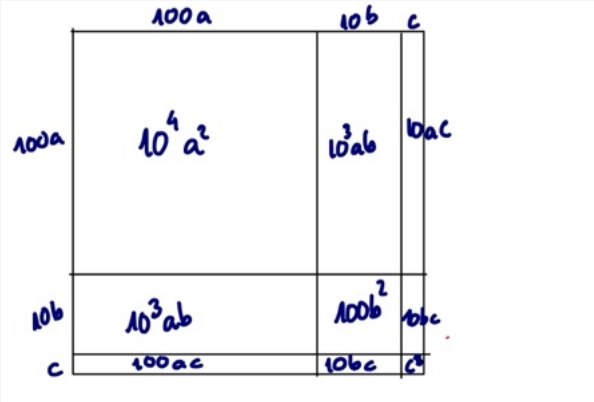
A recipe for generating magic squares of odd size:

Step 1: Write down 1 in the middle of the top row.
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17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

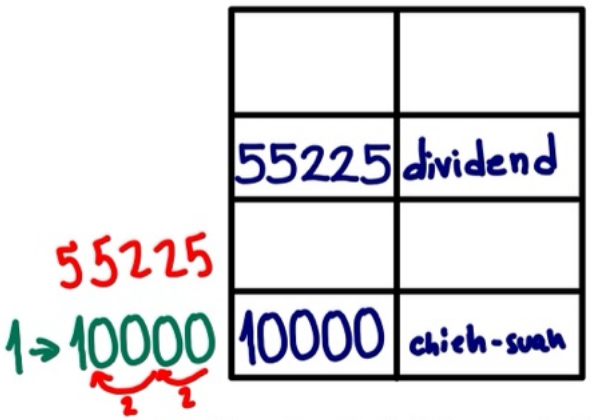
Square root computations

What is the square root of 65,536?



step 3 步[1] 之, 超一等.

This one counting-rod is moved forward (from right to left) by steps of two places each (as far as it can go without transgressing the furthest left digit of the dividend) [This one counting-rod, with its new place-value, is to be called the *Chieh-suan* 借算].



Move this counting-rod to the left by 2 steps of 2 places

Medieval China

Chinese reminder theorem

Example: A number when divided by 3 gives a remainder of 2, when divided by 5 gives a remainder of 1, and when divided by 7 gives a remainder of 6. What is the number

- Sunzi in late 4th-early 5th centuries.
- Qin Jiushao in 13th century, (also gave a general method)
- By the 13th century, this problem appeared in Islamic literature
- This problem also appeared in the work of Leonardo of Pisa (Fibonacci), i the 11th century. Fibonacci states Sunzi's problem and tackles it the same way, which suggests that the problem had travelled.

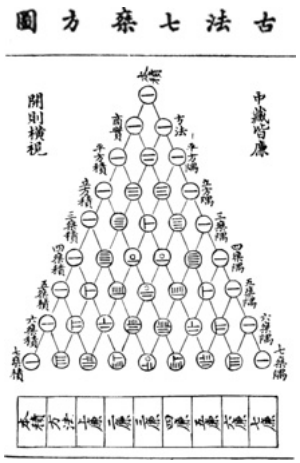
Sunzi rule: Suppose we have an unknown number of objects. If they are counted in threes, 2 are left, if they are counted in fives, 3 are left and if they are counted in sevens, 2 are left. How many objects are there?

Rule: If they are counted in threes, 2 are left: set 140. If they are counted in fives, 3 are left: set 63. If they are counted in sevens, 2 are left: set 30. Take the sum of these [three numbers] to obtain 233. Subtract 210 from this total; this gives the answer.

In general: For each remaining unit when counting in threes, set 70. For each remaining unit when counting in fives, set 21. For each remaining unit when counting in sevens, set 15. If [the sum obtained in this way] is 106 or more, subtract 105 to obtain the answer.

Yang Hui triangle (Pascal's triangle) using rod numerals, as depicted in a publication of Zhu Shijie

1200 CE



The Italian Jesuit Matteo Ricci (left) and the Chinese mathematician Xu Guangqi (right) published the Chinese edition of Euclid's Elements (幾何原本) in 1607. (Wikipedia)





Timeline, summary

