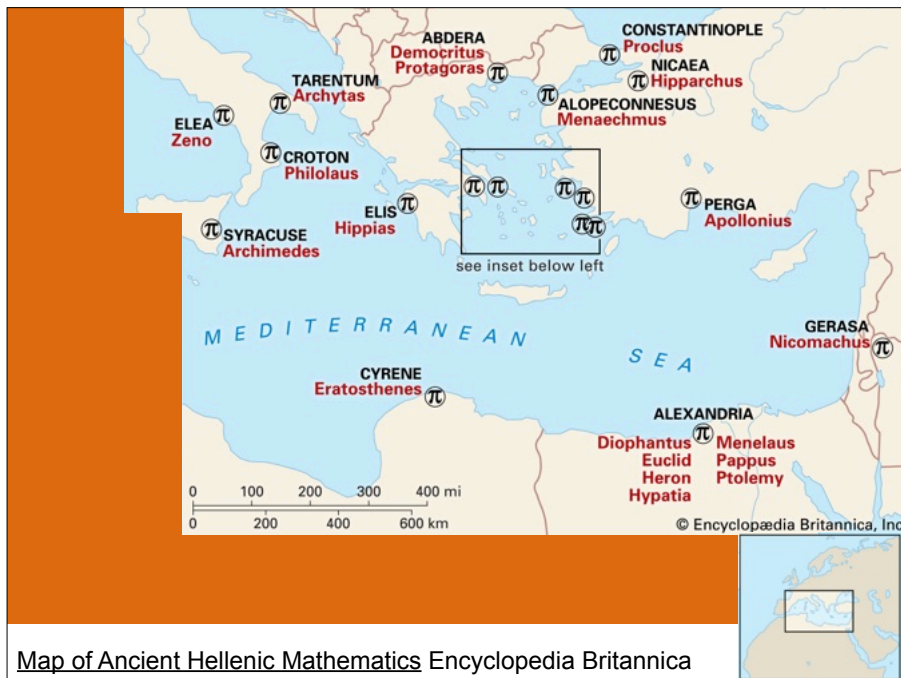


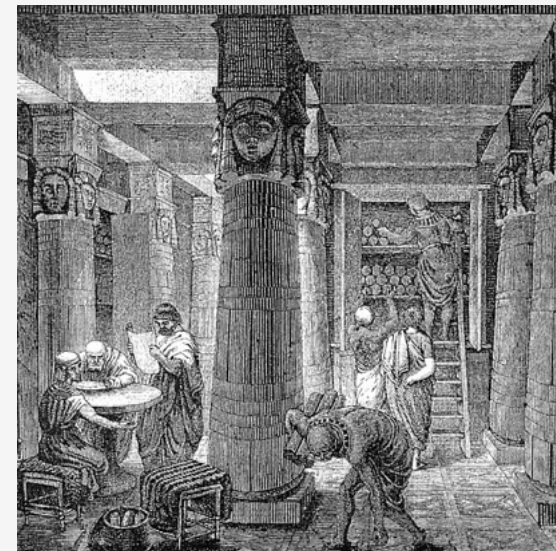
Euclid's Elements

Alexandria as context



It is likely that
Euclid worked
and taught there

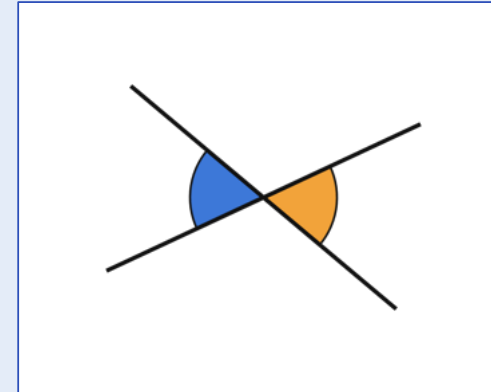
About 500,000
volumes



Nineteenth-century artistic rendering of the Library of Alexandria by the German artist O. Von Corven, based partially on the archaeological evidence available at that time

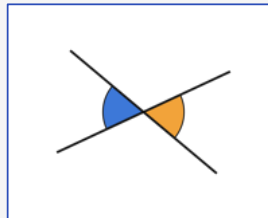
Euclid's Elements: What are they?

The opposite angles (highlighted) look equal. Are they? How do you know?



The opposite angles (highlighted) look equal. Are they? How do you know?
The answer is in Prop. I.15 of Euclid's Elements.

- The blue angle and the top angle make a straight angle, so together they equal two right angles. **(By Prop. I.13)**
- The orange angle and that same top angle also make a straight angle, so together they equal two right angles. **(By Prop. I.13)**
- Therefore, the blue angle plus the top angle equals the orange angle plus the top angle. **(By Common Notion 1)**
- Subtract the top angle from both sides.
- So the blue angle equals the orange angle. **(By Common Notion 3)**



Proposition I.13:

When a straight line crosses another straight line, the two adjacent angles on either side add up to two right angles.

Common Notion 1:

Things equal to the same thing are equal to each other

Common Notion 3:

If equal amounts are subtracted from equal amounts, the remainders are equal.

Every new claim needs a reason.

The reasons are either:

Axioms

Example: If equal amounts are subtracted from equal amounts, the remainders are equal.

Definitions

"Right angle" has a precise meaning.

Statements already proved

Example: When a straight line crosses another straight line, the two adjacent angles on either side add up to two right angles.

Axiomatic method - Euclid's Elements

- Starts with **definitions** and **axioms** (postulates and common notions in Euclid).
- New **propositions** are derived by **logical proof**;
- Each step (new claim) in a proof is **justified** by one of the following types of "reason":
 - Axiom
 - Definition
 - Previously proved proposition
- Note: Nowadays, there are also **undefined terms**.

Euclid's Elements: What is it?

- Greek mathematical treatise
- *Compiled* around 300 BCE
- Attributed to Euclid
- Draws on centuries of accumulated mathematical work.
- Earliest extant mathematical work organized systematically by the axiomatic method.

Euclid's Elements: What does it cover

- **plane geometry** (shapes in the plane)
- **solid geometry** (shapes in space)
- **numbers and ratios** (divisibility, primes, proportions, irrationals) in a geometric way

Euclid's Elements: Quick Overview

- **Definition:** A precise description of what a term means.
- **Postulates:** basic assumptions about geometry. (e.g., "*a straight line can be drawn between any two points*")
- **Common Notions:** intuitively obvious facts about magnitudes (e.g., "*things equal to the same thing are equal to each other*")
- **Propositions:**
 - Theorems:** In right-angled triangles, the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.
 - Constructions:** Construct an equilateral triangle on a given segment.

Axiomatic method: Starting from axioms, new propositions are derived by logical proof; each step is justified by axioms, definitions or previously proved propositions.

Euclid's elements	Today's math
Definitions	Undefined terms Definitions
Common notions Postulates (self evident truths)	Axioms (assumptions wisely chosen)
Propositions Constructions	Propositions Constructions

Axiomatic method: Starting from axioms, new propositions are derived by logical proof; each step is justified by axioms, definitions or previously proved propositions.

If some propositions already proved in Euclid's Elements are added as axioms, what happens?

- A) A contradiction can be deduced from the new (larger) set of axioms
- B) All the propositions in Euclid's Elements can still be proven with the new (larger) set of axioms.
- C) Some, but not all, of the propositions in Euclid's Elements can be proven with the new (larger) set of axioms.

Choose one option and explain your reasoning.

How the Elements reached us. Impact and Variations

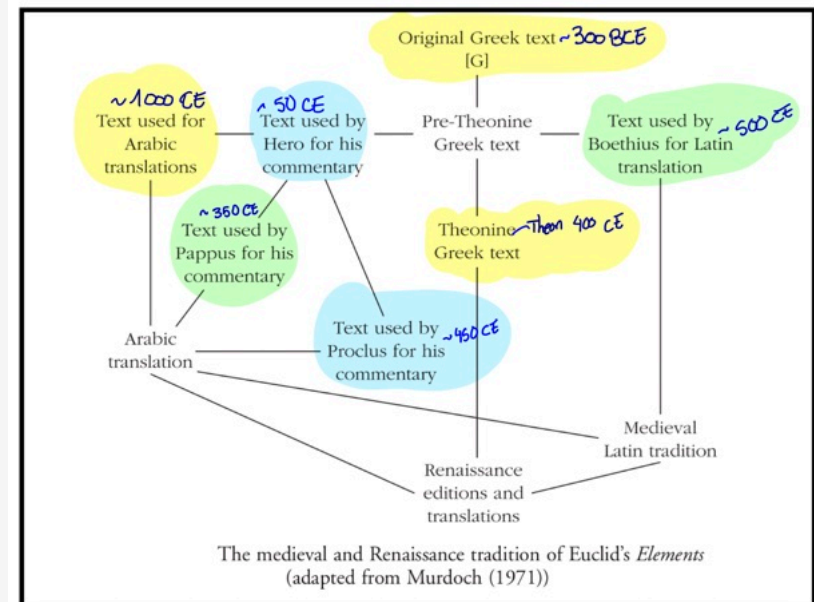
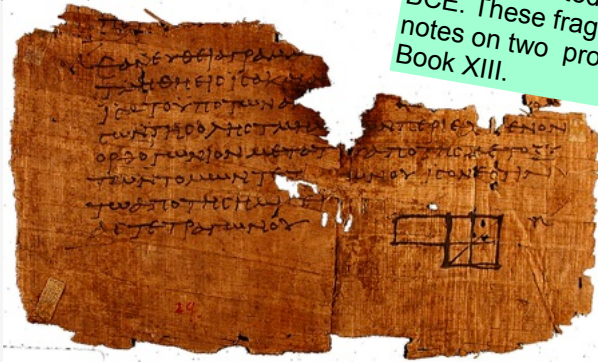


Diagram from the book "Ancient Mathematics" by Serafina Cuomo

One of the Oldest Fragment of Euclid's Elements, dated from 1st century CE,

Proposition II.5: If a straight line is cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole, together with the square on the straight line between the points of the section, is equal to the square on the half.



Also there are fragments found in potsherds discovered in Egypt and dated from 225 BCE. These fragments contain notes on two propositions from Book XIII.

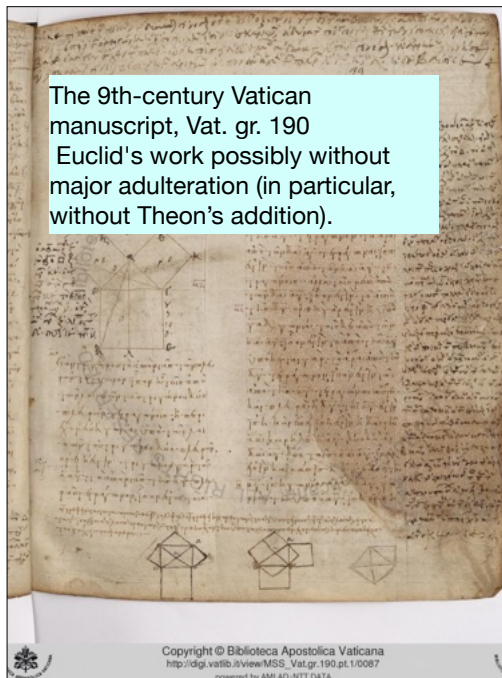
<http://www.math.ubc.ca/~cass/Euclid/papyrus/>



The 9th-century Vatican manuscript, Vat. gr. 190 Euclid's work possibly without major adulteration (in particular, without Theon's note).

Book I proposition 47
The Pythagorean Theorem

https://digi.vatlib.it/view/MSS_Vat.gr.190.pt.1.0087



The 9th-century Vatican manuscript, Vat. gr. 190 Euclid's work possibly without major adulteration (in particular, without Theon's addition).

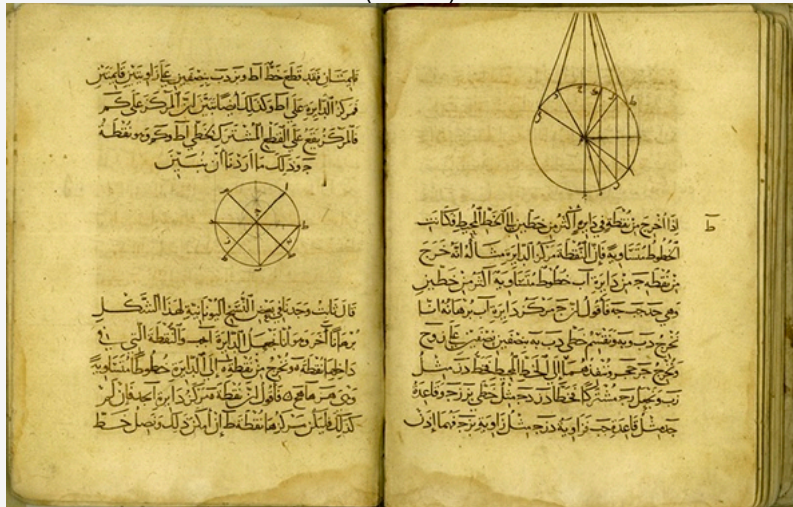
Heath (A History of Greek Mathematics (2 Vols.) (Oxford, 1921).) writes "*.. Thus Theon himself says that he edited the Elements and also that the second part of VI. 33, found in nearly all the MSS., is his addition...*"

This illustrates the many modifications the Elements have endured through the centuries.



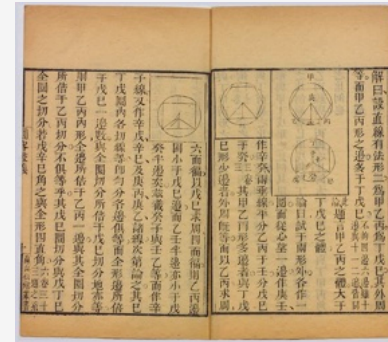
An illumination from a manuscript based on Adelard of Bath's translation of the *Elements*, circa 1309-1316; Adelard's is the oldest surviving translation of the *Elements* into Latin, done in the 12th-century work and translated from Arabic. (Wikipedia)

Early Translation of Euclid's Elements into Arabic (1466)



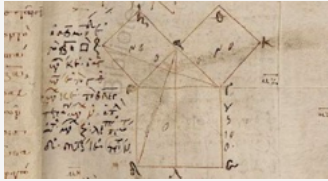
<https://www.maa.org/press/periodicals/convergence/mathematical-treasures-early-translation-of-euclids-elements-into-arabic>

The Italian Jesuit Matteo Ricci (left) and the Chinese mathematician Xu Guangqi (right) published the Chinese edition of Euclid's Elements (幾何原本) in 1607.
(Wikipedia)

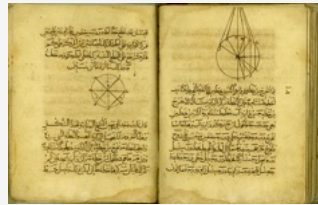


<https://www.c82.net/euclid/>

Euclid's element's



The 9th-century Vatican manuscript, Vat. gr. 190 Euclid's work possibly without major adulteration (in particular, without Theon's note).



were not created
overnight

were not created by
Euclid alone.



Definitions in the Elements

Group activity: Order these four Euclidean definitions from easiest to hardest to visualize and understand.

Then, explain the reasoning behind your order.

Hint: If you didn't already know the terms being defined, could you understand what the object is from the definition alone? Could you draw it?

Definition 1: A **point** is something that has no parts.

Definition 2: A **line** is length with no width.

Definition 10: When one straight line meets another and makes the two adjacent angles equal, each angle is called a **right** angle, and the two lines are said to be **perpendicular** to each other.

Definition 15: A **circle** is a plane figure whose boundary is a single [curved] line, such that all straight lines drawn from one interior point to the boundary are equal.

Definition 1.

A **point** is that which has no part.

Definition 2.

A **line** is breadthless length.

Definition 3.

The **ends** of a line are points.

Definition 4.

A **straight line** is a line which lies evenly with the points on itself.

**First four definitions of Book 1 of
Euclid's Elements.**

Postulates and Common Notions in the Elements

(Recall: Axioms =
postulates + common
notions.)

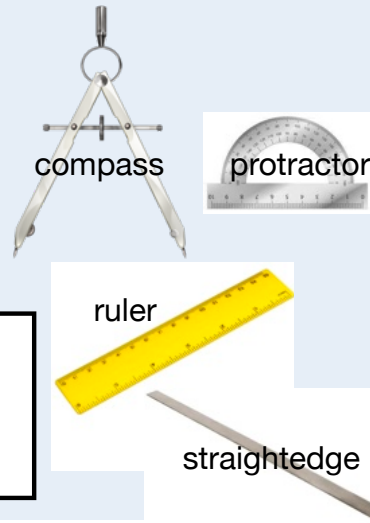
Common notions

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part

The Greek text of J.L. Heiberg translated by Richard Fitzpatrick <https://farside.ph.utexas.edu/Books/Euclid/Elements.pdf>

The first three postulates in the Elements

- Postulate I asserts that it is possible to draw a segment through any two given points.
- Postulate II says that any segment can be extended indefinitely in the same straight line.
- Postulate III states that it is possible to construct a circle with any given center and radius.



These first three postulates are associated with the tools that are used to implement them on a piece of paper.

**Which tool is each of these postulates associated with?
(Select the option with only the essential features)**

Postulate 4 states

“All right angles are equal to one another.” This seems self-evident. Why does Euclid need to write it down as a postulate?

Definition 10: When one straight line meets another and makes the two adjacent angles equal, each of those angles is called a **right** angle.

Euclid's five postulates

1. A straight line can be drawn from any point to any other point.
2. A finite straight line can be extended indefinitely in the same straight line.
3. A circle can be drawn with any center and any radius.
4. All right angles are equal to one another.
5. If a straight line crosses two straight lines and the interior angles on the same side add up to less than two right angles, then the two lines, if extended far enough, will meet on that side.

The first four postulates were straightforward; they were considered "truths". (Recall: for Aristotle, an axiom is a statement worthy of acceptance that is not proved.)

The fifth postulate is long and reads more like a theorem than like an axiom, a self-evident truth. We'll discuss more about this later.

Euclid's Elements

Euclid's Elements are composed of:

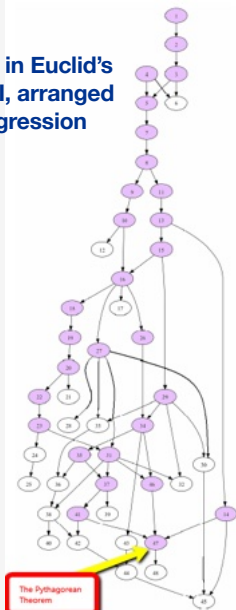
- Definitions
- Axioms
 - Common Notions
 - Postulates
- Propositions
- Proofs of the propositions

The propositions are organized in **logical progression**.

Could you describe, using your own words, what is meant by the statement: "The propositions are organized in logical progression"? What does the order tell you about how a proposition is proved?

. Eyer's graph of Euclid's Book I
<https://www.maa.org/book/export/html/590371>

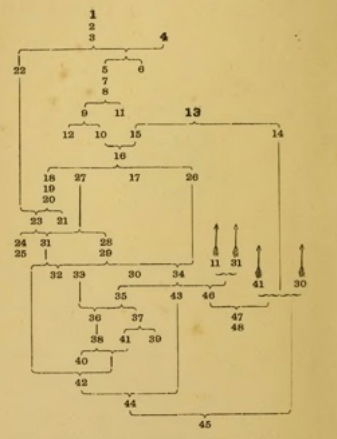
The proposition in Euclid's Elements Book I, arranged in logical progression



The Pythagorean Theorem

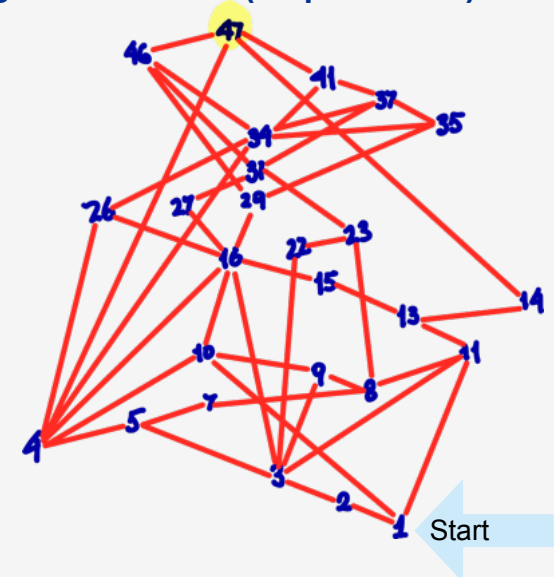
EUCLID, BOOK I.

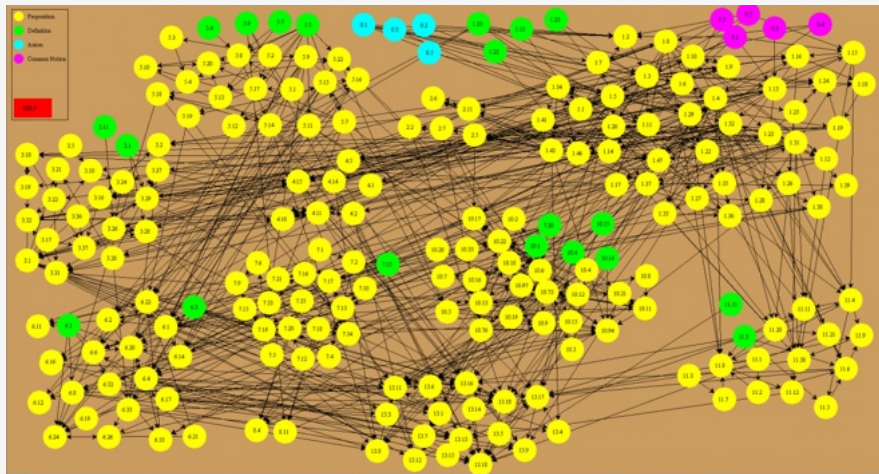
Arranged in Logical Sequence.



From Euclid and His Modern Rivals By Lewis Carroll

The structure of the propositions in Euclid's Elements leading to the Pythagorean Theorem (Proposition 47)





A graph of all 13 books of Euclid's *Elements*

<https://www.maa.org/book/export/html/590371>

Postulate 5'. (Playfair)

For any given point not on a given line, there is exactly one line through the point that does not meet the given line.

Byrne's version

If two straight lines (—) meet a third straight line (—) so as to make the two interior angles (and) on the same side less than two right angles, these two straight lines will meet if they be produced on that side on which the angles are less than two right angles.



Postulate 5.

If a straight line crosses two straight lines and the interior angles on the same side add up to less than two right angles, then the two lines, if extended far enough, will meet on that side.

Proposition I.1 from Euclid's Elements: Construction of an equilateral triangle

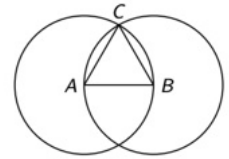
**An equilateral triangle is a triangle that...
(complete the sentence, you have one minute)**

An equilateral triangle is a triangle that has three equal sides.

Proposition 1: Given a segment, it is possible to construct an equilateral triangle that has this segment as one of its sides.

Construction (Part 1 of the proof):

- Draw a circle with center A and distance AB. (Post.3)
- Draw a circle with center B and distance AB. (Post.3)
- Let C be a point where the circles intersect.
- Draw segments AC and BC. (Post.1)
- ABC is the required triangle.



1. Post. 1: A straight line can be drawn from any point to any other point.

3. Post. 3: A circle can be drawn with any center and any radius.

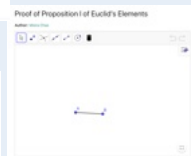
Proposition 1 of Euclid's Elements. Construction of equilateral triangle

Join the lesson at [www.geogebra.org/classroom](https://www.geogebra.org/classroom/zacaty6r) with the code:

ZACA TY6R

Or you can also share the following link with your students:

www.geogebra.org/classroom/zacaty6r



<https://www.geogebra.org/m/nessxht3>

<https://www.geogebra.org/classroom/zacaty6r>

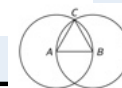
1. **Construction:** In the Geogebra window, construct an equilateral triangle.
2. **Now explain in Vevox: Why does this construction guarantee an equilateral triangle? (Hint: recall the definition of equilateral triangle)** (Use complete sentences)
3. **Optional:** Justify each step of your explanation using the Postulates and Common Notions below.
4. **Super Optional:** Compare the instructions given in 1. with [this one](#), which is a translation of the original Euclid's Element. What are the differences?)

Why does this construction guarantee an equilateral triangle? Justify using the appropriate definitions, postulates and common notions. (Hint: recall the definition of equilateral triangle)

Proposition 1: Given a segment, it is possible to construct an equilateral triangle that has this segment as one of its sides.

Post. 1: A straight line can be drawn from any point to any other point.

Post. 3: A circle can be drawn with any center and any radius.



Construction (Part 1 of the proof):

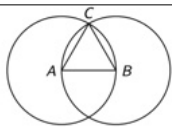
- Draw a circle with center A and distance AB. (Post.3)
- Draw a circle with center B and distance AB. (Post.3)
- Let C be a point where the circles intersect.
- Draw segments AC and BC. (Post.1)
- ABC is the required triangle.

Definition 15: A circle is a plane figure whose boundary is a single [curved] line, such that all straight lines drawn from one interior point to the boundary are equal.

Definition 20: If all three sides are of the same length, then a triangle is **equilateral**.

Common notion 1: Things that are equal to the same thing are also equal to one another.

Proposition 1: Given a segment, it is possible to construct an equilateral triangle that has this segment as one of its sides.



- $AC = AB$, since both are radii of the circle centered at A (Definition 15).
- $BC = AB$, since both are radii of the circle centered at B (Definition 15).
- $AC = BC$, since both equal AB (steps a, b, and Common notion 1).
- By a, b and c and Definition 20 ABC is an equilateral triangle with AB as one of its sides.

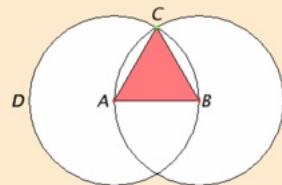
Definition 15: A circle is a plane figure whose boundary is a single [curved] line, such that all straight lines drawn from one interior point to the boundary are equal.

Definition 20: If all three sides are of the same length, then a triangle is equilateral.

Common notion 1: Things that are equal to the same thing are also equal to one another.

Proposition 1: Given a segment, it is possible to construct an equilateral triangle that has this segment as one of its sides.

Let AB be the given finite straight line.



It is required to construct an equilateral triangle on the straight line AB .

Describe the circle BCD with center A and radius AB . Again describe the circle ACE with center B and radius BA . Join the straight lines CA and CB from the point C at which the circles cut one another to the points A and B . LPost.3
LPost.1

Now, since the point A is the center of the circle CDB , therefore AC equals AB . Again, since the point B is the center of the circle CAE , therefore BC equals BA . LDef.15

But AC was proved equal to AB , therefore each of the straight lines AC and BC equals AB .

And things which equal the same thing also equal one another, therefore AC also equals BC . C.N.1

Therefore the three straight lines AC , AB , and BC equal one another.

Therefore the triangle ABC is equilateral, and it has been constructed on the given finite straight line AB . LDef.20
Q.E.F

Text and figure by David Joyce <https://mathcs.clarku.edu/~djoyce/elements/bookI/propI1.html>

Postulates

1. Let it have been postulated¹ to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).¹

Common Notions

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.¹
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part.

15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.

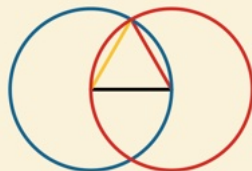
20. And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having three unequal sides.

Proposition 1: Given any finite segment, it is possible to construct an equilateral triangle having that segment as a side.

PROPOSITION I. PROBLEM.



*Q*n a given finite straight line (—) to describe an equilateral triangle.



Describe (—) and (—) (postulate 3.); draw — and — (pof. 1.). then will be equilateral.

For — = — (def. 15.);

and — = — (def. 15.);

∴ — = — (axiom. 1.);

and therefore is the equilateral triangle required.

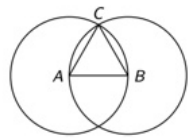
Q. E. D.

Byrne's Euclid

Is every step
appropriately
justified in
Proposition I.1

Proposition 1: Given a segment, it is possible to construct an equilateral triangle that has this segment as one of its sides.

- Denote the segment by AB .
- Draw a circle with center A and distance AB . (Post.3)
- Draw a circle with center B and distance AB . (Post.3)
- Let C be a point where the circles intersect.
- Draw segments AC and BC . (Post 1)



• Claim: ABC is the required triangle.

- $AC = AB$, since both are radii of the circle centered at A (Definition 15).
- $BC = AB$, since both are radii of the circle centered at B (Definition 15).
- $AC = BC$, since both equal AB (steps a, b, and Common notion 1).
- By a, b and c and Definition 20, ABC is an equilateral triangle with AB as one of its sides.

Is every step appropriately justified?

Post 1: A straight line can be drawn from any point to any other point.

Post.3: A circle can be drawn with any center and any radius.

Common notion 1: Things that are equal to the same thing are also equal to one another.

Definition 15: A circle is a plane figure whose boundary is a single [curved] line, such that all straight lines drawn from one interior point to the boundary are equal.

Definition 20: If all three sides are of the same length, then a triangle is **equilateral**.

- Let M be the midpoint of AB
- Let P be the intersection of the perpendicular bisector of AB and the angle bisector at C
- Let U be the foot of the perpendicular from P to line AC
- Let V be the foot of the perpendicular from P to line BC

Claim: The following triangles are congruent.

1. $\triangle AMP \equiv \triangle BMP$

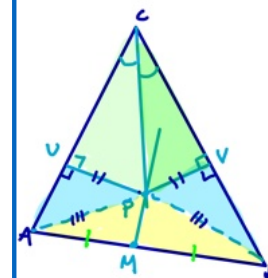
- $AM=BM$ (M is midpoint of AB), $MP=MP$ (common side), angles AMP and BMP are right angles. (SAS)

2. $\triangle PCU \equiv \triangle PCV$

- $PC=PC$ (common side), angle $PCU =$ angle PCV (P lies on the angle bisector at C), angles PUC and PVC are right angles. (AAS)

3. $\triangle APU \equiv \triangle BPV$

- $AP=BP$ (from 1.), $PU=PV$ (from 2.), angles PUA and PVB are right angles. (HL)



Find a statement proven by this argument

Are all triangles isosceles?

Lect 3

Join the lesson at www.geogebra.org/classroom with the code:

ZCH6 CVEE

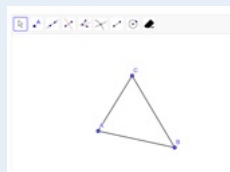
Or you can also share the following link with your students:

www.geogebra.org/classroom/zch6cvee



<https://www.geogebra.org/m/uggnvm6x>

<https://www.geogebra.org/classroom/zch6cvee>

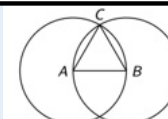


Follow the instructions in the app. Answer in Wooclap:
Based on your GeoGebra construction, what is the flaw in this 'proof' that all triangles are isosceles?

What can you conclude from the "all triangles are isosceles" fallacy?

Proposition 1: Given a segment, it is possible to construct an equilateral triangle that has this segment as one of its sides.

- Denote the segment by AB .
- Draw a circle with center A and distance AB . (Post.3)
- Draw a circle with center B and distance AB . (Post.3)
- Let C be a point where the circles intersect.
- Draw segments AC and BC . (Post 1)



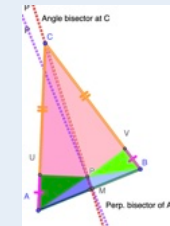
• Claim: ABC is the required triangle.

- $AC = AB$, since both are radii of the circle centered at A (Definition 15).
- $BC = AB$, since both are radii of the circle centered at B (Definition 15).
- $AC = BC$, since both equal AB (steps a, b, and Common notion 1).
- By a, b and c and Definition 20, ABC is an equilateral triangle with AB as one of its sides.

Common notion 1: Things that are equal to the same thing are also equal to one another.

Post 1: A straight line can be drawn from any point to any other point.

Post.3: A circle can be drawn with any center and any radius.



Definition 15: A circle is a plane figure whose boundary is a single [curved] line, such that all straight lines drawn from one interior point to the boundary are equal.

Definition 20: If all three sides are of the same length, then a triangle is **equilateral**.

Proposition 1 of Euclid's Elements. Construction of equilateral triangle

Bad news: There is a gap in the proof of Proposition I.1:
Euclid did not prove that the circles must intersect (remember that one can not "prove by drawing")

How do we deal with this issue nowadays:

Following **Hilbert**, modern geometry adds a **Completeness Axiom**, similar to an axiom you may know from Analysis.

Axiom of Completeness (for a line) which states.

If all points of a straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division.



Number theory in Euclid

There are two distinct types of "quantities":
the continuous (magnitude)

*A quantity is a **magnitude** if it can be measured, and a magnitude is what can be divided into continuous parts. It is a length if continuous in one dimension, a breadth in two, and a depth in three.*

Example: lines, surfaces, bodies and time.
the discrete (number)

***Number** is a plurality measurable by the one, so the one is the measure and number is what is measured.*

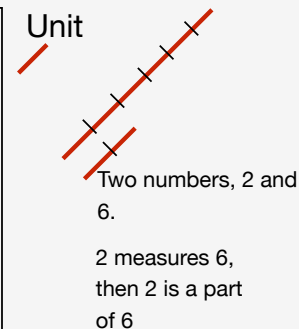
Examples 2, 3, 27 ...

Aristotle (~350 BC)



Book VII. Some Definitions

- A **unit** is (that) according to which each existing (thing) is said (to be) one.
- A **number** is a multitude composed of units.
- A number is **a part of a number**, the less of the greater, when **it measures the greater**;
- A **prime number** is that which is measured by a unit alone.
- Numbers **relatively prime** are those which are measured by a unit alone as a common measure.
- A **composite number** is that which is measured by some number.



Proposition IX.20: The prime numbers are more than any collection of prime numbers.

Which statement best captures Euclid's meaning? Choose the statement that matches Euclid's exact reasoning, not just the modern summary. Explain your choice.

- a) There are infinitely many primes
- b) For any finite list of primes, there exists a prime not on that list
- c) No finite collection contains all primes
- d) There are more primes than composites.

Euclid's Elements

Proposition IX.20: The prime numbers are more than any collection of prime numbers.

What "collection" meant to Euclid

Collection here means a finite group of numbers — Greek mathematicians did not conceive of completed infinite collections.

Euclid never uses the word "infinite"

Two ways of thinking infinity

*The infinite has a **potential** existence but not an **actual** one. It exists in the sense that one thing after another can be taken without end. Aristotle*

Potential infinity: The property that there is always something beyond any given quantity.

Actual infinity: The idea that an infinite collection exists as a completed whole.

Euclid never uses the word "infinite"

Consider the primes $A=7$, $B=11$ and $C=13$.
Define $E = A \cdot B \cdot C + 1$.

G = the smallest prime that divides E .

What is G ? (Hint: Hint: You can find E without computing. Think of even and odd numbers)

Proposition IX.20: The prime numbers are more than any collection of prime numbers.

(Modern) Proof: Assume that p_1 , p_2 , and p_3 are prime.

Suppose that the primes A, B, C are 2, 3, and 5. Let N be as above and let G be the largest prime that divides N . What is G ?

Euclid's Elements

This proposition is not used in the rest of the Elements.

P1	P2	P3	P1.P2.P3+1	FACTORS	FACTORS	FACTORS	FACTORS	FACTORS
2	3	5	31	31				
3	5	7	106	2	53			
5	7	11	386	2	193			
7	11	13	1002	2	3	167		
11	13	17	2432	2	19			
13	17	19	4200	2	3	5	7	
17	19	23	7430	2	5	743		
19	23	29	12674	2	6337			
23	29	31	20678	2	7	211		
29	31	37	33264	2	3	7	11	
31	37	41	47028	2	3	3919		
37	41	43	65232	2	3	151		
41	43	47	82862	2	13	3187		
43	47	53	107114	2	7	1093		
47	53	59	146970	2	3	5	23	71
53	59	61	190748	2	43	1109		

Proposition IX.20: The prime numbers are more than any collection of prime numbers.

- Let A , B , and C be the assigned prime numbers.
- I say that there are more prime numbers than A , B , and C .
- Take the least number DE measured by A , B , and C . Add the unit DF to DE .
- Then EF is either prime or not.
- Let it, first of all, be prime. Thus, the (set of) prime numbers A , B , C , EF , (which is) more numerous than A , B , C , has been found.
- And so let EF not be prime.
- Thus, it is measured by some prime number [Prop. VII.31]. Let it be measured by the prime (number) G .



- I say that G is not the same as any of A, B, C. For, if possible, let it be (the same).
- And A, B, C (all) measure DE.
- Thus, G will also measure DE.
- And it also measures EF.
- (So) G will also measure the remainder, unit DF, (despite) being a number [Prop. 7.28]. The very thing (is) absurd. Thus, G is not the same as one of A, B, C.
- And it was assumed (to be) prime. Thus, the (set of) prime numbers A, B, C, G, (which is) more numerous than the assigned multitude (of prime numbers), A, B, C, has been found. (Which is) the very thing it was required to show.

Euclid's Elements

Proof of Pythagorean Theorem

The structure of the proportions in Euclid's Elements leading to the Pythagorean Theorem (Proposition 47)

The proposition in Euclid's Elements Book I, arranged in logical progression

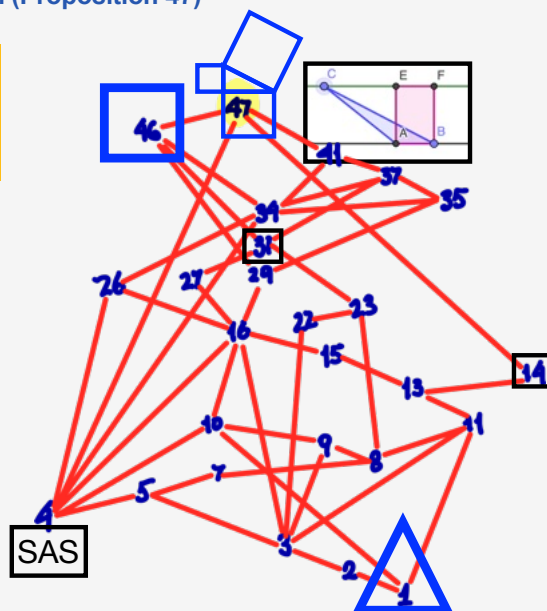
I.4 (SAS): Side angle side.

I.14: If two adjacent angles add up to 180° , their non-shared sides form a straight line.

I.31: Given a line and a point not on it, construct a line through that point parallel to the given line.

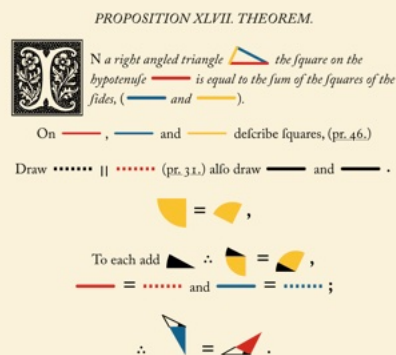
I.41: A parallelogram and a triangle with the same base and between the same parallel lines have areas in the ratio 2:1.

I.46: Given a line segment, construct a square having that segment as one of its sides.

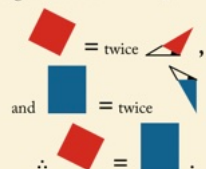


Euclid's Elements Proof of the Pythagorean Theorem

Byrne's Euclid



Again, because — ||



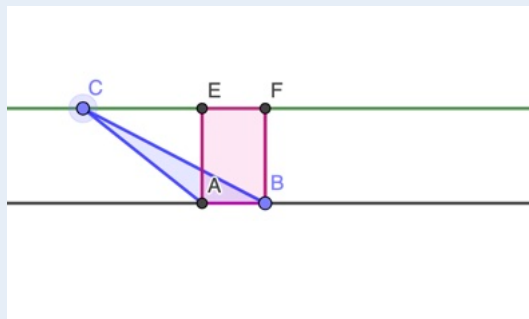
In the same manner it may be shown



Q. E. D.

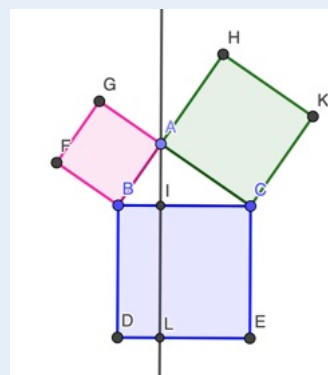
What is the ratio of the areas of triangle ABC and rectangle EFBA?

Does this ratio change when you drag point C along the green line? Explain why or why not.



<https://www.geogebra.org/m/xyfcawtm>

One Step in the Proof of Pythagoras' Theorem

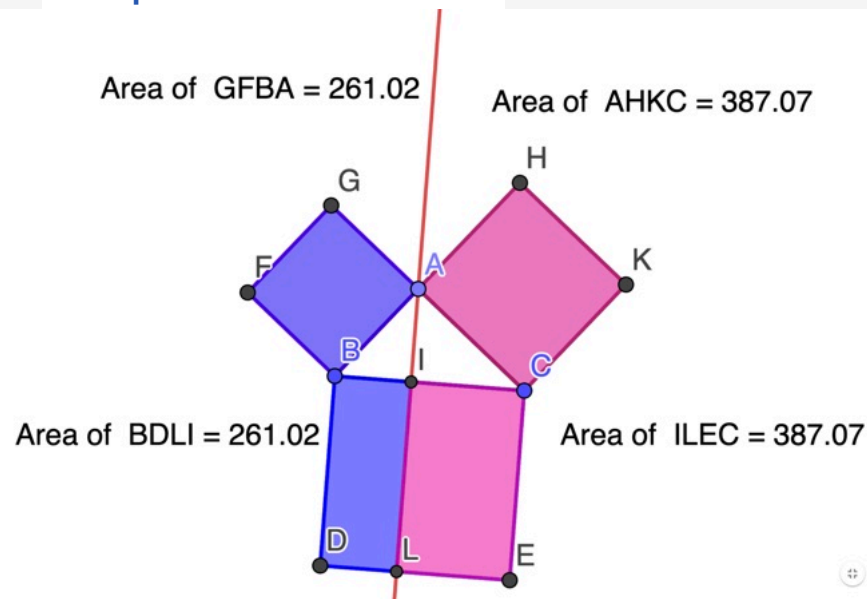


<https://www.geogebra.org/m/arbr5bxv>



Geogebra (QR Code)
Which pairs of regions have equal areas? Do these equalities still hold when you move points A, B and C?

Example



Euclid's Elements Proof of the Pythagorean Theorem: The statement

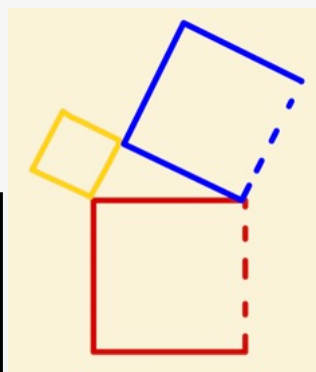
In a right angled triangle the square on the hypotenuse is equal to the sum of the squares of the sides, (and).

In a right angled triangle the square on the hypotenuse is equal to the sum of the squares of the sides, (and).

Euclid's Elements Proof of the Pythagorean Theorem:



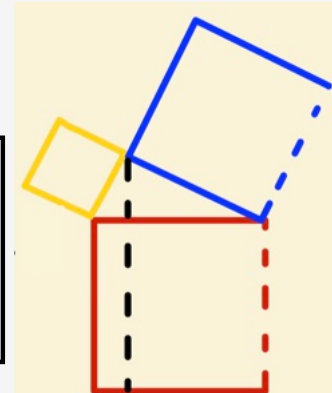
By Prop 46: Given a segment, it is possible to construct a square having that segment as one of its sides.



Euclid's Elements Proof of the Pythagorean Theorem:

Draw || (pr. 31.)

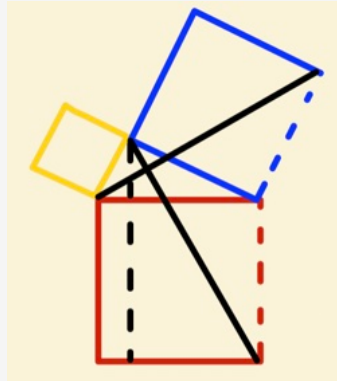
Proposition 31: Given a point and a line not passing through it, it is possible to construct a line through the point that is parallel to the given line.



Euclid's Elements Proof of the Pythagorean Theorem:

draw — and — .

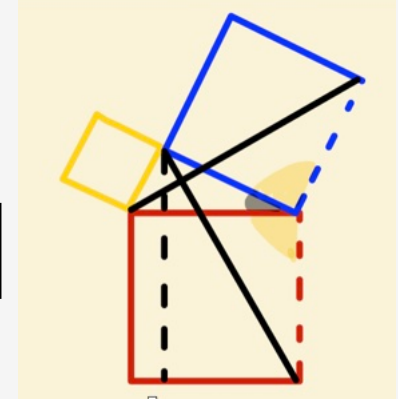
Post. 1: It is possible to draw a straight line from any point to any point



Euclid's Elements Proof of the Pythagorean Theorem:



Postulate 4: All all right angles equal one another.

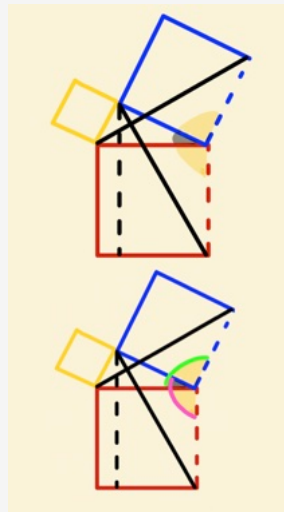


Euclid's Elements Proof of the Pythagorean Theorem:



To each add $\triangle$ $\therefore$ $\triangle = \triangle$,

Common notion 2: If equals are added to equals, the results are equal.



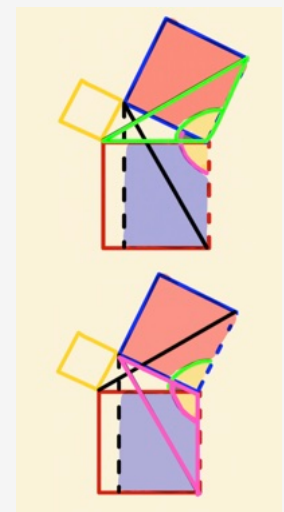
Euclid's Elements Proof of the Pythagorean Theorem:



To each add $\triangle$ $\therefore$ $\triangle = \triangle$,
 $\text{---} = \text{---}$ and $\text{---} = \text{---}$;

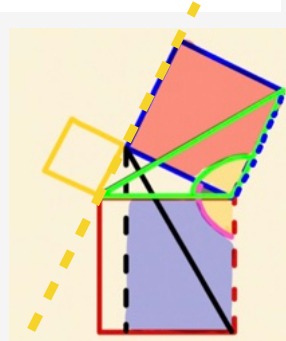
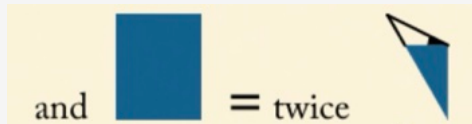
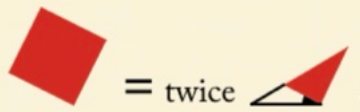


Prop. 4. SAS



Euclid's Elements Proof of the Pythagorean Theorem

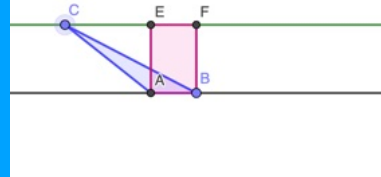
Again, because $\text{---} \parallel \text{---}$



Proposition I.41

If a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle.

Recall

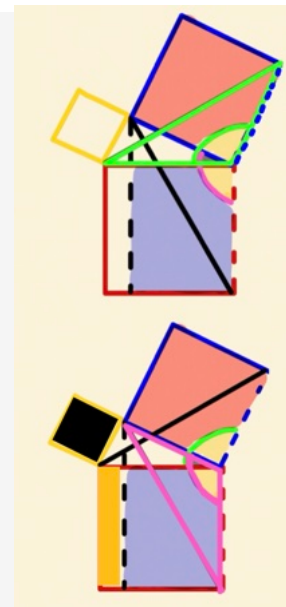


Euclid's Elements Proof of the Pythagorean Theorem

In the same manner it may be shown

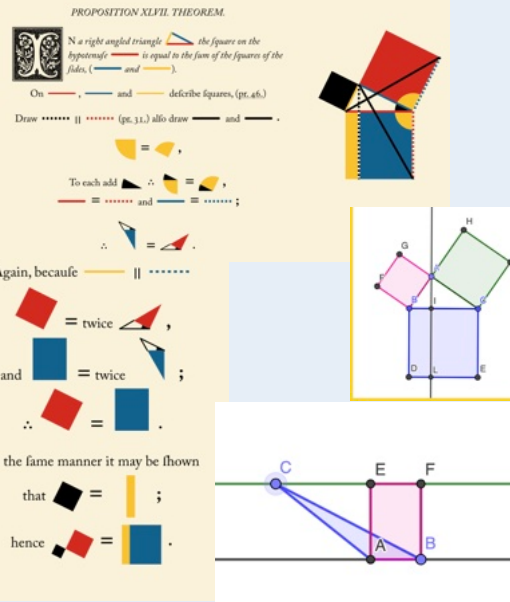
that $\blacksquare = \text{---}$;

hence $\blacksquare = \text{---}$.



Euclid's Elements Proof of the Pythagorean Theorem

Describe how the Pythagorean Theorem is proved in Euclid's Elements using the relationship between triangles and rectangles, and the use of congruent triangles.



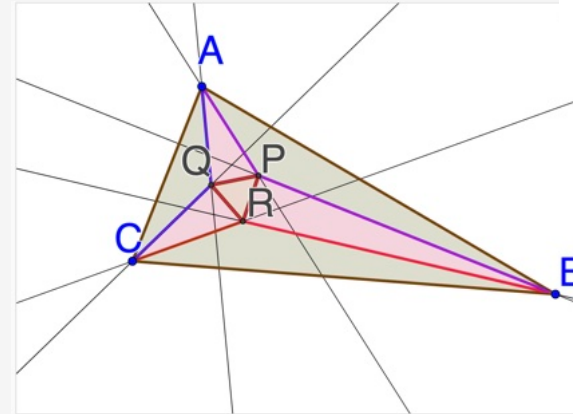
Byrne's Euclid

More About Euclid's Elements

Are there statements that follow from Euclid's postulates and common notions that are not proved in the Elements? Explain.

Morley's trisector theorem, 1899

<https://ggbm.at/FRZ9Nfec>



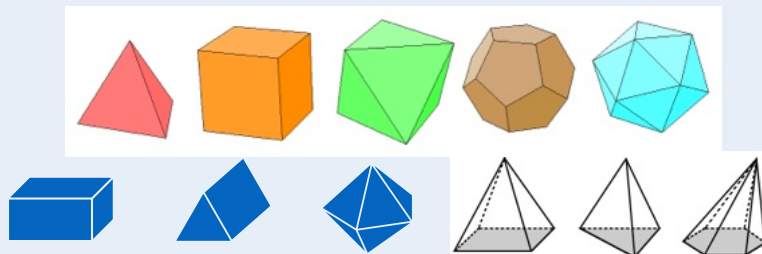
Compute V, E, F and then

$$V - E + F$$

for two or more of the polyhedra below.

What value do you get?

Do you think this will always give the same value for *any* polyhedron? (V is the number of vertices, E is the number of edges, and F is the number of faces.)



Euler's Polyhedral Formula

In a (convex) polyhedron, the following equation holds: $V - E + F = 2$, where V is the number of vertices, E is the number of edges, and F is the number of faces. Descartes (~1600), Euler (~1700)



Full length portraits of Euclid and Ptolemy, each holding a compass; a sphere on a staff is between them.
- Cardano, Girolamo, 1501-1576.

According to Proclus, Ptolemy (the pharaoh) once asked Euclid if there was not a shorter road to the knowledge of geometry than by the study of the Elements, and Euclid replied:

There is no royal road to geometry.



The Philosophers (Ptolemy and Euclid with Their Pupils) Pietro della Vecchia - 1600

What do you think Euclid meant by saying 'There is no royal road to geometry'?

Do you think there's a 'royal road' to geometry today that didn't exist in Euclid's time?

Ptolemy



Tetradrachm of Ptolemy I, British Museum, London

Most of the classical theorems of geometry in 2 and 3 dimensions, as well as many results in number theory are deduced from the axioms in Euclid's Elements. 🤔!!!



Max Ernst, 'Euclid'. 1945. - Fair Use

Lincoln and Euclid's Elements



In Euclid's Elements, how is the statement 'things equal to the same thing are equal to each other' treated?

Clip from the movie Lincoln, 2012.

https://youtu.be/xzc4neprr_c https://youtu.be/xzc4neprr_c

LINCOLN: You're an engineer, you must know Euclid's axioms and common notions.

HOMER BATES: I must've in school, but...

LINCOLN: I never had much of schooling, but I read Euclid, in an old book I borrowed. Little enough ever found its way in here (touching his cranium), but once learnt it stayed learnt.

Euclid's first common notion is this: "Things which are equal to the same thing are equal to each other."

That's a rule of mathematical reasoning. It's true because it works; has done and always will do. In his book, Euclid says this is "self-evident." (A BEAT) D'you see? There it is, even in that two-thousand year old book of mechanical law: it is a self-evident truth that things which are equal to the same thing are equal to each other. We begin with equality. That's the origin, isn't it? That balance, that's fairness, that's justice.

<https://youtu.be/xzc4neprr> https://youtu.be/xzc4neprr_c

Abraham Lincoln, recounted by Rev. Gulliver NYTimes, 1864.

NYTimes - <https://timesmachine.nytimes.com/timesmachine/1864/09/04/82014572.html?pageNumber=5>

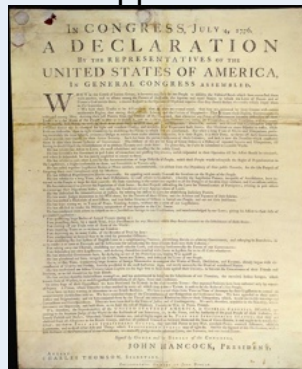
In the course of my law reading I constantly came upon the word "demonstrate". I thought at first that I understood its meaning, but soon became satisfied that I did not. I said to myself, What do I do when I demonstrate more than when I reason or prove? How does demonstration differ from any other proof? I consulted Webster's Dictionary. They told of 'certain proof,' 'proof beyond the possibility of doubt'; but I could form no idea of what sort of proof that was. I thought a great many things were proved beyond the possibility of doubt, without recourse to any such extraordinary process of reasoning as I understood demonstration to be. I consulted all the dictionaries and books of reference I could find, but with no better results. You might as well have defined blue to a blind man. At last I said, - **Lincoln, you never can make a lawyer if you do not understand what demonstrate means; and I left my situation in Springfield, went home to my father's house, and stayed there till I could give any proposition in the six books of Euclid at sight. I then found out what demonstrate means, and went back to my law studies.**

What motivated Lincoln to study Euclid?

Declaration of Independence of the Thirteen Colonies, Paragraph 2, beginning

We hold these truths to be self-evident: that all men are created equal; that they are endowed by their Creator with certain unalienable Rights; that among these are Life, Liberty and the pursuit of Happiness.

In what sense are the Declaration's "self-evident truths" similar to Euclid's axioms? In what sense are they different?



Self-evident truths: Euclid and the Declaration Of Independence

Euclid's Elements	Declaration of Independence
Start with postulates and common notions (axioms): statements accepted without proof. Derive theorems by logical reasoning.	Start with 'self-evident truths' (moral-political principles). Argue for the right to independence by a chain of reasons grounded on those principles.

The Declaration's appeal to 'self-evident truths' has often been read as modeling the axiomatic structure of Euclid's Elements.