

MAT 336 Hellenic Mathematics before Euclid's Elements.

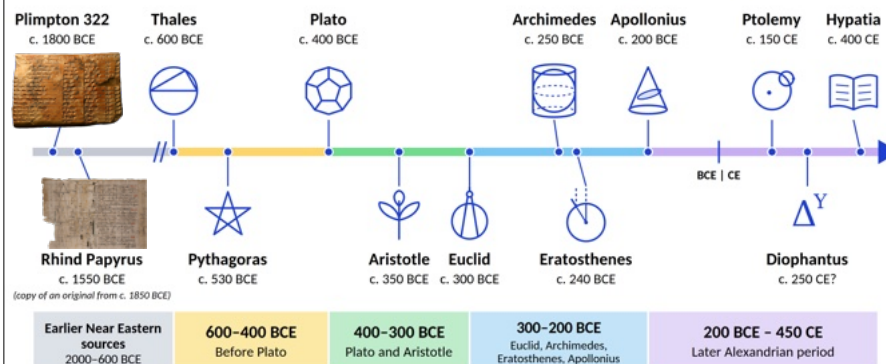


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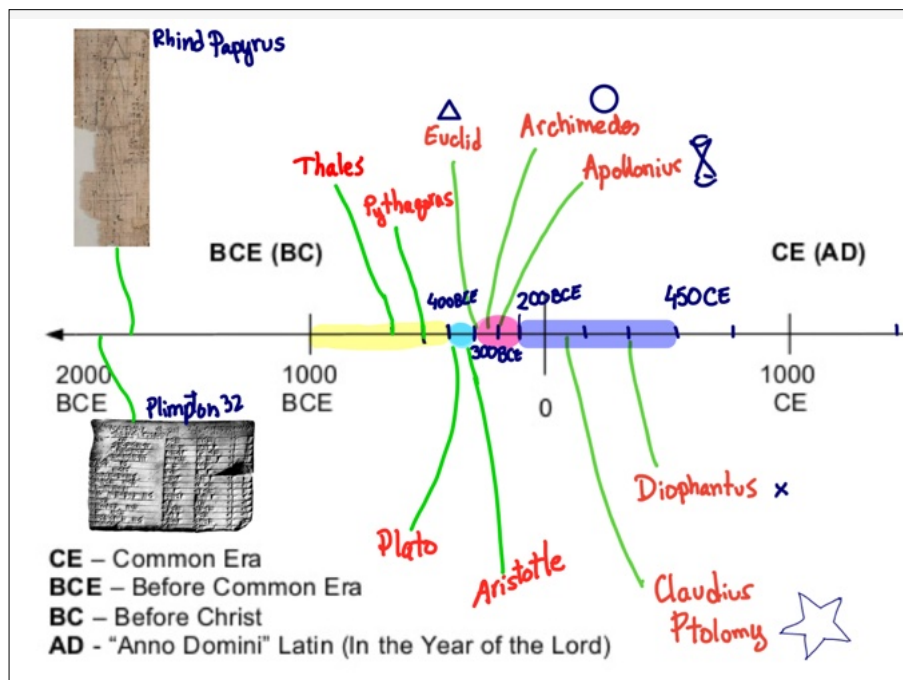
- Thales
- Hippocrates
- Quadrature Hippias

- Introduction
 - Rough timeline of Greek math
 - Map, what when being Greek then
 - Why the proofs were developed in Greece? Arguments
 - The school of Athens - Painting
 - Mesopotamian influence
- The Three Impossible problems of antiquity.
 - Cube doubling → Conic Sections.
 - Incommensurable magnitudes - constructible numbers
- Pythagoras/Pythagoreans
- Plato
- Aristotle
- Attitudes toward math in the Ancient Greek World.
- Zeno's paradoxes. (Parmenides?)

A Timeline of Greek Mathematics



Pythagoras (~600BC?) and the Pythagoreans



Pythagoras (~600BC?) - The Pythagoreans

- **No primary sources**, only accounts written centuries after Pythagoras would have lived.
- Pythagoras probably existed, but almost nothing reliable is known about his life or mathematics.
- The Pythagoreans form a very secretive sect.
 - The mathematical and the mystical were merged.
 - They believed that numbers reveal hidden structure in nature (music, geometry, astronomy).

No primary sources

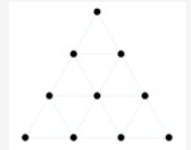
All things are number



Doll by UneekDollDesigns

Pythagorean “mystical” views of number

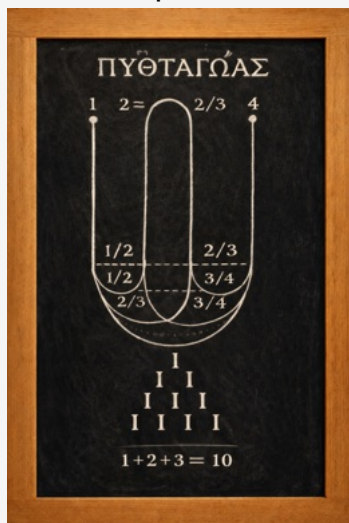
- Number rules the universe.
- **1** = source of all things; neither even nor odd.
- Odd = masculine;
- even = feminine;
- **5** = marriage (**2 + 3**).
- **7** considered sacred: “seven planets,” seven strings on a lyre; Apollo’s day = the 7th.
- **10** called “perfect” because $1 + 2 + 3 + 4 = 10$.
Never meet in groups larger than 10!



Pythagorean Mathematical Music

Whole-number ratios → musical consonance

ChatGPT interpretation



Pythagoras by Raphael in The School of Athens



Pythagorean Mathematical Music

According to a legend (reported much later than Pythagoras, around 200 CE.), Pythagoras discovered the numerical ratios underlying musical consonances after hearing blacksmiths' hammers and investigating their pitches.

- The connection between music and numerical ratios (e.g.,
 - octave = 2:1,
 - fifth = 3:2,
 - fourth = 4:3)was associated with the Pythagorean tradition.
- Related tuning practices predate or extend beyond the Greek and the Pythagoreans.

Pythagoras by Raphael in The School of Athens



The Harmony of the Spheres

5th c. BCE, Pythagoreans: heavenly bodies make sounds as they move, their speeds in musical ratios. No Pythagorean text on this survives; the idea is known through Aristotle.

c. 375 BCE, Plato, *Republic*: a siren on each sphere sings one note. The *Timaeus* builds the world soul from musical ratios.

c. 350 BCE, Aristotle: reports the Pythagorean view, rejects it.

51 BCE, Cicero, *Dream of Scipio*: the spheres make sound; humans are deafened to it.

c. 500 CE, Boethius: *musica mundana*, taught through the Middle Ages.

c. 1597, Shakespeare, *The Merchant of Venice*: our souls hold the harmony; our bodies keep us from hearing it.

1619, Kepler, *Harmonices Mundi*: a "melody" for each planet from its angular speeds at perihelion and aphelion.

1620s–30s, Milton: Latin oration "On the Music of the Spheres"; the image returns in "On the Morning of Christ's Nativity" (1629).

1957, Hindemith, *Die Harmonie der Welt*: an opera about Kepler. By then the phrase is only a metaphor.

Aristotle says the Pythagoreans took the principles of mathematics to be "the principles of all things". Does today's science agree?

Where yes, where no?

Hint: music/acoustics, physical laws as equations, data.

..., the so-called Pythagoreans, who were the first to take up mathematics, not only advanced this study, but also having been brought up in it they thought its principles were the principles of all things.

(...) and numbers seemed to be the first things in the whole of nature, they supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number.

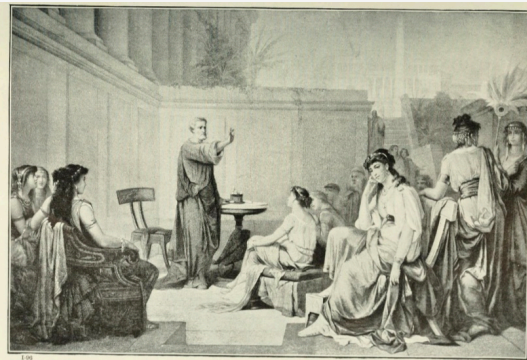
Aristotle, Metaphysics

https://people.bu.edu/wildman/WeirdWildWeb/courses/wphil/readings/wphil_rdg05_metaphysics_entire.htm

The Pythagoreans

Number rules
the universe.

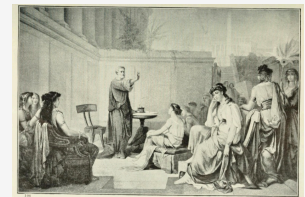
Great influence on
scientists from then on



Pythagoras teaching: Illustration from "The story of the greatest nations", (1910) [Internet Archives](#)

Modern science → mathematics *describes* nature;
the Pythagoreans → nature *is* number.

The Pythagoreans and Science

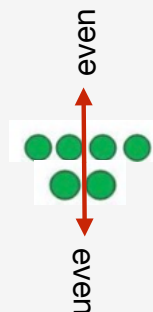


Pythagoras teaching: Illustration from "The story of the greatest nations", (1910) [Internet Archives](#)

Pythagoreans	Later science
Harmonious intervals = simple ratios: 2:1, 3:2, 4:3	Acoustics: the same ratios, as frequencies
The heavens as a musical scale	Kepler: Platonic solids universe model failed period and distance to Sun relation ($T^2 \propto a^3$) held
10 perfect, 7 sacred, 5 marriage	No meanings attached to numbers
Diagonal and side of a square: no common measure ($\sqrt{2}$ incommensurable with 1)	$\sqrt{2}$ another real number
Nature <i>is</i> number	Mathematics <i>describes</i> nature

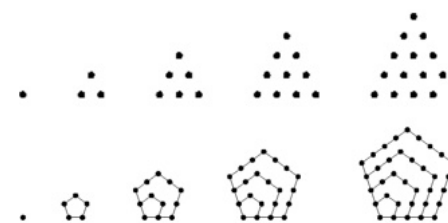
The Pythagoreans

- Recall Dichotomy between odd and even.
- Pythagoreans probably represented numbers with pebbles.
 - Example:** A number is **even** if it can be represented by a configuration of pebbles that than be divided into two equal parts. Otherwise is odd. (Well, 1 was not considered odd, nor even)
 - Proof: $4+2$ is even.
 - Proof: An even sum of odd numbers is even. (Similar to the proof of $4+2$ is even.)



The Pythagoreans - Figurate numbers

The *gnomon* is the piece of the figure that needs to be added to a given figurate numbers in order to get the next greater figurate number.



- What is the 6th triangular number?
- (If you have time) What is the 6th pentagonal number?
- (Question for you to think about later) Can you find a formula for the n-th triangular number? What about the n-th pentagonal number?

The Pythagorean theorem and the attribution to Pythagoras

Possible path of attribution:

verse about “famous diagram” →
 3–4–5 rule →
 hypotenuse theorem,
 amplified by a later tradition that increasingly credited discoveries to Pythagoras.

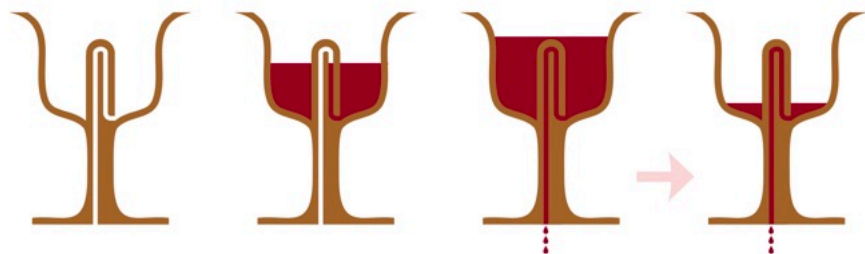
- c. 1800 BCE, Mesopotamia:** the [rule](#) in use. No proof.
- c. 300 BCE, Euclid I.47:** first extant [theorem](#), proved deductively from axioms
- Probably 4th–1st c. BCE:** A verse attributed to “Apollodorus” has Pythagoras sacrifice oxen for a “famous diagram.”
- 1st c. BCE:** Vitruvius attributes to Pythagoras the 3–4–5 [rule](#); Cicero doubts the oxen story.
- c. 100–250 CE:** Plutarch is unsure which discovery is related to the oxen story; Diogenes Laertius identifies the discovery with the hypotenuse [theorem](#).
- 5th c. CE, Proclus:** reports the attribution to Pythagoras, but does not credit him with a proof. Contrasts with Euclid’s proof of the [theorem](#).
- Renaissance:** Pythagorean legends flourish, but Pythagoras is not yet primarily known for this [theorem](#).

- 1) What does the Pythagorean theorem story suggest about how mathematical ideas acquire names and credit? Use one detail from what we just saw.
- 2) (optional) Does it matter who first discovered an idea? Why or why not? Who gets credit when an AI finds it?

Most scholars will agree that there was a Pythagorean school of philosophy from the sixth until probably the fourth century BC, that they were involved in politics and that they had certain beliefs about life and the universe, including perhaps the tenet that ‘everything is number’, or that number holds the key to understanding reality. But most scholars today also think, for instance, that Pythagoras never discovered the theorem that bears his name.

Cuomo, S. 2001. *Ancient Mathematics, Science of Antiquity*, Routledge.

Cross section of a Pythagorean cup being filled: at B, it is possible to drink all the liquid in the cup; but at C, the siphon effect causes the cup to drain



https://commons.wikimedia.org/wiki/File:Physagorian_Pythagoras_Greedy_Tantalus_cup_05.svg

French manuscript from 1512/1514, showing Pythagoras turning his face away from fava beans in revulsion



French early 16th century. <http://www.nga.gov/content/ngaweb/Collection/art-object-page.86044.html>

No beans for the Pythagoreans!

Authors discuss many reasons

- Beans were associated with souls of the dead.
- Beans were linked with generation and decay (and connected with life and reincarnation)
- beans were used in voting (white and black beans). Avoiding beans could symbolize avoiding politics or public life.

Ancient sources agree on one fact:
The Pythagorean community had a rule not to eat beans.

The Discovery of $\sqrt{2}$: Drama Among the Pythagoreans?

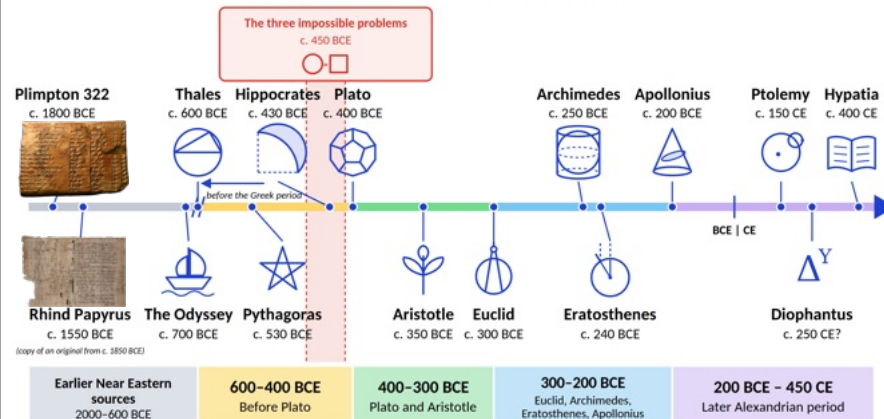
- **Discovery:** The incommensurability of the square's diagonal with its side was likely discovered in the 5th century BCE.
- **Hippasus of Metapontum** (possibly 5th century BCE) is described by Aristotle as an early Pythagorean, though not in connection with incommensurability.
- **The legend:** Later writers (3rd–4th century CE) claim that Hippasus revealed a Pythagorean secret—sometimes linked to incommensurability—and was punished by drowning.
- **Historically unverified:** the sources for the legend are late, inconsistent, and not considered reliable evidence.

Hippasus of Metapontum

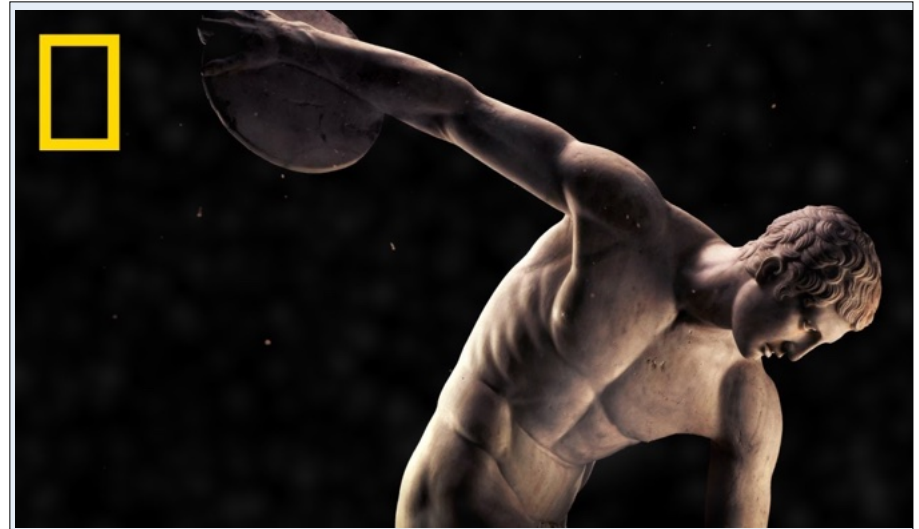


Hippasus, engraving by Girolamo Olgiati, 1580

A Timeline of Greek Mathematics

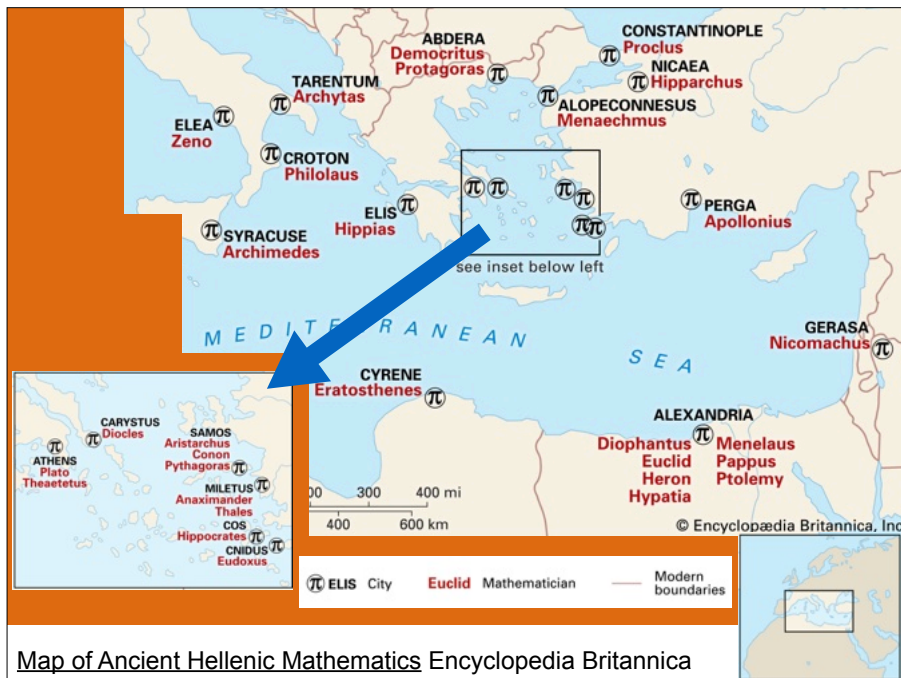


Hellenic Mathematics: Timeline, context, main ideas and sources Proof, Persuasion, Mathematical Reasoning

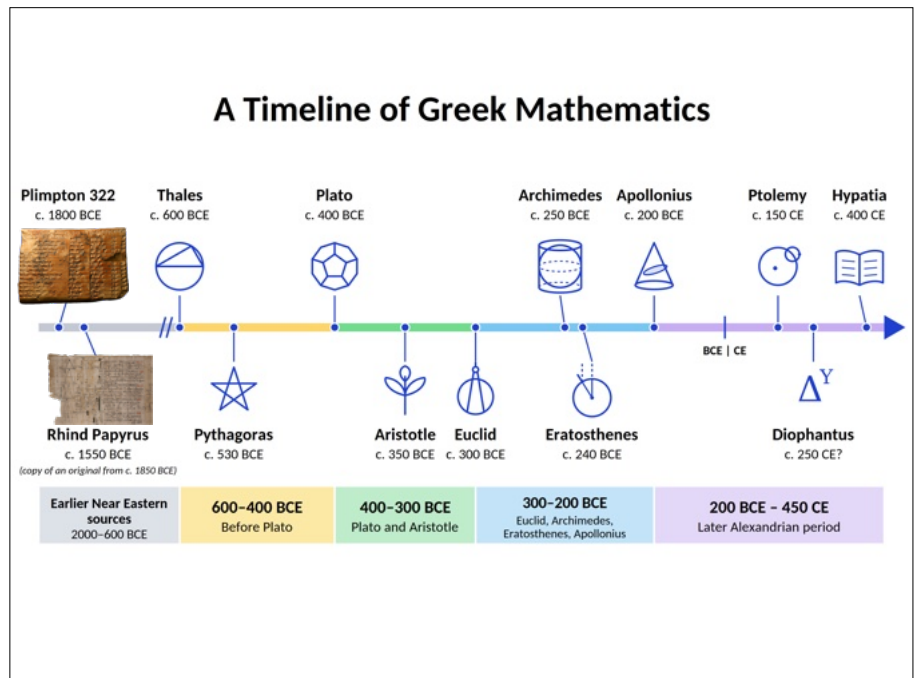


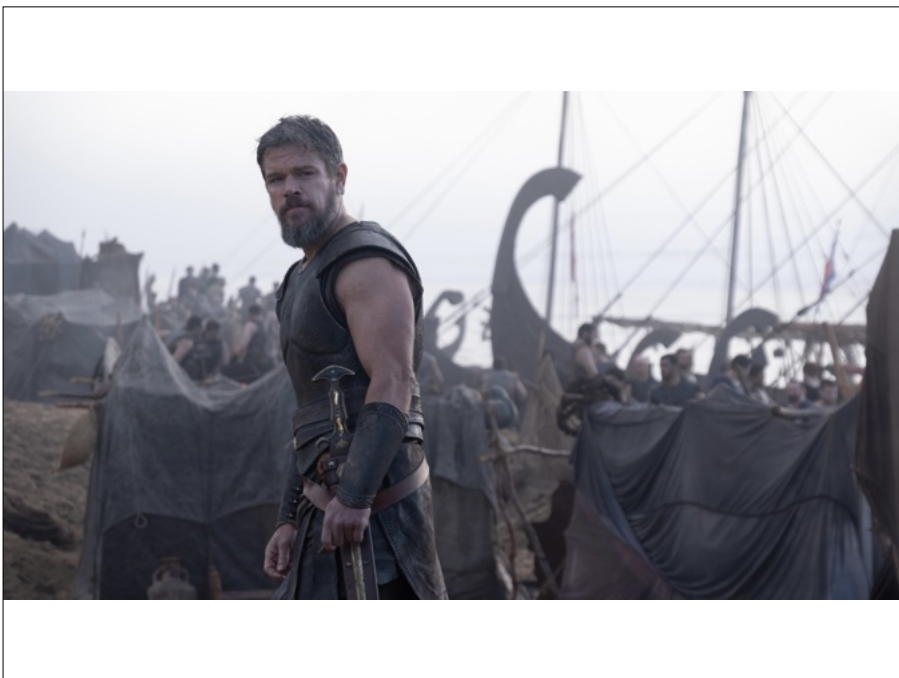
Ancient Greece 101 | National Geographic, <https://youtu.be/5bDrYTXQLu8>

1. What did the video emphasize? Get concrete answers (e.g. philosophy)
2. (If you have time) What picture of ancient Greece is this video constructing? What does it include, omit, or simplify?



Map of Ancient Hellenic Mathematics Encyclopedia Britannica





Odyssey 4.411–413

The verses tell how Proteus, the shape-shifting old man of the sea, counted his seals by fives:

*“First he will enumerate the seals and pass along them; but when he has **counted them all by fives** and inspected them, he will lie down among them, like a shepherd among his flocks.”*

The word πεμπάζειν (“to five”), used by Homer (Od. iv. 412) in the sense “to count,” would appear to be a relic of a base 5 system of reckoning.



Menelaus Binding Proteus - Giulio Bonasone - Italian, 1498 - 1580
<https://www.philamuseum.org/objects/24946>

Greek or Hellenic Mathematics (Between ~600 BCE and 300 AD)

- In mathematics, the question
“**why?**”
- was now asked together with the older question
“**how?**”

Greek or Hellenic Mathematics (Between ~600 BCE and 300 AD) Primary Sources

- No tablets or papyri or ...
- Only copies of copies of copies of...
 - Restored versions of texts of Euclid, Archimedes, Apollonius and other.
- To study how this mathematics was developed we rely on these copies and remarks of writers of the time.

Greek or Hellenic Mathematics

Two important aspects

1. Emphasis on proofs
2. Involvement with specific, challenging problems.
 - need to decide which new arguments counted as genuine proof
 - in this way, the problems led to
 - new discoveries
 - increasingly sophisticated ideas about proofs

Public life - Sound arguments

Mathematics was a public activity:

- played out **in front of an audience**.
- fulfilled functions that were **significant at a communal level**,
 - counting revenues,
 - measuring out land or
 - exploring the limits of persuasive speech.

Cuomo, Serafina. Ancient mathematics. Routledge, 2001.

G. E. R. Lloyd argued that debate in assemblies, courts and the agora fostered the demand for rigorous argument, and eventually for proof.

G. E. R. Lloyd, Magic, Reason and Experience, 1979



By Louis Loeb - <http://www.alexanderstomb.com/>, Public Domain, <https://commons.wikimedia.org/w/index.php?curid=649344>

The paragraph below (by S. Cuomo) discusses a trial

The jurors were called upon to be not just witnesses, but also calculators – in principle, a *logistes* [public auditor of accounts] could tell with absolute certainty whether the account presented to him worked out or not, whether everything added up. It was this kind of persuasiveness, mathematical persuasiveness, that both speakers [the two opposing sides] wished to claim for their arguments.

The author continues explaining:

(...) Along with its practical functions, public counting was associated with political accountability, and in fourth-century legal speeches it seems increasingly to symbolize the role itself of the Athenian citizen.

Cuomo, Serafina. Ancient mathematics. Routledge, 2001.

Groups of 3–4. Produce one written answer per question.

1. What were the jurors using calculation to decide? Why could calculation make an argument more persuasive?
2. What does this tell you about the role of mathematics in ancient Greece?

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Takeaway 1: In the trial

- Jurors checked whether each side's accounts added up.
- A calculation could persuade because any citizen could check it.

Takeaway 2: Mathematics in Athens

- Some mathematical activity was public: ordinary citizens checked accounts.
- Counting was tied to political accountability and citizenship.
- Mathematical certainty carried weight in public argument, which is one possible root of Greek proof.

Compare

- Mesopotamia: much calculation done by trained scribes, for administration.
- Athens: calculation checked by citizens, in public.
- In both: mathematics could support accountability and authoritative decisions.

Mesopotamian Influence on Greek Mathematics - Current Scholarly Understanding

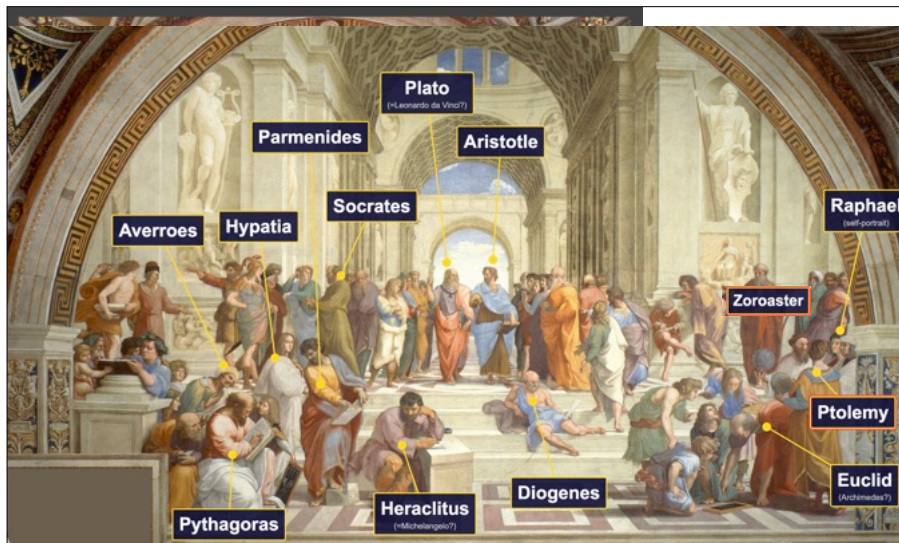
- Greek mathematics did not develop in isolation; it emerged within a broader eastern Mediterranean intellectual world.
- Greek mathematical astronomy shows clear dependence on Babylonian numerical methods.
- Near Eastern computational techniques likely circulated widely before the classical Greek period.
- Certain problem traditions (e.g., algebraic-type problems) likely moved across cultures.



Vigorous Debates

The school of Athens, by Raffaele

Podcast: <https://www.bbc.co.uk/programmes/b00j7txt>



You can place the labels The school of Athens, by Raffaele yourself in [this link](#)

The School of Athens - Raphael, 1509–1511, Vatican

- The painting gathers thinkers from different centuries into one imagined scene.
- The setting is imagined classical architecture inspired by Roman forms, not an actual Greek building.
- The composition is symmetrical, but the left and right sides differ in mood and activity.
- **The vanishing point is between Plato and Aristotle.**
- Two views of reality. Plato points up (ideal forms); Aristotle holds his hand flat (the observable world). Plato holds the *Timaeus*, the dialogue that contains the Platonic solids.
- **The mathematicians are placed closest to the viewer:** Pythagoras (left) and Euclid (right) are prominently placed at the front. Raphael may be making a statement about the status of mathematics.
- A tablet near Pythagoras includes numerical ratios and a harmonic diagram.
- The man bending with a compass is usually identified as Euclid (sometimes Archimedes). He is demonstrating a construction to students.
- Ptolemy (astronomer) holds a terrestrial globe.
- Students are physically gathered around the mathematicians, emphasizing teaching.
- **Mathematics is depicted as active, demonstrative, and teachable.**
- **Question: Who is missing?**

Numbers as a Gift from the Gods

Prometheus at Rockefeller Center, sculpture by Paul Manship, 1933



Paul Manship, Public domain, via Wikimedia Commons

Prometheus Bound By Aeschylus

PROMETHEUS: Yes, and from it [the fire] they'll learn many arts [...] hear what wretched lives people used to lead, how babyish they were – until I gave them intelligence, **I made them masters of their own thought.** [...] they knew nothing of making brick-knitted houses the sun warms, nor how to work in wood. They swarmed like bitty ants in dugouts in sunless caves. They hadn't any sure signs of winter, nor spring flowering, nor late summer when the crops come in. All their work was work without thought, until I taught them to see what had been hard to see: where and when the stars rise and set. **What's more, I gave them the numbers, chief of all the stratagems.** And the painstaking, putting together of letters: to be their memory of everything, to be their Muses' mother, their handmaid! [...]

Codex-Style Vessel CE 700–750.

Two teaching scenes with Itzam—an elderly creator/atlas-like deity—
instructing four students



<https://kimbellart.org/collection/ap-200404>

Itzam ID cues: Aged face; netted headdress with a brush tucked in.

Scene 1: Itzam uses a pointer and a folded codex; a speech thread with **bar-and-dot numerals suggests arithmetical/calendrical calculation.**

Scene 2: Itzam taps the ground while addressing two students; spoken glyphs on a speech thread.

Takeaway: Elite scribal education in mathematics/calendar lore framed as instruction from a creation deity.

Birds, by Aristophanes performed in 414 BC

A new city has to be founded from scratch. The main character, Peisthetaerus, is visited by various people who offer their services.

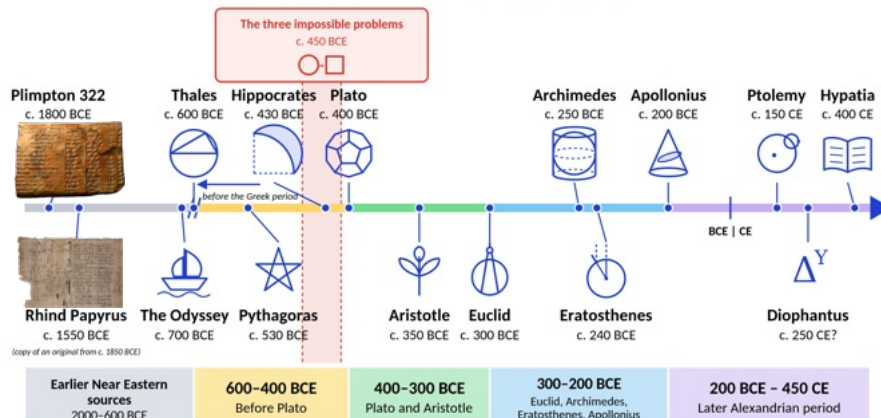
METON : With the straight rod I measure out, that **so the circle may be squared**; and in the centre a market-place; and streets be leading to it straight to the very centre; just as from a star, though circular, straight rays flash out in all directions.

PEISTHETAERUS : **Why, the man's a Thales!**

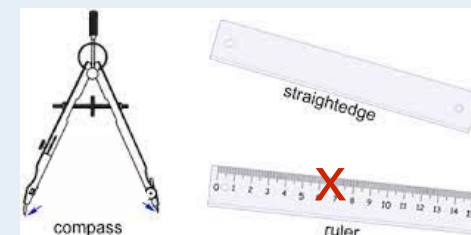
Translation Cuomo, Serafina. Ancient mathematics. Routledge, 2005.

Impossible problems of the Antiquity ~5th century BCE

A Timeline of Greek Mathematics



Greek geometers limited constructions to a straightedge and a compass. Why only these two tools? What's special about them? Hint: Think about geometry.

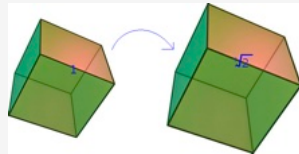


If cube C has side length 1 inch, what is the side length of a cube with twice the volume of C?

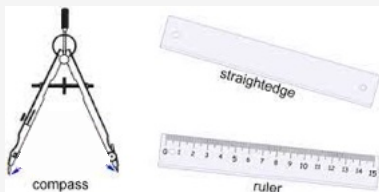
Doubling of the cube

Three “impossible” problems

Doubling a cube: Given a cube, **construct** another cube with twice its volume.



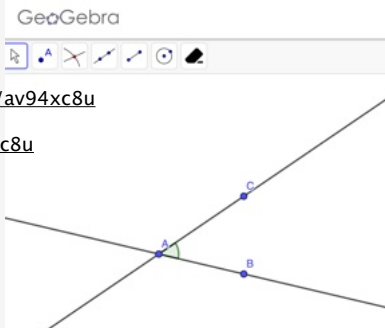
“Construct” means “construct using only straightedge and compass



These problems originated around 400 BCE

Trisection of the angle

Bisect the angle using only straightedge and compass. Hint: Use symmetry.



Classroom: <https://www.geogebra.org/m/av94xc8u>

App: <https://www.geogebra.org/m/av94xc8u>

Join the lesson at www.geogebra.org/classroom with the code:

YTNE C4WC

Or you can also share the following link with your students:

www.geogebra.org/classroom/ytne4wc



Three “impossible” problems



Trisecting an angle:

Given an angle between two straight lines, **construct** two lines that divide the angle into three equal parts.

“Construct” means “construct using only straightedge and compass



These problems originated around 400 BCE

Trisecting an angle

Constructing an angle α from the angle 3α

equivalent

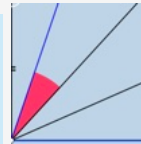
Constructing a segment of length $\cos(\alpha)$ from a segment of length $\cos(3\alpha)$.

Since $\cos(3\alpha) = 4 \cos^3(\alpha) - 3 \cos(\alpha)$.

Let $x = \cos \alpha$, $a = \cos 3\alpha$ (a is known/constructible).

Then

$$4x^3 - 3x - a = 0$$



Thus, trisecting an angle is equivalent to solving certain cubic equation

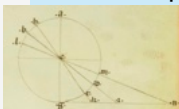


Diagram from 14th-century manuscript copy of Ptolemy's Almagest, folio 22 recto. Manuscript owned and digitized by gallica.bnf.fr / Bibliothèque nationale de France.

It is likely that the question of trisection of angles arose when trying to construct a table of chords for astronomical purposes. (A chord for the angle 3° can be constructed so it would have been natural to try to get a chord for the angle 1° from the chord for the angle 3°).

Quadrature of the circle

Quadrature of the rectangle: Construct (with only straightedge and compass) a square with same area as a given rectangle

Quadrature of the rectangle

Author: Moira Chas



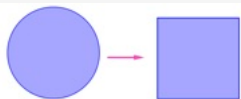
<https://www.geogebra.org/m/4dyz52jz>

If a circle has radius 1 inch, what is the side length of a square that has the same area as the circle?

Three “impossible” problems

Squaring a circle is closely related to finding its area

Squaring a circle: Given a circle, **construct** a square with the same area.

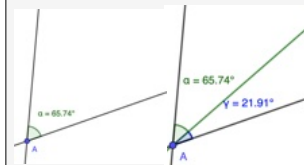
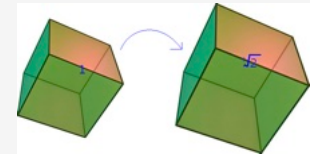


“Construct” means “construct using only straightedge and compass

These problems originated around 400 BCE

Three “impossible” problems

Doubling a cube: Given a cube, **construct** another cube with twice its volume.



Trisecting an angle: Given an angle between two straight lines, **construct** two lines that divide the angle into three equal parts.

“Construct” means “construct using **only straightedge and compass**

Squaring a circle: Given a circle, **construct** a square with the same area.



These problems originated around 400 BCE

The key mathematical idea behind impossibility

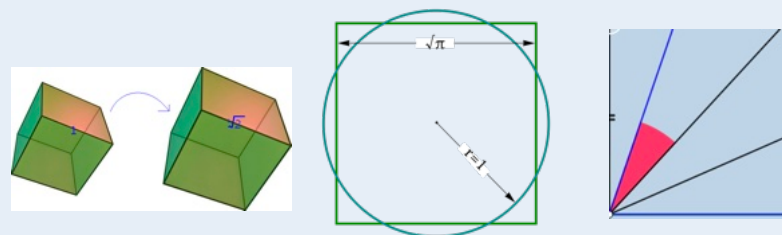
- **Constructible numbers** are those obtained by the combining the following operations
 - addition
 - subtraction
 - multiplication
 - division
 - square roots
- **Straightedge-compass constructions → only constructible numbers**
- The numbers π , $\sqrt[3]{2}$, and $\cos(\alpha)$ (for “most” angles α) are not constructible

In your opinion, which of the three classical problems was the hardest to prove impossible — and why?

The three problems are :

- Squaring the circle
- Trisecting an angle
- Doubling the cube

Using only straightedge and compass.



The mathematical problems that the Greeks tackled, and indeed the whole geometrical bias of so much Greek mathematics, both **stimulated and was stimulated by construction methods**, and the question naturally arose as to which geometrical problems can be solved by the basic **line-and-circle** constructions, and which ones cannot.

The History of Mathematics: A Source-Based Approach, Vol. 1, (2014) June Barrow-Green, Jeremy Gray, Robin Wilson.

Takeaway: Problems shaped the tools, and the tools shaped the problems.

Can you think of a practical technology that may trace back to line-and-circle constructions?

Hint: Think parabola, ellipse, triangles/triangulation.

Examples of practical applications

Constructions → conics (parabola / ellipse / hyperbola)

- Parabola → headlights, satellite dishes, solar ovens.
- Ellipse → planetary/satellite orbits; whispering galleries.
- **AI**: camera calibration and conic/ellipse detection in computer vision.

Proof culture → surveying / architecture / engineering

- Triangulation → land surveys, mapmaking, GPS logic.
- Geometry of structure → arches, trusses; constraints in CAD.
- **AI**: formal verification/specs for algorithms; constraint reasoning in planning and robotics.

Constructibility → algebra & number theory → real-world tech

- Public-key crypto (RSA, ECC) → HTTPS, banking, messaging (protects AI models/ data).
- Error-correcting codes → QR codes, Wi-Fi, storage (reliable datasets/training).
- **AI**: hashing/modular arithmetic for indexing, sharding, and fast retrieval.

Obsession with Proofs and Abstract constraints (line & circle) → powerful ideas
→ tools embedded in today's communications, navigation, engineering, AI, ...

What reason gave Plato to the Delians that for the oracle of the god?

In his work entitled *Platonicus* Eratosthenes says that, when the god announced to the Delians by oracle that to get rid of a plague they must construct an altar double of the existing one, their craftsmen fell into great perplexity in trying to find how a solid could be made double of another solid, and they went to ask Plato about it. He told them that the god had given this oracle, not because he wanted an altar of double the size, but because he wished, in setting this task before them, to reproach the Greeks for their neglect of mathematics and their contempt for geometry.

Theon of Smyrna

What reason gave Plato to the Delians that for the oracle of the god?

In his work entitled *Platonicus* Eratosthenes says that, when **the god announced to the Delians by oracle that to get rid of a plague they must construct an altar double of the existing one**, their craftsmen fell into great **perplexity** in trying to find how a solid could be made double of another solid, and **they went to ask Plato** about it. He told them that the god had given this oracle, not because he wanted an altar of double the size, but because he wished, in setting this task before them, to **reproach the Greeks for their neglect of mathematics and their contempt for geometry**.

Theon of Smyrna

The legend is doubtful but the **Delian problem was studied in Plato's Academy**.

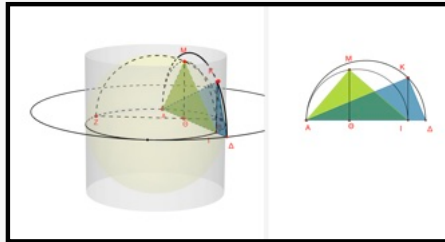
Plato insisted on an exact solution accomplished using only **ruler and compass**.

Suppose you can construct two mean proportionals between any lengths a and b . How could you construct $2^{1/3}$.

Hint: From the definition of double mean proportional, you get two equations. Combine them to eliminate x and find a relationship between y , a , and b . Then choose specific values for a and b .

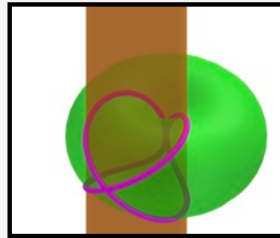
Double mean proportional: Given two lengths a and b , find x and y such that $a/x = x/y = y/b$

Reconstruction of Archytas method to find the double mean proportional I.



<https://www.geogebra.org/m/tg4b5hmu#material/QgpK29uA>

Reconstruction of Archytas method to find the double mean proportional II.



<https://www.geogebra.org/m/cc9szphi>

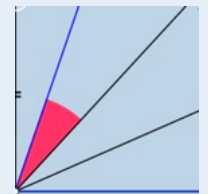
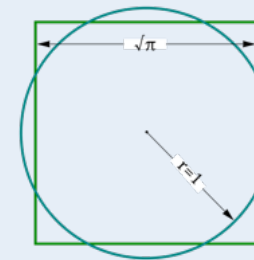
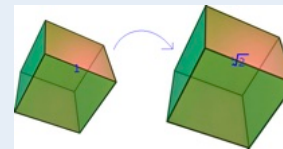
Double mean proportional: Given two lengths a and b , find x and y such that $a/x = x/y = y/b$

All three problems were eventually proven impossible.
Who proved each one impossible, and in what year?
You may use Google and AI to answer this question.

The three problems are :

- Squaring the circle
- Trisecting an angle
- Doubling the cube

Using only straightedge and compass.

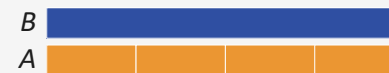


Incommensurable magnitudes

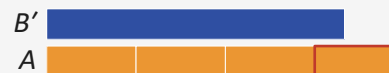
The Greeks on Measuring, numbers and quantities

A quantity **A measures** a quantity **B** of the same kind if **B** is exactly a whole number of copies of **A**.

A is a **part** of **B**; **B** is a **multiple** of **A**.



A measures B: 4 copies fill B exactly.



A does not measure B': 3 copies fall short, 4 overshoot.

Recall: 1 does not measure $\sqrt{2}$

Aristotle distinguishes two kinds of quantity: magnitude and number. In your own words, define each and give an example of each. What is the key difference between them?

There are two distinct types of “quantities”:

Aristotle (~350 BC)



the continuous (magnitude)

A quantity is a **magnitude** if it can be measured, and a magnitude is what can be divided into continuous parts. It is a length if continuous in one dimension, a breadth in two, and a depth in three.

Example: lines, surfaces, bodies and time.

the discrete (number)

Number is a plurality measurable by the one, so the one is the measure and number is what is measured.

Examples 2, 3, 27 ...

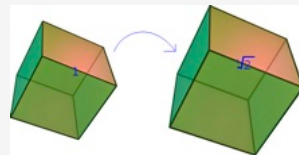
Aristotle, Metaphysics V.13 and X (Ross translation))

Number → discrete quantity, multiple of units

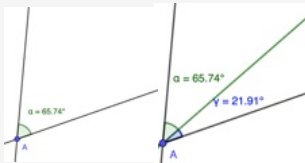
Magnitude → continuous quantity

Three “impossible” problems

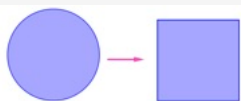
Doubling a cube: Given a cube, **construct** another cube with twice its volume.



Trisecting an angle: Given an angle between two straight lines, **construct** two lines that divide the angle into three equal parts.



Squaring a circle: Given a circle, **construct** a square with the same area.



“Construct” means “construct using only straightedge and compass

These problems originated around 400 BCE

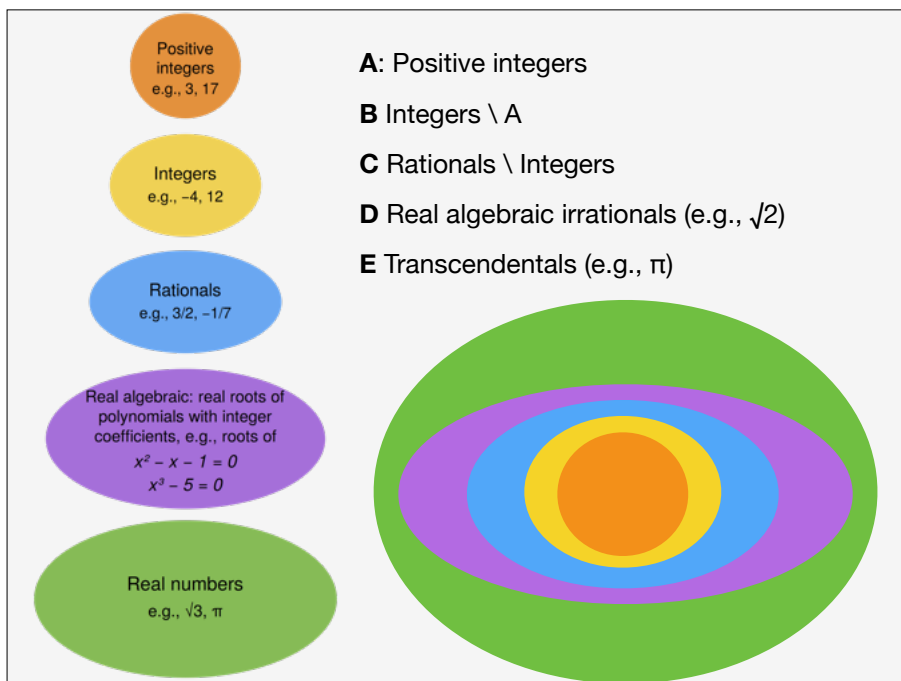
Positive integers
e.g., 3, 17

Integers
e.g., -4, 12

Rationals
e.g., $3/2$, $-1/7$

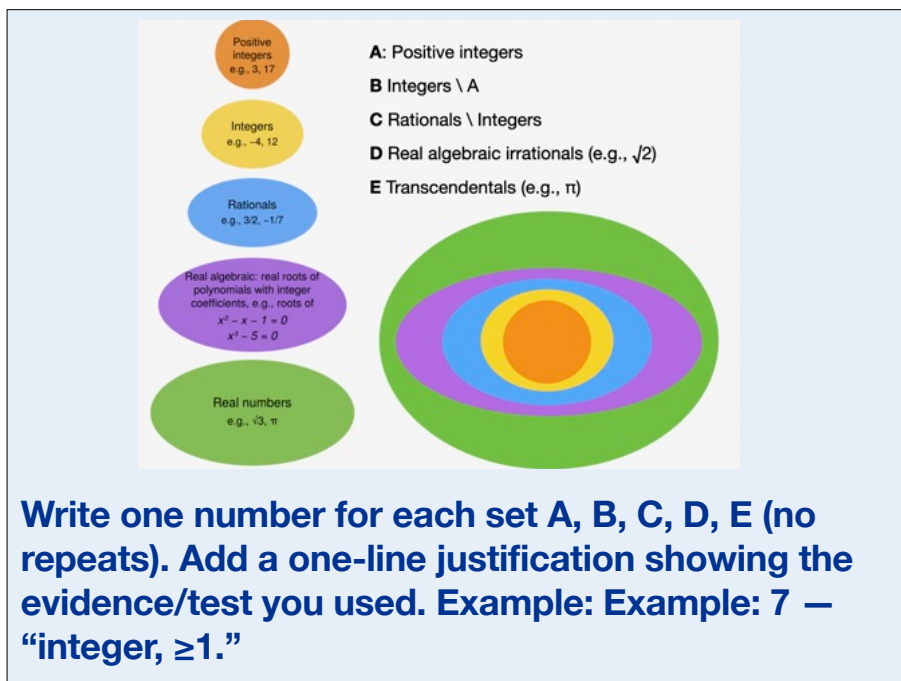
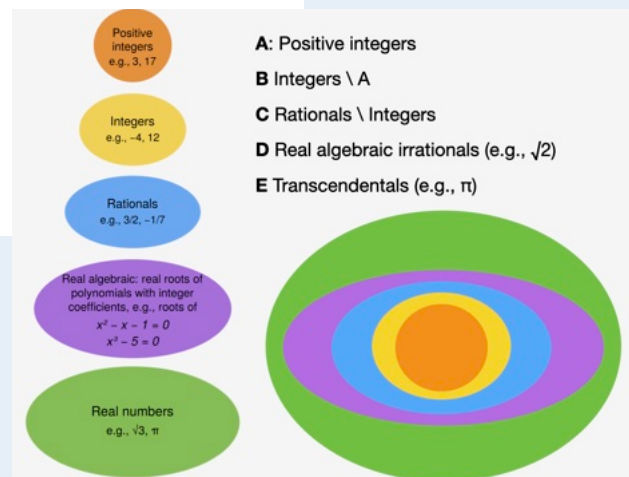
Real numbers
e.g., $\sqrt{3}$, π

Real algebraic: real roots of integer-coefficient polynomials,
e.g., roots of
 $x^2 - x - 1 = 0$, $x^3 - 5 = 0$

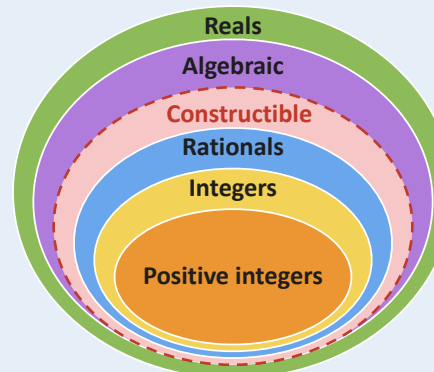


In which of the sets A,B,C, D,E is each of the following numbers?

- i. e
ii. 0
iii. $2^{1/3}$
iv. $-11/5$



Which of these are constructible?



Give a one-line reason for each.

$(\sqrt{2}+\sqrt{3})^{1/3}$	$2^{1/4}$
$(\sqrt{2}+\sqrt{3})^{1/8}$	$-11/5$
$1+(5/2)^{1/2}$	π
	$2^{1/3}$

Constructible: built from 1 using +, -, $\times$, $\div$ and square roots.

$\sqrt{2}$ is not rational

No pattern!



<https://cosmosmagazine.com/mathematics/the-square-root-of-2/>

Holy beard of Zeus! It's expanding irrationally and infinitely! Credit: Jeffrey Phillips

Plato ~ 400 BC



Plato (~400 BC)

- 'greatly advanced mathematics in general, and geometry in particular, because of his zeal for these studies' (Proclus)
- was an **important influence** upon the **mathematicians** of his time, by inspiration and direction.
- the works of Plato are some of our fullest and **best source of information about the mathematical developments at that time.**
- many scholars credit Plato with laying the conceptual foundations of Western thought.

Plato (~400 BC)



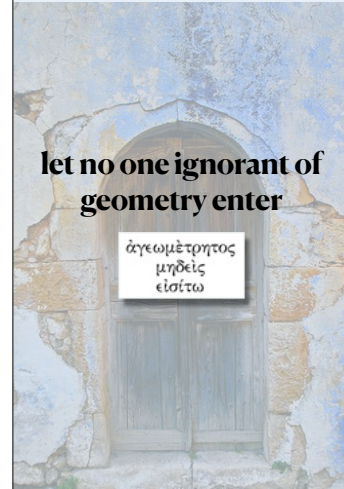
- Emphasized distinction between pure math and “merely useful”
- Math education is essential for the mind (of an elite)



Plato (~400 BC)

In the philosophical school of thought of Plato's (429–348) academy, **mathematics was shaped as ideal case of purely deductive science**, which has influenced the development of this science enormously up to the present day. Plato argued that **mathematics had an intermediate position between the realm of mere ideas and the world of empirical objects.**

5000 Years of Geometry Mathematics in History and Culture
By Christoph J. Scriba, Peter Schreiber · 2015



Prompt (at least one per group):

- 1. Why would Plato post 'Let no one ignorant of geometry enter'?**
- 2. What qualities was he actually seeking in students?**
- 3. Elitist gatekeeping or reasonable prerequisite—defend your position.**

There is no direct historical or archaeological evidence of such inscription but it reflects Plato's authentic emphasis on the importance of geometric reasoning.



Plato (~400 BC)

Plato greatly advanced mathematics in general and geometry in particular because of his zeal for these studies. It is well known that his writings are thickly sprinkled with mathematical terms and that he everywhere tries to arouse admiration for mathematics among students of philosophy.

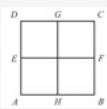
Proclus (~400 CE, Platonist philosopher)



Plato (~400 BC)

In sum, what information we can get from Plato is fascinating and rich, and fraught with problems. Beyond issues of detailed reconstructions and precise attributions, which may never be solved with any certainty, **he definitely is testimony to the vitality of mathematics at the time, and to the interest that mathematical issues aroused in the educated public at large.**

Cuomo, Serafina. Ancient mathematics. Routledge, 2005.



Meno by Plato (~400 BC)



The dialogue has three characters: the philosopher Socrates (Plato's teacher), Meno (a young aristocrat), and a slave boy. Socrates and Meno are discussing the nature of virtue, and whether it can be taught. In so doing, they move on to the more general question of whether a knowledge of anything can be taught, and if so how. In order to show that true knowledge can be elicited from someone who initially holds false beliefs, Socrates holds a mathematical discussion with the slave boy to show that knowledge is recollection.

The purpose of the mathematical discussion was to elicit from the slave boy the solution of the following problem:
Find a square whose area is twice that of a given square.

Meno 82b–85b: Socrates questions Meno's slave boy about doubling a square. "Area" is modern language; the text says "a space of so many feet," meaning square feet.

- Socrates draws a square of side 2 feet, area 4. The boy says a square twice its size has area 8
- The boy proposes side 4 (for double area, 8). Socrates shows this gives four copies of the original, area 16.
- He then proposes side 3, which gives area 9, still not 8. He admits he doesn't know. For Socrates this perplexity (aporia) is progress.
- Socrates builds the 4×4 square from four copies of the original and draws a diagonal in each copy.
- The diagonals enclose a square made of four half-squares, twice the original's two, so its area is 8.
- So the side of the doubled square is the diagonal of the original.

Plato's purpose is philosophical, not mathematical. The episode is meant to show that learning is recollection: the boy was never taught geometry, yet with questioning he "finds" the answer.

The text never says the diagonal is incommensurable. At 84a Socrates tells the boy that if he can't count the length, he should point to it. Some read this as a hint that the length can't be given as a number. That reading is an interpretation, not a claim in the text.

A second passage at 86e–87b uses a geometric problem about inscribing an area as a triangle in a circle. Socrates uses it to illustrate reasoning "from a hypothesis." How to reconstruct that problem is still debated.

More about Plato

Plato required geometry not as a practical skill but as proof that you could reason from fixed principles to necessary conclusions without relying on your senses.

A geometric truth holds universally — not because you measured triangles, but because it follows logically from definitions. For Plato, this was the model for all genuine knowledge.

The inscription "Nobody ignorant of geometry can enter here", whether historical or not, signals that the Academy was not a place for people who confused opinion with knowledge. Geometry was the entrance requirement because it trained the mind to operate in the realm of eternal truths rather than the unreliable world of appearances.

Aristotle ~ 350 BCE

Aristotle (~350BCE) Wrote about the structure of mathematical knowledge.



- An **axiom** is a statement worthy of acceptance: a starting point that is not proved.*.
- Example: when equals taken from equals the remainders are equal.

*worthy of acceptance: roughly, what we would now call self-evident.

Aristotle (~350BCE) Wrote about the structure of mathematical knowledge.



- Through deductive reasoning, **necessary conclusions** follow from accepted **starting points**. (Note: starting points=axioms)
- Demonstration produces knowledge of **why** something must be so.

<https://plato.stanford.edu/entries/aristotle-mathematics/>

Aristotle distinguishes two kinds of quantity: magnitude and number. In your own words, define each and give an example of each. What is the key difference between them?

There are two distinct types of “quantities”:

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Example: lines, surfaces, bodies and time.

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***Number** is a plurality measurable by the one, so the one is the measure and number is what is measured.*

Examples 2, 3, 27 ...

Aristotle, Metaphysics V.13 and X (Ross translation))

An old question in modern terms: Is it possible to find integers p and q such that $\sqrt{2} = p/q$? In other words, is $\sqrt{2}$ rational?

Here is Aristotle's take

...when something impossible results from the assumption of its contradictory; e.g. **that the diagonal of the square is incommensurate with the side, because odd numbers are equal to evens if it is supposed to be commensurate. One infers syllogistically that odd numbers come out equal to evens, and one proves hypothetically the incommensurability of the diagonal, since a falsehood results through contradicting this.**

Aristotle - The Organon ANALYTICA PRIORIA Book 1 Part 23- ~350 BCE

Proof of the irrationality of $\sqrt{2}$ discussed by Aristotle

It is clear then that the ostensive syllogisms are effected by means of the aforesaid figures; these considerations will show that reductions ad also are effected in the same way. For all who effect an argument per impossibile infer syllogistically what is false, and prove the original conclusion hypothetically when something impossible results from the assumption of its contradictory; e.g. **that the diagonal of the square is incommensurate with the side, because odd numbers are equal to evens if it is supposed to be commensurate. One infers syllogistically that odd numbers come out equal to evens, and one proves hypothetically the incommensurability of the diagonal, since a falsehood results through contradicting this.** For this we found to be reasoning per impossibile, viz. proving something impossible by means of an hypothesis conceded at the beginning. Consequently, since the falsehood is established in reductions ad impossibile by an ostensive syllogism, and the original conclusion is proved hypothetically, and we have already stated that ostensive syllogisms are effected by means of these figures, it is evident that syllogisms per impossibile also will be made through these figures. Likewise all the other hypothetical syllogisms: for in every case the syllogism leads up to the proposition that is substituted for the original thesis; but the original thesis is reached by means of a concession or some other hypothesis. But if this is true, every demonstration and every syllogism must be formed by means of the three figures mentioned above. But when this has been shown it is clear that every syllogism is perfected by means of the first figure and is reducible to the universal syllogisms in this figure.

Aristotle - The Organon ANALYTICA PRIORIA Book 1 Part 23- ~350 BCE

Reasoning pattern: Proof by contradiction:

assume opposite →

derive impossibility →

conclude what we wanted to prove.

Some reasons why $\sqrt{2}$ irrationality matters

Proof by contradiction: A reasoning pattern

(Uses: diagnosis, debugging, risk, legal).

Frameworks have limits: find something not fitting →
recognize limits → build better systems

(Uses: floating-point, measurement, model failure).

Paradigm shock: a core belief shatters → the field
rebuilds.

(Uses: Progress of science)

Zeno (~450BCE) and Paradoxes

The android is the one in grey, wearing pajamas-like clothes.
Why do you think the android self-destructs at the end of the clip?
(Optional) What does this suggest about how the android's
reasoning differs from human reasoning?



https://youtu.be/EzVxsYzXl_Y?si=gfZ16Y6C5nQYfjcl

Paradox

Greek

- **para**: distinct from
- **doxa**: opinion, belief.

Colloquial

A statement contrary to common belief or expectation

Logical/Philosophical

A statement that starts from acceptable premises and follows sound reasoning, yet reaches a self-contradictory or illogical conclusion.

Which type of paradox destroyed the android?



I am a barber.
I shave anyone in my town who does not shave himself, and no one else.

Do I shave myself?

Barber paradox

Analogous to Russell's paradox

Analogous the paradoxes "*This sentence is false*" and the one in the Star Trek clip

Liar paradox

I am a liar

Is my previous sentence true or false?



Zeno's Dichotomy paradox

The first (argument) is the one which declares movement to be impossible because, however near the mobile is to any given point, it will always have to cover half, and then the half of that, and so on without limit before it gets there.

Aristotle - Physics

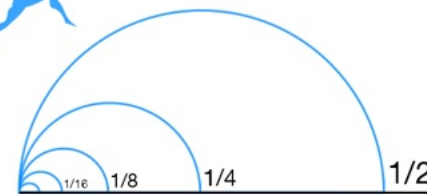
Read Aristotle's explanation. Then discuss:
What has to happen every time you try to reach a point?
Why does this create a logical problem?

Zeno's Dichotomy paradox

To go from A to B, you have to

- go midway to $A_{1/2}$,
- midway from A to $A_{1/2}$, $A_{1/4}$,
- and midway from A to $A_{1/4}$, $A_{1/8}$,
- and so on.

So you have to pass through infinite points in finite time, which is impossible.



By Miranche - Own work, CC BY-SA 4.0, <https://commons.wikimedia.org/w/index.php?curid=55976282>

How many Zenos does it take to change a light bulb?

Infinite. One to screw it in halfway, one to screw it in half of what's left, another to screw it in half of what remains, and so forth...

Zeno showing the Doors to Truth and False



Youths the Doors to Truth and False (Veritas et Falsitas) by Pellegrino Tibaldi

Zeno's Achilles and the Tortoise paradox

Zeno's logic seems sound, but we know Achilles actually catches the tortoise.



Video from Open University <https://www.youtube.com/watch?v=skM37PcZmWE>

Zeno's logic seems sound, but we know Achilles actually catches the tortoise.

- 1 What assumption about space and time makes Zeno's argument work? (Hint: What does he keep dividing?)
- 2 How is the Achilles paradox similar to the Dichotomy paradox we discussed earlier?
- 3: Zeno assumes infinite steps = infinite time. Why would ancient Greeks find this assumption convincing? Do you agree with Zeno? Why or why not?

Zeno's Achilles and the Tortoise paradox

"In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead."

Aristotle, Physics, VI:9

Zeno's Arrow Paradox



- At any instant in time, an arrow in flight occupies a precise position in space.
- At that instant, it's not moving (motion requires time to pass).
- If it's motionless at every instant, when does it move?
- The paradox: If time is composed of instants, and the arrow is frozen at each instant, motion is impossible.

Is movement impossible?

- *Nude descending a staircase*, by Marcel Duchamp 1912
- Provoked an scandal when first shown.
- The object at each moment is at a fixed position (according to Zeno at rest) how can it then move?



Why are paradoxes important in mathematics?
(Hint: What happens when mathematicians try to resolve a paradox? What might they discover or create in the process?)