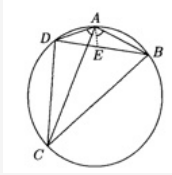
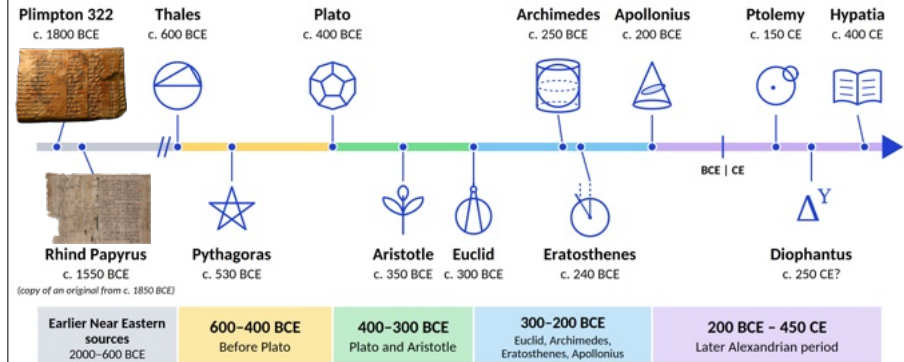


## MAT 336 Hellenic Mathematics After Euclid



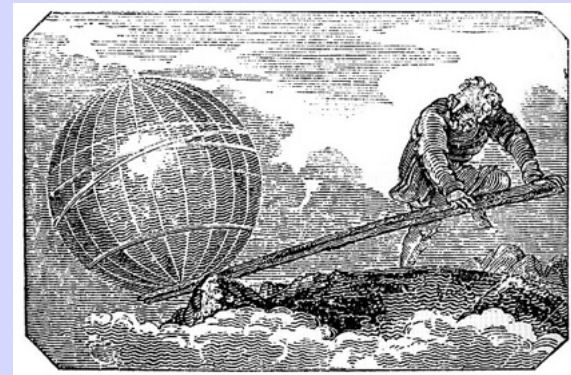
- Archimedes
- Archimedes and  $\pi$
- Archimedes: Area of the circle and a right triangle
- Archimedes and large numbers
- The Archimedes Palimpsest
- Archimedes' favorite result
- Mean proportionals, conic sections, and Apollonius
- Archimedes and the quadrature of the parabola
- Archimedes and mirrors (with conic sections)
- Stories about Archimedes
- Archimedes and the law of the lever
- Archimedes' spiral
- Apollonius
- Eratosthenes, the librarian of Alexandria
- Claudius Ptolemy
- Diophantus and the birth of literal algebra

## A Timeline of Greek Mathematics



## Archimedes

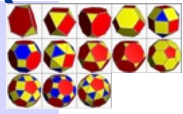
## Archimedes (~250 BCE)



**GIVE ME A PLACE TO STAND  
AND I WILL MOVE THE EARTH**

## Archimedes (~250BCE)

Derived and computed approximations to:



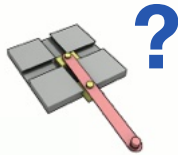
- Area of the circle
- Surface area and volume of the sphere
- Area of an ellipse.
- Area under a parabola

Ideas of calculus

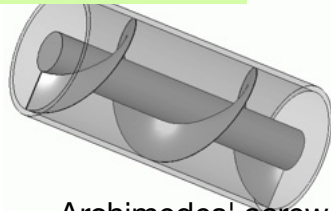
- Infinitesimals
- Method of exhaustion

Dealt with infinity!!!

- Approximation of  $\pi$
- Created a number system to deal with **arbitrary** large numbers

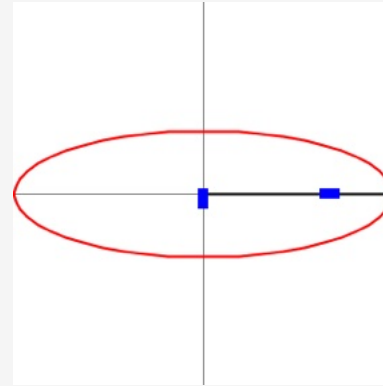


- Applied mathematics to physics
- Explanation of principle of the lever
- Innovative machines

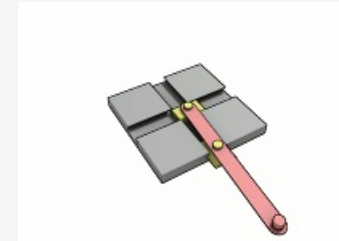


Archimedes' screw ?

## Trammel of Archimedes



Animation of the trammel of Archimedes by Alastair Rae for Wikimedia Commons



## Archimedes' screw



Stunteltje, (looped gif by Jahobv)

Archimedes  
and  $\pi$

## Archimedes' Bounds for $\pi$ (radius = 1)

**Idea:** Computed the perimeter of inscribed and circumscribed regular n-gons

$$n = 6 \rightarrow 12 \rightarrow 24 \rightarrow 48 \rightarrow 96$$

Each doubling narrows the bounds for  $\pi$

Used 96-sided polygons for his final bounds.

Archimedes Proposition: The **ratio of the circumference of any circle to its diameter** as less than  $3 \frac{1}{7}$  but greater than  $3 \frac{10}{71}$ .

In modern symbols

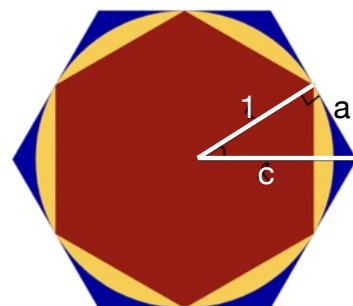
$$3 \frac{10}{71} < \frac{\text{circumference}}{\text{diameter}} < 3 \frac{1}{7}$$

## Archimedes' Method for Approximating $\pi$

A circle has **radius 1**.

Two regular hexagons are drawn:

- **Inscribed hexagon:** vertices on the circle
- **Circumscribed hexagon:** sides tangent to the circle



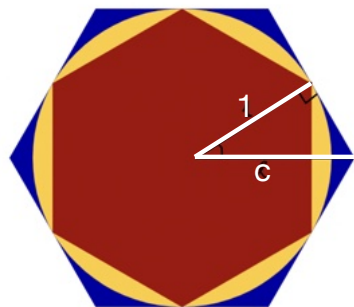
1. What is the perimeter of the inscribed hexagon?
2. What is the perimeter of the circumscribed hexagon? (Hint: What is the relation between a and c?)  
Compute without using trigonometry, (as Archimedes did.)

## Archimedes' Method for Approximating $\pi$

A circle has **radius 1**.

Two regular hexagons are drawn:

- **Inscribed hexagon:** vertices on the circle
- **Circumscribed hexagon:** sides tangent to the circle



1. What is the perimeter of the inscribed hexagon?
2. What is the perimeter of the circumscribed hexagon? (Hint: study the angles in the three triangles in the figure below)

$$\begin{aligned} (c/2)^2 + 1^2 &= c^2 \\ c^2 &= 4/3 \end{aligned}$$

## Archimedes' Bounds for $\pi$ (radius = 1)

**Idea:** Computed the perimeter of inscribed and circumscribed regular n-gons

$$n = 6 \rightarrow 12 \rightarrow 24 \rightarrow 48 \rightarrow 96$$

Each doubling narrows the bounds for  $\pi$

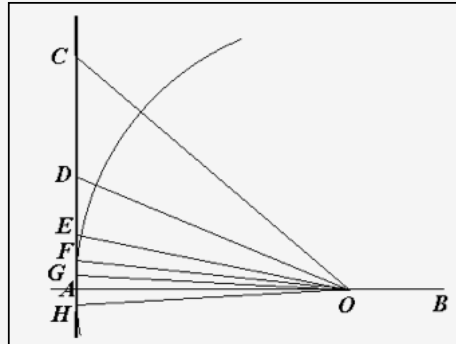
Used 96-sided polygons for his final bounds.

Archimedes Proposition: The **ratio of the circumference of any circle to its diameter** as less than  $3 \frac{1}{7}$  but greater than  $3 \frac{10}{71}$ .

In modern symbols

$$3 \frac{10}{71} < \frac{\text{circumference}}{\text{diameter}} < 3 \frac{1}{7}$$

## Archimedes approximation of $\pi$



O – center of the circle

A, B – ends of the diameter (AB passes through O)

AC – tangent at A

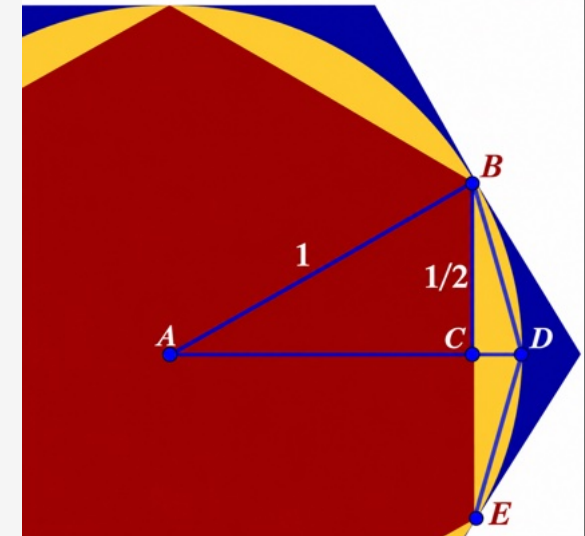
C – point on the tangent where  $\angle AOC = 30^\circ$  (circumscribed hexagon)

D, E, F, G – successive intersections of the tangent AC with lines from O that bisect the previous central angles (12-, 24-, 48-, 96-gons), each nearer A.

## Archimedes ~250BCE and the approximation of $\pi$



<https://www.geogebra.org/m/ajamkye2>



**Educated guess:**  
**Why do Archimedes**  
**stop at 96 sides?**  
**Why not 192 or 384?**

## Archimedes approximation of $\pi$ Recall the alphabetic number system

| units     | $\alpha$ | $\beta$  | $\gamma$  | $\delta$   | $\epsilon$ | $\varsigma$ | $\zeta$    | $\eta$   | $\theta$ |
|-----------|----------|----------|-----------|------------|------------|-------------|------------|----------|----------|
|           | 1        | 2        | 3         | 4          | 5          | 6           | 7          | 8        | 9        |
| tens      | $\iota$  | $\kappa$ | $\lambda$ | $\mu$      | $\nu$      | $\xi$       | $\omicron$ | $\pi$    | $\chi$   |
|           | 10       | 20       | 30        | 40         | 50         | 60          | 70         | 80       | 90       |
| hundreds  | $\rho$   | $\sigma$ | $\tau$    | $\upsilon$ | $\phi$     | $\chi$      | $\psi$     | $\omega$ | $\eta$   |
|           | 100      | 200      | 300       | 400        | 500        | 600         | 700        | 800      | 900      |
| thousands | $\alpha$ | $\beta$  | $\gamma$  | $\delta$   | $\epsilon$ | $\varsigma$ | $\zeta$    | $\eta$   | $\theta$ |
|           | 1000     | 2000     | 3000      | 4000       | 5000       | 6000        | 7000       | 8000     | 9000     |

Source: Wikimedia

## Archimedes

- omits actual calculations and only gives results.
- gives two fractional approximations to  $\sqrt{3}$  and square roots of other whole numbers

See Heath, The works of Archimedes <https://archive.org/details/worksofarchimede00000arch/>

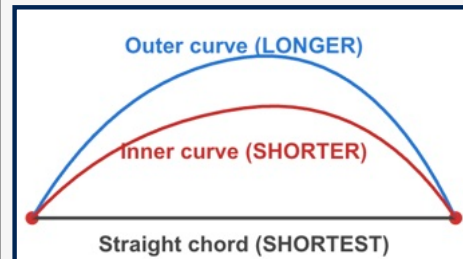
Archimedes Proposition: The **ratio of the circumference of any circle to its diameter** as less than  $3 \frac{1}{7}$  but greater than  $3 \frac{10}{71}$ .

## What did Archimedes mean by the *length of a curve*?

**Archimedes Length Ordering axioms (in modern language).**

**Axiom 1:** Of all lines which have the same extremities, the straight line is the least.

**Axiom 2:** Of other lines in a plane with the same endpoints and concave in the same direction, that which is **closer to the straight line** (or lies entirely between **the other line** and the **straight line**) is the shorter.

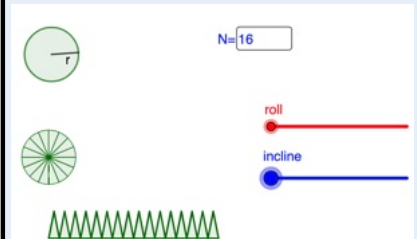


Archimedes treated the **length of a curve** as a **magnitude**—something you can compare (greater/equal/less) but not compute exactly.

## Archimedes, area of circle and (certain) right triangle

<https://www.geogebra.org/m/fg9d4ueb>

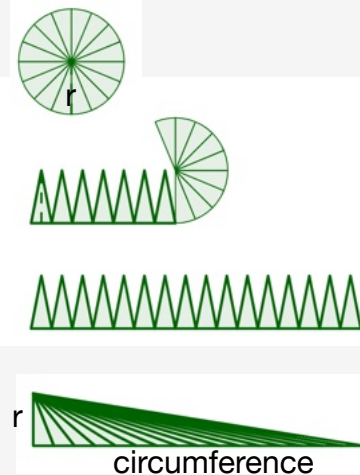
1. As the number of sectors grows, what are the lengths of the legs of the limiting right triangle?
2. Based on the Geogebra app, what relationship do you expect between the area of the original circle and the area of this limiting right triangle?
3. Is the app a proof that the circle and the triangle have the same area? If not, what is missing?



## Archimedes (~250BCE) on the Measurement of a Circle

For I realized that just as every **circle equals a triangle** having as its base the **circumference** of the circle and **altitude equal** to the [distance] from the center to the circle [that is, the **radius**]...

This shows that the constants for computing circumference and area of the circle are the same,  $\pi$ .



## Archimedes' favorite result

1. A plane cuts a sphere of radius  $r$  at height  $h$  above its center, forming a circle. Find its area.
2. The same plane cuts a cylinder with a cone removed, forming a ring. Find its area.
3. Compare the two areas. What can you conclude about the volumes of the hemisphere and the cylinder with the cone removed?

$C$  = volume of cylinder (height  $r$ , base radius  $r$ )

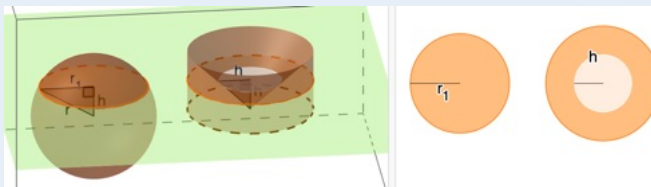
$S$  = volume of sphere (radius  $r$ )

$O$  = volume of cone (height  $r$ , radius base  $r$ ) =  $C/3$



<https://www.geogebra.org/m/faxwvhsg>

**Archimedes (~250BCE)**



## Archimedes (~250BCE)



Hemisphere =  $C - O = 2C/3$

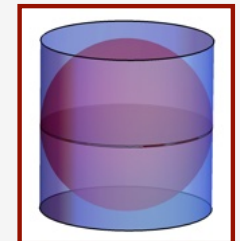
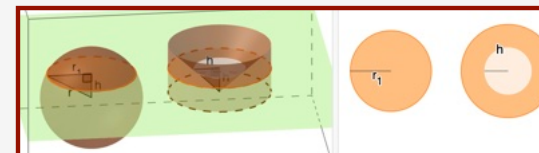
Doubling:

**Sphere =  $2/3$ (Cylinder circumscribing sphere)**

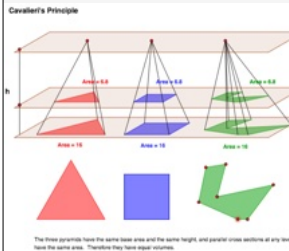
$C$  = volume of cylinder (height  $r$ , base radius  $r$ )

$S$  = volume of sphere (radius  $r$ )

$O$  = volume of cone (height  $r$ , radius base  $r$ ) =  $C/3$

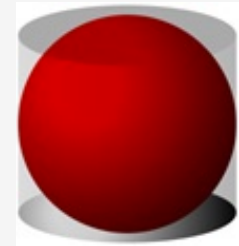


**Cavalieri's principle**  
(although was known by  
Chinese mathematicians  
before Cavalieri stated it)



**Two solids such that  
the section of one by  
each horizontal plane  
bears a fixed ratio to  
the section of the  
other by the same  
plane have volumes  
in that same ratio.**

*And although he made many excellent discoveries, he is said to have asked his kinsmen and friends to place over the grave where he should be buried a cylinder enclosing a sphere, with an inscription giving the proportion by which the containing solid exceeds the contained.*



Plutarch (AD 45-120), Parallel Lives: Marcellus



A tomb in Syracuse in the Necropolis of Grotticelli referred to affectionately (or deceptively) as "[Archimedes' Tomb](https://www.math.nyu.edu/~corres/Archimedes/Tomb/TombIllus.html)", but known to be of Roman origin dating at least two centuries after the death of Archimedes.

# Archimedes and large numbers

## Archimedes The sand reckoner (~250BCE)

Created a system  
to deal with **very**  
large numbers

$$1,373,943\frac{33}{64} = M^{\rho\lambda\zeta} \gamma\alpha\mu\gamma' \lambda\gamma' \xi\delta''.$$

Archimedes found an upper bound ( $10^{63}$ ) for the number of grains of sand that would fill a sphere whose radius was equal to the distance from the sun to the fixed stars.

To work with very large numbers, he imagined a "doubled class" of numbers of eight numerals (instead of the four the the Greek ciphered system)

| units     | α    | β    | γ    | δ    | ε    | ς    | ζ    | η    | θ    |
|-----------|------|------|------|------|------|------|------|------|------|
|           | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    |
| tens      | ι    | κ    | λ    | μ    | ν    | ξ    | ο    | π    | ρ    |
|           | 10   | 20   | 30   | 40   | 50   | 60   | 70   | 80   | 90   |
| hundreds  | σ    | τ    | υ    | φ    | χ    | ψ    | ω    | η    | θ    |
|           | 100  | 200  | 300  | 400  | 500  | 600  | 700  | 800  | 900  |
| thousands | α    | β    | γ    | δ    | ε    | ς    | ζ    | η    | θ    |
|           | 1000 | 2000 | 3000 | 4000 | 5000 | 6000 | 7000 | 8000 | 9000 |

Wikimedia

*Some of the numbers to which I have given a name [...] surpass not only the number of grains of sand that could fill the Earth [...] but even the number of grains of sand that could fill the universe itself.*

Archimedes, Sand-Reckoner

# The Archimedes' Palimpsest

## The Archimedes' Palimpsest

- **10th century:** Scribe in Constantinople copied Archimedes texts
- **13th century:** Christian monks needed vellum for a prayer book — washed off Archimedes, wrote prayers over it
- Named **Palimpsest** (manuscript with text written over earlier, erased text)
- **Discovered 1906** in a monastery in Constantinople. **reappeared in the 1970's**, (in the hands of a French family that had bought it in Istanbul in the 1920s) **resurfaced** at Christie's auction **1998** (badly damaged)
- Anonymous collector bought for **\$2m** → deposited at **Baltimore's Walters Art Museum**
- **Importance:** Revealed *The Method* — Archimedes' techniques for discovering geometric results using physical reasoning before formal proof.

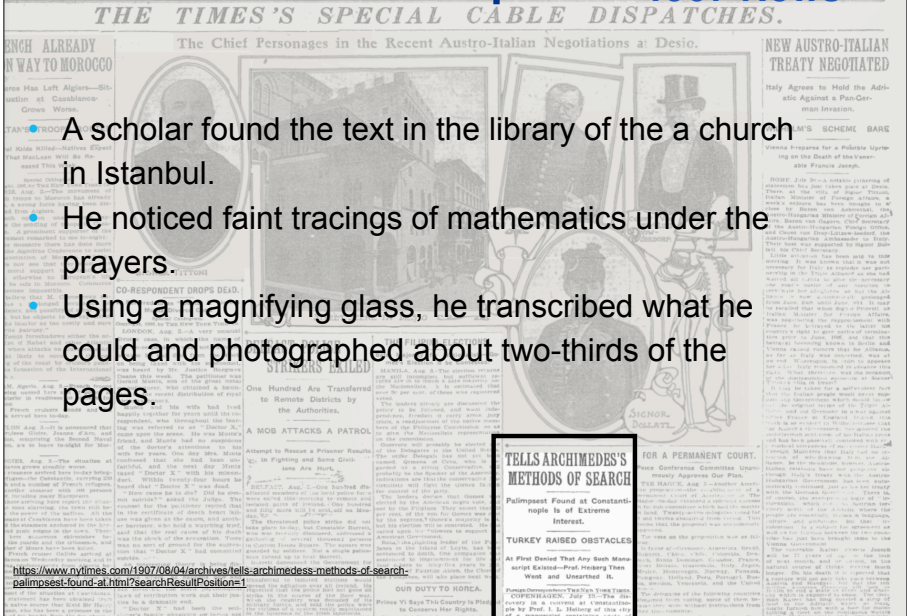
Vellum = writing material made from treated calfskin; expensive and reusable

The Method uses physical reasoning — imagining geometric shapes balanced on a lever and comparing cross-sections — to discover results before proving them rigorously.

## The Archimedes' Palimpsest contains

- The Equilibrium of Planes
- Spiral Lines
- The Measurement of the Circle  $\rightarrow \pi$
- Sphere and Cylinder  $\rightarrow$  Today
- On Floating Bodies  $\rightarrow$  Eureka
- The Method of Mechanical Theorems  $\rightarrow$  Today
- The Stomachion  $\rightarrow$  Puzzle

## The Archimedes' Palimpsest · 1907 News

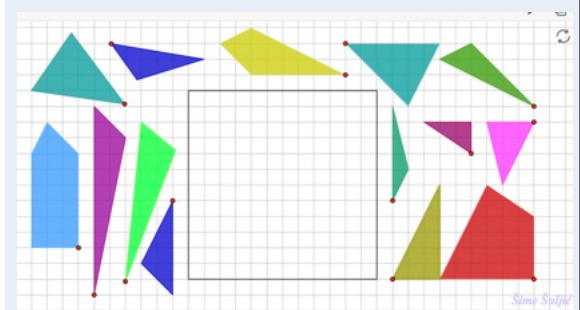


In the introductory letter to *The Method*, written to Eratosthenes, the chief librarian at the Library in Alexandria, Archimedes described his purpose in writing this text:

Since, as I said, I know that you are diligent, an excellent teacher of philosophy, and greatly interested in any mathematical investigations that may come your way, I thought it might be appropriate to write down and set forth for you in this same book a certain special method, by means of which **you will be enabled to recognize certain mathematical questions with the aid of mechanics**. I am convinced that this is no less useful for finding the proofs of these same theorems. For some things, which first became clear to me by the mechanical method, were afterwards proved geometrically, because their investigation by the said method does not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge. . . . I now wish to describe the method in writing, partly because I have already spoken about it before, . . . partly because I am convinced that it will prove very useful for mathematics; in fact, I presume there will be some among the present as well as future generations who by means of the method here explained will be enabled to find other theorems which have not yet fallen to our share.

**Wild guess: How many ways to form a square do you think there are?**

<https://www.geogebra.org/material/copy/id/pteravjV>



## The Stomachion: Archimedes' Forgotten Work

**A puzzle well known in Antiquity:** 14 irregular pieces → how many ways to form a square?

**Archimedes** presents systematic ideas of counting all these ways

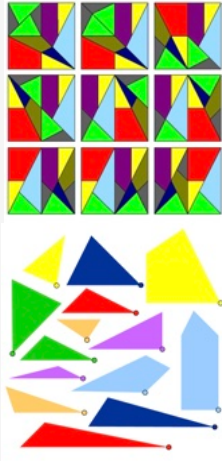
**2006 answer to Archimedes' question (interpreted):** **17,152** configurations (required systematic counting of *all* possibilities)

**Why ignored?** Only references until the discovery of the Palimpsest.

**Why important?** A treatise on **combinatorics** — a field that didn't develop until the rise of **computer science**

Far ahead of its time

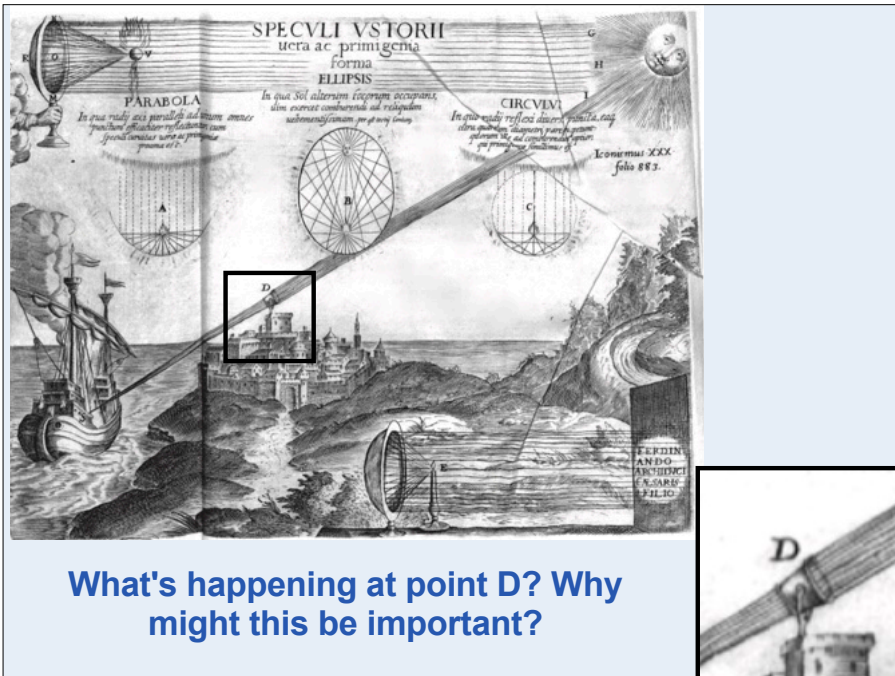
**Challenge:** How many ways can *you* arrange these pieces into a square?



<https://www.geogebra.org/m/d9thyxv>

<https://pi.math.cornell.edu/~mec/GeometricDissections/1.2%20Archimedes%20Stomachion.html>

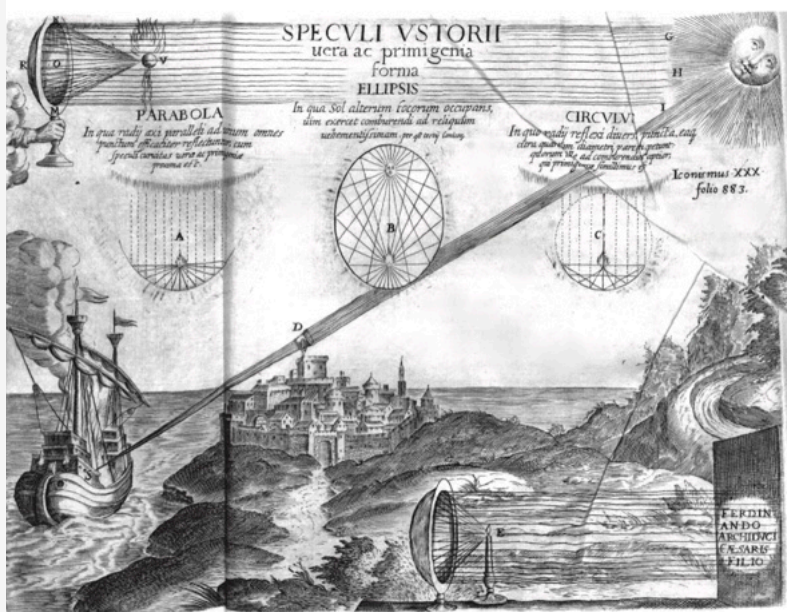
## A story about Archimedes' and mirrors (with conic sections)



**What's happening at point D? Why might this be important?**

## Archimedes' mirror





Setting fire to a ship using the sun and a mirror. Engraving from a book by our buddy Athanasius Kircher, 1671. (Wellcome Collection)

One of the many stories about Archimedes tells that he managed to destroy the Roman fleet at the siege of Syracuse in 213 BC by the application of directed solar radiant heat.

Many studies debunked this...

# Stories about Archimedes

**Eureka!**





Death of Archimedes (Mosaic, Municipal Gallery Francfort)

## Archimedes last words



Cassus: *"The head and not the lines"*

Georgios Pachymeris: *"heat me (my head) do not destroy the lines"*  
(drawing)

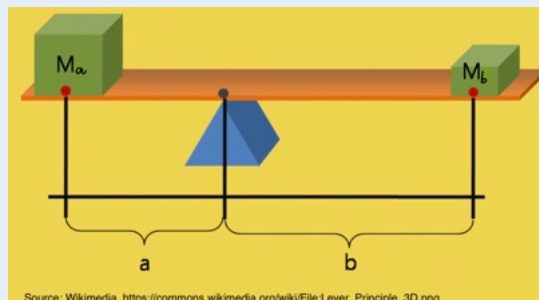
Tzetzes: *"Stay away man from my diagram"*

Unknown: *"Don't disturb my circles"*

# Archimedes and the Law of the Lever

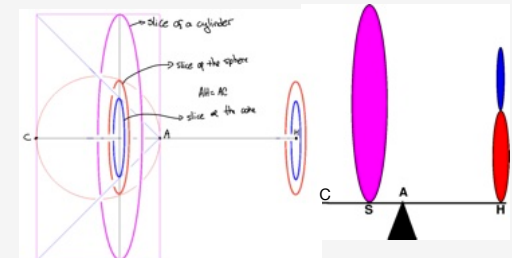
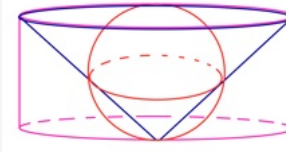
## Archimedes Law of the Lever

**Magnitudes are in equilibrium at  
distances reciprocally  
proportional to their weights.**



**Write down a  
formula for  
Archimedes Law of  
the Lever in terms of  
the weights  $M_a$  and  
 $M_b$  and the  
distances  $a$  and  $b$**

$$\text{Vol}(\text{Sphere}) + \text{Vol}(\text{Cone}) = \frac{1}{2}\text{Vol}(\text{Cylinder})$$



- Applied Cavalieri's principle, imagining the sections of a region balanced about a fulcrum.
- If each pair of corresponding sections balance at distances  $a$  and  $b$ , then the bodies themselves will balance at these distances
- The balancing force of the cylinder is the same if all of its mass is concentrated at its center of mass, which is the midpoint of  $AC$ .

### Example of Archimedes' Method

**Archimedes Law of the Lever:**  
Magnitudes are in equilibrium at  
distances reciprocally proportional to  
their weights.

# Archimedes' Spiral

## Archimedean Solids

<https://www.geogebra.org/m/rVx87NXP>

### Archimedean spiral

If a straight line drawn in a plane revolve at a uniform rate about one extremity which remains fixed and return to the position from which it started, and if, at the same time as the line revolves, a point move at a uniform rate along the straight line beginning from the extremity which remains fixed, the point will describe a spiral ( $\epsilon\lambda\iota\xi$ ) in the plane.



( $\epsilon\lambda\iota\xi$ ) Helix

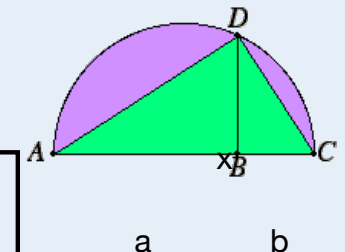
# Mean Proportional

## Conic Sections

### Proposition VI.13 of Euclid's elements

To find a mean proportional to two given segments.

**Prove that BD is the mean proportional between AB and BC. In other words, show that  $AB : BD = BD : BC$ .**



A positive number  $x$  is the **mean proportional** of two (positive) numbers  $a$  and  $b$  if  $a/x = x/b$ .

Join the lesson at [www.geogebra.org/classroom](https://www.geogebra.org/classroom/y9rmrh6) with the code:

**Y9RM RHN6**

Or you can also share the following link with your students:

[www.geogebra.org/classroom/y9rmrh6](https://www.geogebra.org/classroom/y9rmrh6)

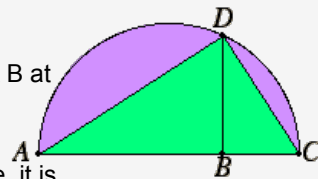


<https://www.geogebra.org/classroom/eulogm>  
<https://www.geogebra.org/m/helndk>

## Proposition VI.13 of Euclid's elements

To find a mean proportional to two given straight lines.

- Let AB and BC be the two given straight lines.
- It is required to find a mean proportional to AB and BC.
- Place them in a straight line, and describe the semicircle ADC on AC. Draw BD from the point B at right angles to the straight line AC, and join AD and DC. (I.11)
- Since the angle ADC is an angle in a semicircle, it is right. (III.31)
- Since, in the right-angled triangle ADC, BD has been drawn from the right angle perpendicular to the base, therefore BD is a mean proportional between the segments of the base, AB and BC. (VI.8, Corollary)
- Therefore a mean proportional BD has been found to the two given straight lines AB and BC**



A positive number  $x$  is the **mean proportional** of two (positive) numbers  $a$  and  $b$  if  $a/x = x/b$ .

## From mean proportional to conic sections

- Hippocrates of Chios (~400 BCE)** reduced the problem of **doubling the cube** to finding **two mean proportionals** between two given lengths  $a$  and  $b$ .  
In modern terms, the unknowns  $x$  and  $y$  satisfy:  $a : x = x : y = y : b$ , or equivalently:

$$1. x^2 = a y$$

$$2. y^2 = b x$$

$$3. x y = a b.$$

- Menaechmus (~350 BCE)** discovered **conic sections** while seeking a geometric **construction** of these two mean proportionals.

**Doubling the cube → Two mean proportionals → Conic sections**

<https://www.geogebra.org/m/pgu4vvtj>



### Conic sections

1. When  $\alpha = 40^\circ$  and  $\beta = 35^\circ$ , the intersection is a hyperbola.

Find all pairs  $(\alpha, \beta)$  that produce a hyperbola.

2. For which values of  $(\alpha, \beta)$  is the intersection an ellipse?

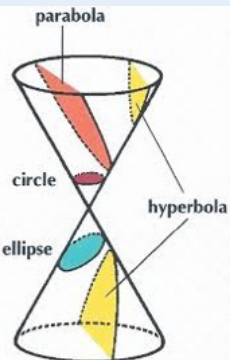
3. For which values of  $(\alpha, \beta)$  is the intersection a circle?

4. For which values of  $(\alpha, \beta)$  is the intersection a parabola?

5. Can you find any other type of curve formed by this intersection?

Compare the angle of the plane with the cone's axis ( $\alpha$ ) to the angle of the cone's side with the axis ( $\beta$ )

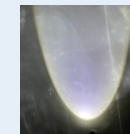
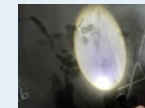
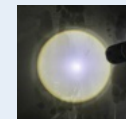
Find all the values you can — the GeoGebra app will help you visualize and test your answers.



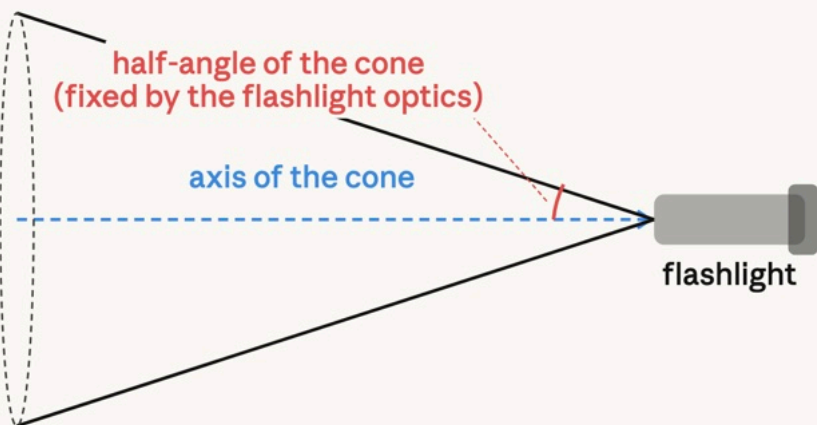
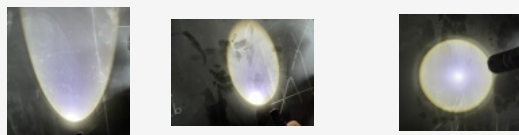
<http://wilson.coe.uga.edu/emt725/Conics%20Images/Conics.html>

1. What curves do you see in the demonstration?

2. Explain why we see those curves.



**Hint: there should be a cone in the explanation**

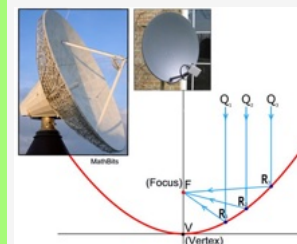


## Apollonius of Perga (~200BCE) Studied parabolas. Current applications of properties parabolas



A parabolic dish is a surface with a cross-sectional shape of a parabola -paraboloid- used to direct light or sound waves. This concept is used by

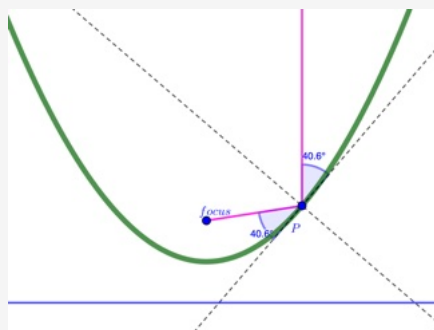
- Radio telescope antennas
- satellite dishes
- parabolic reflector,
- car's headlights and in spotlights.



<https://mathbitsnotebook.com/Algebra2/Quadratics/QDParabolaApplied.htm>

## How a Parabolic Mirror Focuses Light

<https://www.geogebra.org/m/fkqe4v5m>



- <https://www.geogebra.org/m/jeqqdnkh>  
(spherical mirror for comparison)

# Apollonius (~200 BCE)

# Apollonius of Perga (~200BCE)

- Alexandria
- worked out more fully and generally than in the writings of others
- studied and later taught under the followers of Euclid

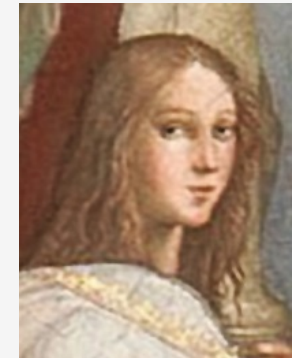
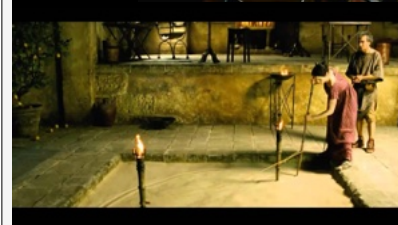


Conics - 8 books

*"the most and prettiest of these theorems are new, and it was their discovery which made me aware that Euclid did not work out the syntheses of the locus with respect to three and four lines, but only a chance portion of it, and that not successfully; for it was not possible for the said synthesis to be completed without the aid of the additional theorems discovered by me."*

Hypatia (~400 CE) was a philosopher, astronomer, and mathematician who lived in Alexandria.

In the clip below from movie Agora (a fictionalized version of her life), she starts by discussing the assumption that the orbits of the earth around the sun are circles.



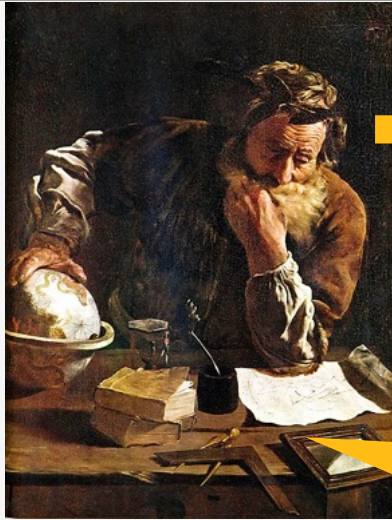
Detail from The School of Athens by Raffaello Sanzio.



Hypatia arrives to the conclusion that the orbits follow another curve and not a circle. What is that curve? What is the property of that curve that she describes?

<https://youtu.be/bUUB1u8xJA4>

Eratosthenes, the  
librarian of Alexandria  
(~200 BCE)



**Archimedes Thoughtful** (also known as Portrait of a Scholar) by Domenico Fetti, 1620



**Erastosthenes Teaching in Alexandria**  
By Bernardo Strozzi - Musée des beaux-arts de Montréal, Public Domain, <https://commons.wikimedia.org/w/index.php?curid=42582581>

"The proofs then of these theorems I have written in this book and now send to you (that is, Eratosthenes)" (letter from

## A well at Kom Ombo, 50 km north of Syene



Image credit: <https://medium.com/roaming-physicist/looking-down-a-well-in-egypt-to-measure-the-size-of-the-earth-5e12d2d1b2>

### Syene (Aswan)

On the summer solstice at noon

→ Sun is (almost) directly overhead

→ No shadow (or negligible)

### Alexandria

On the summer solstice at noon

→ shadow is shortest.

**Key idea: Same Sun, same day:**  
no shadow in Syene vs. small shadow in Alexandria → gives angle difference.

On the summer solstice, the Sun reaches its maximum height at noon in the Northern Hemisphere.

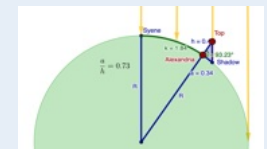
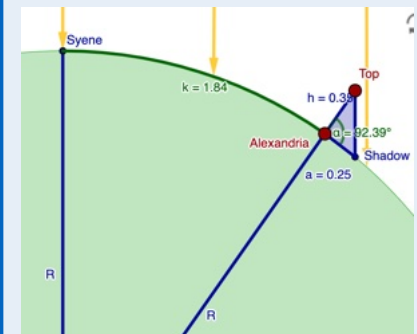
The small triangle with vertices Alexandria, Shadow and Top is not a right-angled triangle but you may assume it is (If the picture was to scale, the error from such an approximation would be less than any other error in the measurements).

1. Note that the angles  $\beta$  and  $\gamma$  are equal. Explain why.

2. Find the diameter of the circumference of the earth in terms of  $h$  and  $a$ , the two legs of the small triangle. (Hint: Start by finding  $R$ , the radius of the "green" circle above, in terms of  $a$  and  $h$ ).

3. Why can we assume that the rays of the sun are parallel?

## Eratosthenes on the circumference of the earth



Discussion in groups. Answer in Wooclap.  
<https://www.geogebra.org/m/ty5ych75>

1/50 of a circle  $\leftrightarrow$  5000 stadia ( $\sim$ 800 km)  
 $\therefore$  1 circle  $\leftrightarrow$   $50 \times 5000$  stadia  
 $= 250\,000$  stadia ( $\sim$ 40 000 km)

Angle from lengths of the pole and its shadow:  
 1/50 of a circle ( $\sim 7^\circ$ )

Parallel sun rays

Pole's shadow

Pole at Alexandria

Well at Syene (Aswan)

Alternate interior angles are equal

Centre of the Earth

Alexandria

5000 stadia

Syene

# Claudius Ptolemy (100 – 170 CE)

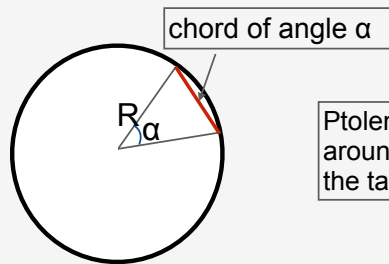


## Claudius Ptolemy Table of Chords

The idea of **angles** seems to have appeared in Greece between 6th and 5th century BCE.

**Trigonometry** is the branch of mathematics that studies relationships between side lengths and angles of triangles.

Ptolemy made a table of correspondence between angles and their chords. (This is the earliest form of trigonometry we know of)



Ptolemy's treatise on astronomy, written around 150 CE, explains how to construct the table.

## Claudius Ptolemy ~100AD

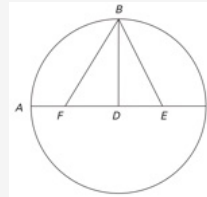
- Worked with a circle of **radius 60** (because he did calculations in base 60) **Question: Why base 60?**

- Computed chord(36°). How?

Main idea: computed chord tables of certain angles using

- propositions from Euclid's elements
- square roots (or approximations)

In the Almagest, Ptolemy announces that he is going to employ in general the base 60 system, in order to avoid "**the embarrassment of the fractions**"



Example: Suppose that E bisects DC

Find F so EF=EB

Then (prove it!) DF is the side of a decagon inscribed on a circle with radius DC.

Ptolemy computed  $DF = \text{chord}(36^\circ)$ :

$$DF^2 = (EF - ED)^2 = (EB - ED)^2 = ((BD^2 + ED^2)^{1/2} - ED)^2 = ((60^2 + 30^2)^{1/2} - 30)^2 = (37;4,55)_{60}$$

## Claudius Ptolemy ~100AD

- regular dodecagon  $\rightarrow$  chord( $30^\circ$ )
- regular decagon  $\rightarrow$  chord( $36^\circ$ )
- chord( $36^\circ$ ) and chord( $30^\circ$ )  $\rightarrow$  chord( $6^\circ$ )
- Bisecting chord( $3^\circ$ )  $\rightarrow$  chord( $1^\circ 30'$ )
- Upper and lower values of the chord( $1^\circ$ )
- Finally, a table of 360 entries, with chords of at  $30'$  increments up to  $180^\circ$ .

| παρα-<br>γωνία | εὐθείαν | ἡχοειδῶν |
|----------------|---------|----------|
| α'             | α       | α        |
| β'             | β       | β        |
| γ'             | γ       | γ        |
| δ'             | δ       | δ        |
| ε'             | ε       | ε        |
| ζ'             | ζ       | ζ        |
| η'             | η       | η        |
| θ'             | θ       | θ        |
| ι'             | ι       | ι        |
| κ'             | κ       | κ        |
| λ'             | λ       | λ        |
| μ'             | μ       | μ        |
| ν'             | ν       | ν        |
| ξ'             | ξ       | ξ        |
| ο'             | ο       | ο        |
| π'             | π       | π        |
| ρ'             | ρ       | ρ        |
| σ'             | σ       | σ        |
| τ'             | τ       | τ        |
| υ'             | υ       | υ        |
| φ'             | φ       | φ        |
| χ'             | χ       | χ        |
| ψ'             | ψ       | ψ        |
| ω'             | ω       | ω        |

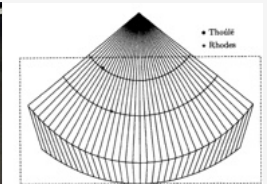
Beginning of Ptolemy table of chords

<https://www.wibourhall.org/pdfs/HeibergAlmagest/Complete.pdf>

## Claudius Ptolemy ~100AD



Ptolemy's world map, reconstituted from Ptolemy's Geography (circa 150) in the 15th century, indicating "Sinae" (China) at the extreme right, beyond the island of "Taprobane" (Ceylon or Sri Lanka, oversized) and the "Aurea Chersonesus" (Southeast Asian peninsula).



Ptolemy method of making (The History of Mathematics A Brief Course - Roger Cooke - 2011)

One of the first scholars to look at the problem of representing large portions of the Earth's surface on a flat map.

# Diophantus (~200 CE) and The birth of literal algebra

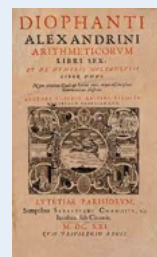
## Diophantus - (possibly ~250AD) - Alexandria

Work by Diophantus (died in about 280 B.C.), translated from Greek into Latin by Claude Gaspard Bachet de Méziriac. This edition of the book was published in 1621

- Lived in Roman Egypt.
- Little is known about his life.
- Best known for his **Arithmetica**: A collection of problems about finding numbers satisfying specified conditions.



**Solve Problem I.2 in Diophantus' Arithmetica: To split a given number (100) in two parts having a given difference (40).**



**Problem I.2 from Diophantus' Arithmetica: It is required to divide a proposed number into two numbers having a given ratio.**

Now, let it be proposed to divide 100 into two numbers having a triple ratio.

Let the smaller number be one "unit".

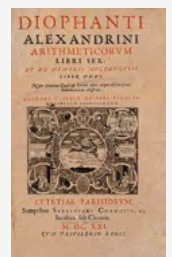
Then the larger is that unit plus 40.

Together they make:  $2 \text{ units} + 40 = 100$ .

Subtract 40 from both sides:  $2 \text{ units} = 60$ .

So each unit is 30.

Therefore, the two numbers are: 30 and 70.



### Problem I.1: To divide the prescribed number (arithmos) into two numbers in an excess that's given.

In fact, let the given arithmos be 100, and the excess  $\overset{\circ}{M}$  40. To find the arithmoi. Let the smaller be assigned  $\varsigma$  1.

Therefore, the larger will be  $\varsigma$  1  $\overset{\circ}{M}$  40. Therefore, together they become  $\varsigma$  2  $\overset{\circ}{M}$  40. But  $\overset{\circ}{M}$  100 are given. Therefore,  $\overset{\circ}{M}$  100 are equal to  $\varsigma$  2  $\overset{\circ}{M}$  40.

And from similars are similars, I subtract from 100,  $\overset{\circ}{M}$  40 [and from the 2 arithmoi and of 40 units similarly 40 units].

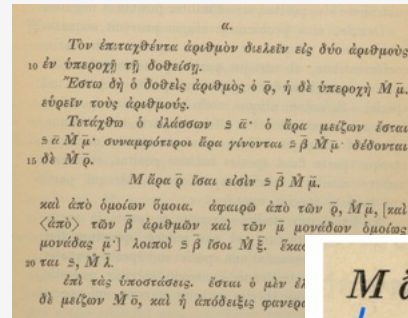
Remainders  $\varsigma$  2 equal  $\overset{\circ}{M}$  60. Therefore each  $\varsigma$  becomes  $\overset{\circ}{M}$  30.

For the actualities. The smaller will be  $\overset{\circ}{M}$  30 and the larger  $\overset{\circ}{M}$  70, and the demonstration is obvious.

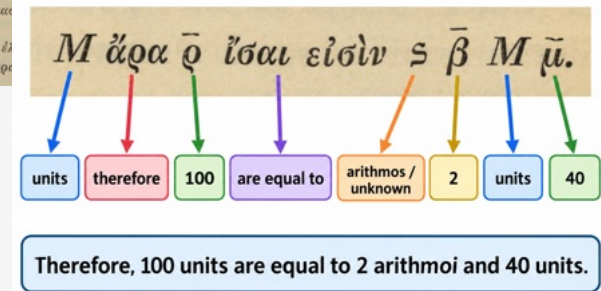
From <https://web.calstatela.edu/faculty/hmendel/Ancient%20Mathematics/Diophantus/Arithmetica/Diophantus.I.1-6.html>



### Diophantus' Arithmetica Problem I.1: To divide a given number into two numbers having a given difference.



$$100 = 2x + 40$$



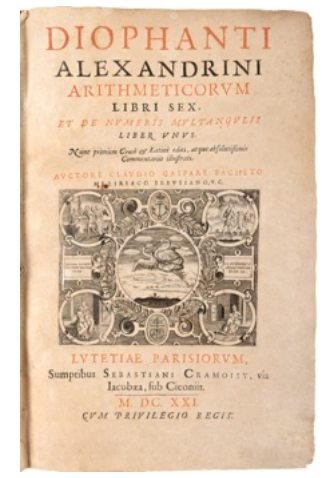
## Arithmetica - What it is

- Mathematics about numbers themselves, not numbers of things.
- A collection of subtle, sometimes complicated problems, carefully thought through.
- Two scholarly interpretations:
  - His approach seems new, perhaps because so much earlier mathematics is lost.
  - Its isolation is overstated (e.g. A Greek papyrus written a century earlier -Papyrus Michigan 620- solves number problems with a symbol for the unknown.)

## Arithmetica - What it is not

- Written in the formal proposition-and-proof style of Greek mathematics.
- Euclidean “number theory.”
- A collection of problems motivated primarily by practical applications.
- A work with obvious Neopythagorean mystical interests.

Work by Diophantus translated from Greek into Latin by Claude Gaspard Bachet de Méziriac. This edition of the book was published in 1621



If Diophantus had not existed, no historian of ancient Greek mathematics would have invented him. Unlike the formal geometric treatises of Euclid, Archimedes and Apollonius, taken as representative of the characteristically Greek style of mathematics, Diophantus' Arithmetica falls within the category of arithmetic theory.

Knorr, Wilbur. Books IV to VII of Diophantus' Arithmetica: In the Arabic Translation Attributed to Qustā Ibn Lūqāby Jacques Sesiano. The American Mathematical Monthly, vol. 92, no. 2, 1985, pp. 150–54. JSTOR, <https://doi.org/10.2307/2322659>

### Notation in The Arithmetica - by Diophantus

**Example:**  $x^3 - 2x^2 + 10x - 1 = 5$  would be written  
 $\text{Κ}^{\Upsilon}\bar{\alpha} \varsigma\bar{\iota} \wedge \Delta^{\Upsilon}\bar{\beta} \overset{\circ}{\text{Μ}}\bar{\alpha} \quad \tau\alpha\upsilon\tau\alpha \text{ ἴσα } \overset{\circ}{\text{Μ}}\bar{\epsilon}$

| Greek name                       | English name                   | Modern notation                                | Abbreviated symbol    |
|----------------------------------|--------------------------------|------------------------------------------------|-----------------------|
| ἀριθμός                          | number (unknown)               | x                                              | ς                     |
| δύναμις                          | power                          | x <sup>2</sup>                                 | Δ <sup>Υ</sup>        |
| κύβος                            | cube                           | x <sup>3</sup>                                 | Κ <sup>Υ</sup>        |
| δυναμοδύναμις                    | power-power                    | x <sup>4</sup>                                 | Δ <sup>Υ</sup> Δ      |
| δυναμόκυβος                      | power-cube                     | x <sup>5</sup>                                 | ΔΚ <sup>Υ</sup>       |
| κυβόκυβος                        | cube-cube                      | x <sup>6</sup>                                 | Κ <sup>Υ</sup> Κ      |
| μονάς                            | unit(s)                        | constant term                                  | Μ̇                    |
| λείψις                           | lacking / deficiency           | negative (lacking) term, e.g. -16x             | ∧ (Tannery-like form) |
| ὀμώνυμον μέρος (e.g. ἀριθμοστόν) | inverse number / inverse power | 1/x, 1/x <sup>2</sup> , 1/x <sup>3</sup> , ... | × mark                |

### Problem V.2 from Diophantus' Arithmetica: To find three numbers in geometric proportion, each of which plus a given number is a square.

Let the given number be 20.  
 Since 16 + 20 = 36 is a square, take the extremes 16 and x<sup>2</sup>; the mean is then 4x.  
 It remains that x<sup>2</sup> + 20 and 4x + 20 must be squares.  
 Their difference is x<sup>2</sup> - 4x = x · (x - 4). Half the difference of the factors is 2, and its square, 4, is set equal to the smaller: **4x + 20 = 4.**

Diophantus calls this *ἄτονος*, “*absurd*”, because the 4 should be greater than 20, and he goes back and chooses a different square instead of 16.

In our notation:  
**x = -4**  
 A negative number appears, and is rejected.

(Solution written in more modern language)

### Problem II.8 from Diophantus' Arithmetica: To divide a proposed square (number) into two squares.

## Problem II.8 from Diophantus' Arithmetica: To divide a proposed square (number) into two squares.

Now, let it be proposed to divide 16 into two squares.

And let the first be assigned to be 1 Power. Therefore, the other will be 16 units lacking 1 Power. Thus, 16 units lacking 1 Power should be equal to a square.

I form the square from (a side of) whatever (multitude) of Numbers lacking as many units as there are in the side of the 16 units. Let it be 2 Numbers lacking 4 units. Therefore, the square itself will be 4 Powers, 16 units lacking 16 Numbers.

These are equal to 16 units lacking 1 Power.

Let the lacking be added in common and likes from likes. Thus, 5 Powers are equal to 16 Numbers, and the Number becomes 16 5ths.

The one will be 256 25ths, the other 144 25ths, and the two added together make 400 25ths, that is 16 units, and each of them is a square.

Now, let (it be proposed) again to divide the square (number) 16 into two squares.

Let, again, the side of the first be assigned to be 1 Number, and (the side) of the other whatever (multitude) of Numbers lacking as many units as there are in the side of the divided (square). Well, let it be 2 Numbers lacking 4 units. Therefore, one of the squares will be 1 Power, and the other, 4 Powers, 16 units lacking 16 Numbers. Well then, I want the two, when added together, to be equal to 16 units. Thus, 5 Powers, 16 units lacking 16 Numbers are equal to 16 units, and the Number becomes 16 5ths.

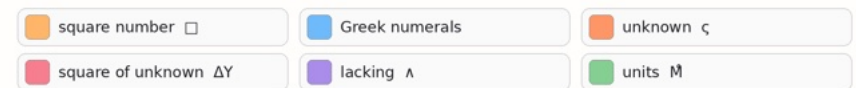
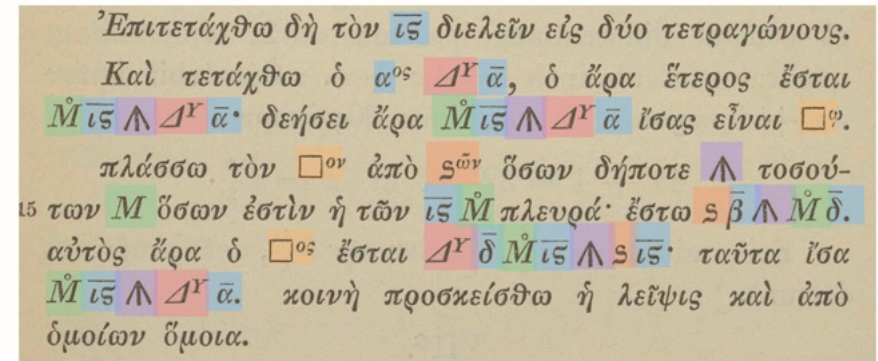
The side of the first will be 16 5ths, therefore (the first) itself will be 256 25ths; and (the side) of the second 12 5ths, therefore (the second) itself will be 144 25ths.

And the proof is obvious.

Christianidis & Oaks, The Arithmetica of Diophantus (2023).

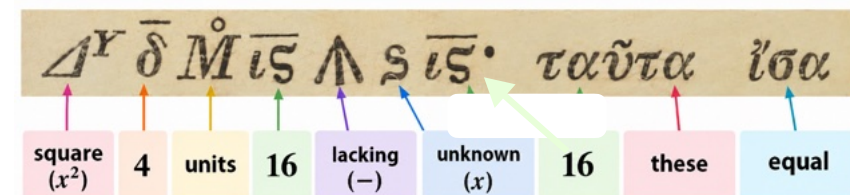
From Heath - <https://archive.org/details/in.ernet.dli.2015.154147>

## Problem II.8 from Diophantus' Arithmetica: To divide a proposed square (number) into two squares.



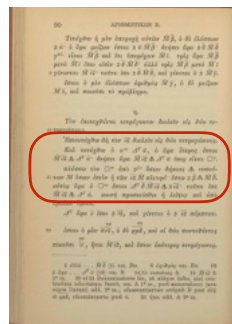
Diophantus, Arithmetica II.8 • Tannery, p. 90

<https://poclib.org/viewer/image/934841556/103/#docAnchor>  
Diophanti Quae Exstant Omnia Continens. In Aedibus B. G. Teubneri, 1893-Tannery translation

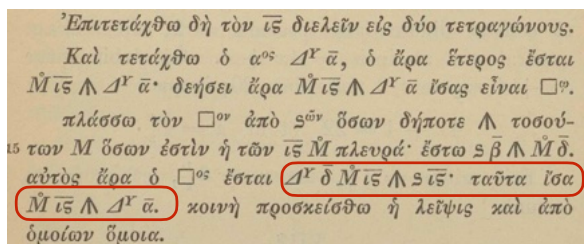
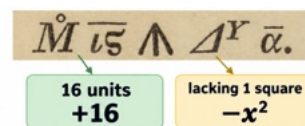


$$4x^2 + 16 - 16x = \dots$$

Diophanti Quae Exstant Omnia Continens. In Aedibus B. G. Teubneri, 1893.



<https://poclib.org/viewer/image/934841556/103/#docAnchor>



## Problem II.8 from Diophantus' Arithmetica: To divide a proposed square (number) into two squares.

Given square number 16.

$x^2$  one of the required squares. Therefore  $16 - x^2$  must be equal to a square.

Take a square of the form  $(mx - 4)^2$ ,  $m$  being any integer and 4 the number which is the square root of 16, e.g. take  $(2x - 4)^2$ , and equate it to  $16 - x^2$ .

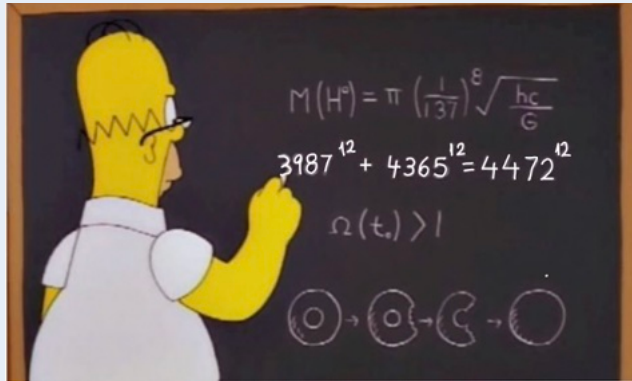
Therefore  $4x^2 - 16x + 16 = 16 - x^2$ ; or  $5x^2 = 16x$ , and  $x = 16/5$ .

The required squares are therefore 256/25, 144/25.

From Heath - <https://archive.org/details/in.ernet.dli.2015.154147>

**Fermat wrote in the margin of this problem**

Use a gadget (phone, computer..) that can only use numbers of 10 digits to compute first  $3987^{12}+4365^{12}$  and second,  $4472^{12}$ . Have we (jointly with Homer Simpson) disproved Fermat Last Theorem? Why or why not?



Compute  $3987^{12}+4365^{12}$  and  $4472^{12}$ .

Have we (jointly with Homer Simpson) disproved Fermat Last Theorem? Why or why not?

2026

$$3987^{12}+4365^{12} = 6.397665635 \times 10^{43}$$

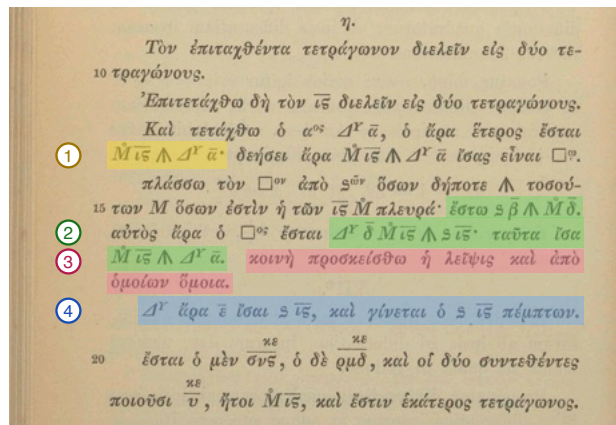
$$4472^{12} = 6.3976656348 \times 10^{43}$$

2019ish

$$4472^{12} = 6.397665635E43$$

$$3987^{12}+4365^{12} = 6.397665635E43$$

1. Yellow:  $\dot{M} \bar{\iota} \varsigma \mathfrak{m} \Delta^{\wedge} Y \alpha$ , the other square  $(16 - x^2)$ .
2. Green: the side  $\varsigma \beta \mathfrak{m} \dot{M} \delta$ , then the equation  $\Delta^{\wedge} Y \delta \dot{M} \bar{\iota} \varsigma \mathfrak{m} \varsigma \iota \varsigma$  ταῦτα ἴσα  $\dot{M} \bar{\iota} \varsigma \mathfrak{m} \Delta^{\wedge} Y \alpha$ .
3. Pink: the rule, κοινὴ προσκείσθω ἡ λείψις καὶ ἀπὸ ὁμοίων ὅμοια.
4. Blue:  $\Delta^{\wedge} Y \bar{\alpha} \rho \alpha \epsilon$  ἴσαι  $\varsigma \iota \varsigma$ , καὶ γίνεται ὁ  $\varsigma \iota \varsigma$  πέμπτων  $(x = 16/5)$ .

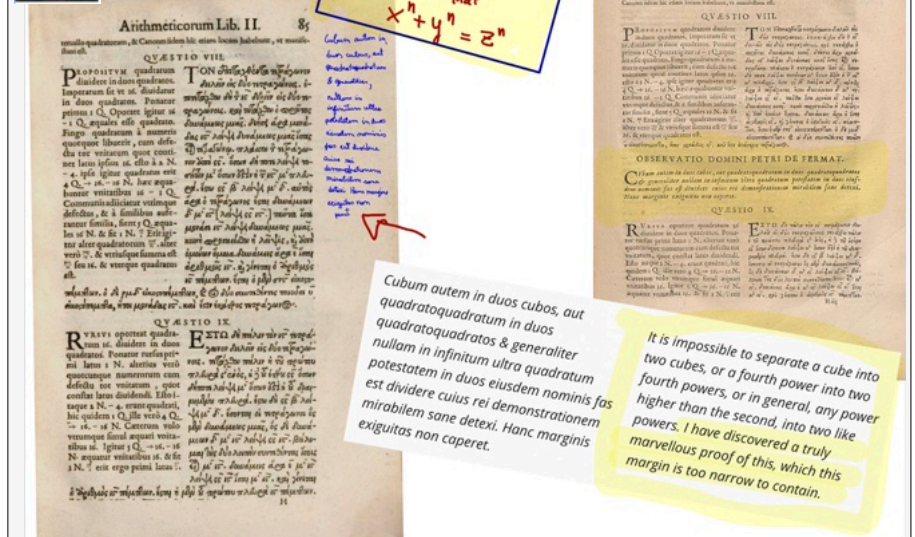


Tannery's printed edition (1893). Paul Tannery, Diophanti Alexandrini opera omnia, vol. I (Leipzig: Teubner, 1893)



Fermat  
~1600

Fermat last theorem  
If  $n \geq 3$ , there are no positive integers  $x, y$  and  $z$  such that  $x^n + y^n = z^n$





Fermat last theorem  
Is  $n \geq 3$ , there are no <sup>positive</sup> integers  
 $x, y$  and  $z$  such that  
$$x^n + y^n = z^n$$

Andrew Wiles  
presented a proof in 1993 - 1994