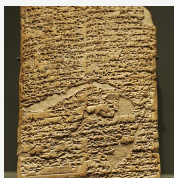


Mathematics in Mesopotamia: Plimpton 322



- Tables of Reciprocals
- Plimpton 322
- Solutions of equations

Mesopotamia and the Pythagorean theorem

Mesopotamia and the Pythagorean theorem

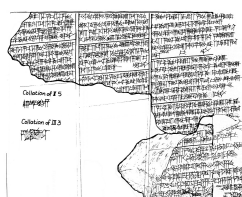
- The theorem is ascribed to Pythagoras by Proclus (Greek, 5th century) and several later commentators (based on Proclus)
- This is one of many improbable legends ascribed to Pythagoras in late Antiquity
- Old Babylonian texts (c. 2000 BC) show the principle was already known

Pythagoras Emerging from the Underworld, 1662



By Salvator Rosa, Public Domain, <https://commons.wikimedia.org/w/index.php?curid=7119956>

<https://badwcal-eb01.srv.mwn.de/library/BM.96957>



See Three Old Babylonian Methods for Dealing with "Pythagorean" Triangles Author(s): Eleanor Robson Source: Journal of Cuneiform Studies, 1997, Vol. 49 (1997), pp. 51-72

Mesopotamia and the Pythagorean rule

A Babylonian tablet from around 2000 BCE asks for the diagonal of a rectangle whose sides are $2/3$ and $1/6$. The scribe gives the following procedure

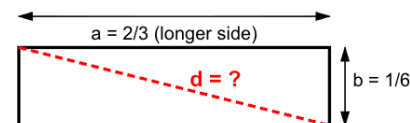
- Square the shorter side, $1/6$, to obtain $1/36$.
- Take the reciprocal of the longer side, $2/3$, which is $3/2$, and multiply it by $1/36$.
- The result is $1/24$.
- Take half of this value to get $1/48$.
- Add this to $2/3$.
- The result is $11/16$, which is the diagonal.
- Square the shorter side, b : b^2
- Take the reciprocal of the longer side, a , which is $1/a$, and multiply it by b^2 .
- The result is $(1/a) \times b^2 = b^2/a$
- Take half of this value: $(b^2/a)/2 = b^2/(2a)$
- Add this to a : $a + b^2/(2a)$
- This gives the approximation for the diagonal: $d \approx a + b^2/(2a)$

$$d = (a^2 + b^2)^{1/2}$$

$$\approx a + (1/2) b^2 (1/a) = a + b^2/(2a)$$

This is a particular case of the approximation when $b \ll a$

$$d = (a^2 + b^2)^{1/2} \approx a + b^2/(2a)$$



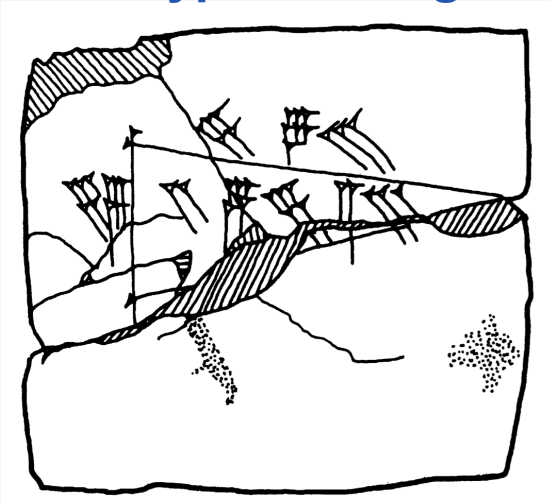
Plimpton 322: Motivation and overview

**Without overthinking,
quickly draw a triangle
(paper or tablet).**

List features common to most drawings (besides the obvious fact that they are triangles)

A Babylonian typical triangle

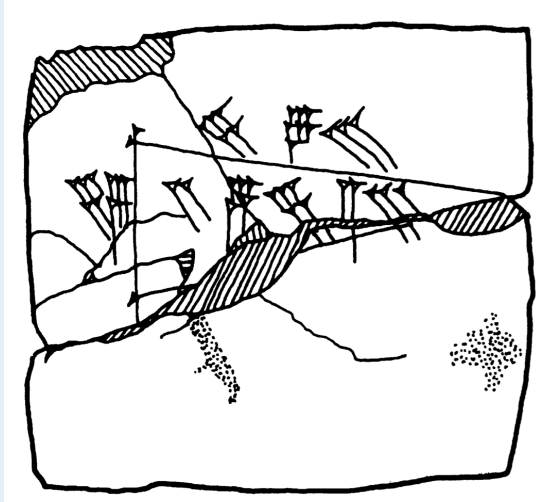
The point of triangle drawing activity is to make us aware of our biases.



M 29-15-709 (obverse). Drawing by the Eleanor Robson - Words and Pictures: New Light on Plimpton 322 - The American Mathematical Monthly, Feb., 2002, Vol. 109, No. 2 (Feb., 2002), pp. 105-120

M 29-15-709 (obverse). Drawing by the Eleanor Robson - Words and Pictures: New Light on Plimpton 322 - The American Mathematical Monthly, Feb., 2002, Vol. 109, No. 2 (Feb., 2002), pp. 105-120

If you had found this triangle drawing in a Mesopotamian tablet, what would you have assumed about it?"



M 29-15-709 (obverse). Drawing by the Eleanor Robson - Words and Pictures: New Light on Plimpton 322 - The American Mathematical Monthly, Feb., 2002, Vol. 109, No. 2 (Feb., 2002), pp. 105-120

<https://www.nytimes.com/2010/11/23/science/23babylon.html?smid=url-share>



Plimpton 322

An Exhibition That Gets to the (Square) Root of Sumerian Math - NYTimes - Nov 22, 2010

- First Western owner, George A. Plimpton bequeathed to Columbia University in the mid-1930s.
- Surviving correspondence shows that he bought the tablet for \$10 from a dealer called Edgar J. Banks.
- Banks told him it came from an archaeological site called Senkereh in southern Iraq, whose ancient name was Larsa
- Approximate date of the tablet: 1800 BCE.

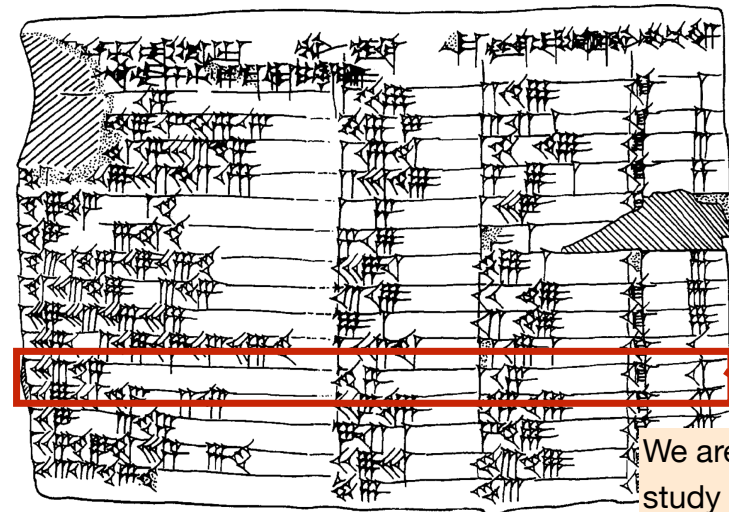
To understand Plimpton 322, in what order should we investigate the following:

- A. How were the numbers calculated (algorithm or procedure)?
 - B. What mathematical relationships do the numbers exhibit?
 - C. What is the purpose of the table?
 - D. Who made it?
- Explain your choice.

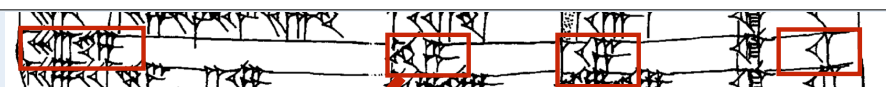


An Exhibition That Gets to the (Square) Root of Sumerian Math - NYTimes - Nov 22, 2010

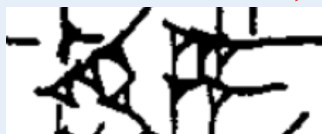
Plimpton 322 - Drawing by Eleanor Robson



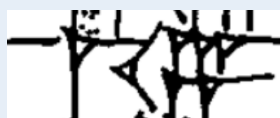
Plimpton 322 (Robson, Eleanor. "Neither Sherlock Holmes nor Babylon: A Reassessment of Plimpton 322." Historia Mathematica 28.3 (2001): 167-206.)



A



B



Express A or B (or both) in Hindu Arabic numerals

Plimpton 322 Transcription (with Minor Corrections)

$(d/l)^2$ or $(s/l)^2$	Short side s	Diagonal d	Row
$(59;0;15)_{60}$	$(1;59)_{60}$	$(2;49)_{60}$	1
$(56;56;58;14;50;6;15)_{60}$	$(56;7)_{60}$	$(3;12;1)_{60}$	2
$(55;7;41;15;33;45)_{60}$	$(1;16;41)_{60}$	$(1;50;49)_{60}$	3
$(53;10;29;32;52;16)_{60}$	$(3;31;49)_{60}$	$(5;9;1)_{60}$	4
$(48;54;1;40)_{60}$	$(1;5)_{60}$	$(1;37)_{60}$	5
$(47;6;41;40)_{60}$	$(5;19)_{60}$	$(8;1)_{60}$	6
$(43;11;56;28;26;40)_{60}$	$(38;11)_{60}$	$(59;1)_{60}$	7
$(41;33;45;14;3;45)_{60}$	$(13;19)_{60}$	$(20;49)_{60}$	8
$(38;33;36;36)_{60}$	$(9;1)_{60}$	$(12;49)_{60}$	9
$(35;10;2;28;27;24;26;40)_{60}$	$(1;22;41)_{60}$	$(2;16;1)_{60}$	10
$(33;45)_{60}$	$(45)_{60}$	$(1;15)_{60}$	11
$(29;21;54;2;15)_{60}$	$(27;59)_{60}$	$(48;49)_{60}$	12
$(27;0;3;45)_{60}$	$(7;12;1)_{60}$	$(4;49)_{60}$	13
$(25;48;51;35;6;40)_{60}$	$(29;31)_{60}$	$(53;49)_{60}$	14
$(23;13;46;40)_{60}$	$(56;56)_{60}$	$(53)_{60}$	15

**Plimpton 322:
What mathematical
relationships do the
numbers exhibit?**

Short side s	Diagonal d	Row	$d^2 - s^2$
119	169	1	14,400
3367	4825	2	11,943,936
4601	6649	3	23,040,000
12709	18541	4	182,250,000
65	97	5	5,184
319	481	6	129,600
2291	3541	7	7,290,000
799	1249	8	921,600
481	769	9	360,000
4961	8161	10	41,990,400
45	75	11	3,600
1679	2929	12	5,760,000
161	289	13	57,600
1771	3229	14	7,290,000
56	106	15	8,100

The columns labeled by *Short side s* , *Diagonal d* and *Row* are three of the four columns of Plimpton 322.

The orange column ($d^2 - s^2$) wasn't on the original Plimpton 322 tablet, it was added to help you spot a pattern.

What kind of numbers do you see in this column?

Plimpton 322 (with first column removed and two extra columns at the end) in Hindu Arabic number system

Short side s	Diagonal d	Row	$d^2 - s^2$	$(d^2 - s^2)^{1/2}$
119	169	1	14,400	120
3367	4825	2	11,943,936	3,456
4601	6649	3	23,040,000	4,800
12709	18541	4	182,250,000	13,500
65	97	5	5,184	72
319	481	6	129,600	360
2291	3541	7	7,290,000	2,700
799	1249	8	921,600	960
481	769	9	360,000	600
4961	8161	10	41,990,400	6,480
45	75	11	3,600	60
1679	2929	12	5,760,000	2,400
161	289	13	57,600	240
1771	3229	14	7,290,000	2,700
56	106	15	8,100	90

The numbers in the added column ($d^2 - s^2$) are all perfect squares!

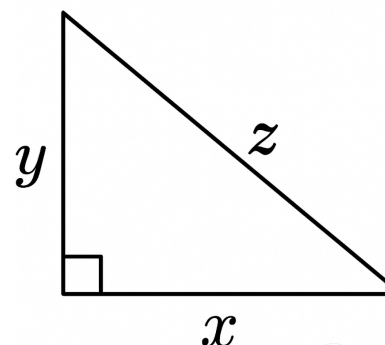
Recall: a **perfect square** is an integer whose square root is also an integer.

Plimpton 322 (with two extra columns) in Hindu Arabic number system

$(d/l)^2$ or $(s/l)^2$	Short side s	Diagonal d	Row	$d^2 - s^2$	$(d^2 - s^2)^{1/2}$
(1).9834028	119	169	1	14400	120
(1).9491586	3367	4825	2	11943936	3456
(1).9188021	4601	6649	3	23040000	4800
(1).8862479	12709	18541	4	182250000	13500
(1).8150077	65	97	5	5184	72
(1).7851929	319	481	6	129600	360
(1).7199837	2291	3541	7	7290000	2700
(1).6927094	799	1249	8	921600	960
(1).6426694	481	769	9	360000	600
(1).5861226	4961	8161	10	41990400	6480
(1).5625	45	75	11	3600	60
(1).4894168	1679	2929	12	5760000	2400
(1).4500174	161	289	13	57600	240
(1).4302388	1771	3229	14	7290000	2700
(1).3871605	56	106	15	8100	90

What is the purpose of Plimpton 322?

Pythagorean triples



A **Pythagorean triple** is a set of three positive integers, **x**, **y** and **z** with no common factors that satisfy the equation $x^2 + y^2 = z^2$.

“No common factors” means that the largest positive integer that divides **x**, **y**, and **z** is 1.

A **Pythagorean triple** is a set of three positive integers, **x, y and z** that satisfy the equation $x^2 + y^2 = z^2$.

Find one Pythagorean triple that is not a multiple of (3, 4, 5) or (5, 12, 13).

1. What the numbers exhibit: Pythagorean triples

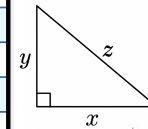
Fact: The rows correspond to integer right triangles.

Historical question: Was the tablet intended as a table of such triples?

(d/l)2 or (s/l)2	Short side s	Diagonal d	Row	$d^2 - s^2$	$(d^2 - s^2)^{1/2}$
(1).9834028	119	169	1	14400	120
(1).9491586	3367	4825	2	11943936	3456
(1).9188021	4601	6649	3	23040000	4800
		18541			
		97			
		481			
		3541			
		1249			
		769			
		8161			
		75			
(1).984168	1670	2000	4	5760000	2400

Entries are generated by pairs (p, q), with no common divisor, not both odd and such that $p > q$.
 $L = 2pq$,
 $D = p^2 + q^2$
The remaining leg is $p^2 - q^2$

Pythagorean triples: positive integers x, y, and z, such that $x^2 + y^2 = z^2$.



2. A modern trigonometric reinterpretation

Mathematically valid: the entries can be described using modern trigonometric ratios.

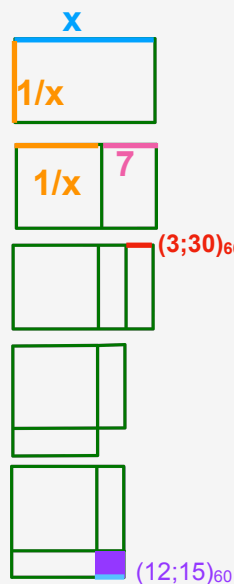
Historically disputed: did the Babylonians conceive or use the table in this way?

(d/l)2 or (s/l)2	Short side s	Diagonal d	Row	$d^2 - s^2$	$(d^2 - s^2)^{1/2}$
(1).9834028	119	169	1	14400	120
(1).9491586	3367	4825	2	11943936	3456
(1).9188021	4601	6649	3	23040000	4800
(1).8862479	12709	18541	4	182250000	13500
(1).8150077	65	97	5	5184	72
(1).7851929	319	481	6	129600	360

Trigonometric table: if Columns L and D contain the Legs and Diagonals of right-triangles, then the values in the first column are $\tan^2$ or $1/\cos^2$. The acute angles of the triangles decrease by approximately 1°

Recall the problem: A number and its reciprocal differ by 7. Your task: form a quadratic equation whose solution gives that number.

Recall: Find a number and its reciprocal whose difference is 7
(Solve the Babylonian way)



1. You: break in half the 7 by which the reciprocal exceeds its reciprocal, and $(3;30)_{60}$ (will come up).
2. Multiply $(3;30)_{60}$ by $(3;30)_{60}$ and $(12;15)_{60}$ (will come up).
3. Append [1 00, the area,] to the $(12;15)_{60}$ which came up for you and $(1 12;15)_{60}$ (will come up).
4. What is [the square-side of 1] 12;15? $(8;30)_{60}$.
5. Put down $[(8;30)_{60}$ and] $(8;30)_{60}$, its equivalent, and subtract $(3;30)_{60}$, the takiltum-square, from one (of them); append $(3;30)_{60}$ to one (of them).
6. One is 12, the other is 5.
7. The reciprocal is 12, its reciprocal 5.

Recall $x^2 - 7x - 1 = 0$.
Take $y = 1/x \rightarrow y^2 + 7y - 1 = 0$

$$(y + 7/2)^2 = (y^2 + 7y) + (7/2)^2 = 1 + (7/2)^2$$

Then $y + 7/2 = (1 + (7/2)^2)^{1/2}$

Completing the square

$(1; 33, 45)_{60}$	$(0;45)_{60}$	$(1;15)_{60}$	11	Row 11
	45	75		

$a = (1;33,45)_{60} \rightarrow$ "area"

$s = (0;45)_{60} \rightarrow$ "short side"

$d = (1;15)_{60} \rightarrow$ "diagonal"

The teacher sets up the igi-igibi problem

- From s , I compute $2 \cdot s = (1;30)_{60}$.
- I speak the **igi-igibi problem**: "Find a number and its reciprocal; their difference is $(1;30)_{60}$."

The student's solution

- Halve the difference: $(1;30)_{60} \div 2 = s = (0;45)_{60}$.
- Make a square and append one: $1 + s^2 = a = (1;33,45)_{60}$.
- Take the square-side of a : $\sqrt{a} = d = (1;15)_{60}$.
- Take the two numbers:
 - $x = d + s = (2; 0)_{60}$, and
 - $1/x = d - s = (0;30)_{60}$.

Note: $x \times 1/x = 1$ (In the tablets, 1 and 60 look the same).

This is a simplified version of a problem that can be set up with Row 11, consistent with modern scholarship.

Words and Pictures: New Light on Plimpton 322 <https://doi.org/10.1080/00029890.2002.11919845>

3. Historical interpretation: a teaching/ problem-setting resource

$(d/l)^2$ or $(s/l)^2$	Short side s	Diagonal d	Row	$d^2 - s^2$	$(d^2 - s^2)^{1/2}$
(1).9834028	119	169	1	14400	120
(1).9491586	3367	4825	2	11943936	3456
(1).9188021	4601	6649	3	23040000	4800
(1).8862479	12709	18541	4	182250000	13500
(1).8150077	65	97	5	5184	72
(1).7851929	319	481	6	129600	360
(1).7199837	2291	3541	7	7290000	2700
(1).6927094	799	1249	8	921600	960
(1).6426604	481	769	9	260000	500
(1).6125396	439	701	10	200000	440
(1).5824192	399	629	11	150000	380
(1).5523008	359	559	12	100000	310
(1).5221836	319	489	13	70000	250
(1).4920676	279	419	14	50000	210
(1).4619528	239	349	15	30000	160
(1).4318392	199	279	16	10000	100
(1).4017268	159	209	17	0	0
(1).3716156	119	139	18	0	0
(1).3415056	79	69	19	0	0
(1).3113968	39	9	20	0	0

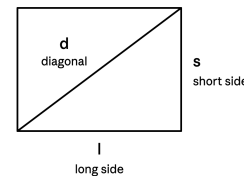
Teaching tool: The tablet gives examples (or starting values) for generating exercises (igi-igibi problems), possibly used by teachers to create math problems for students.

Plimpton 322: How were the numbers calculated (algorithm or procedure)? Who made it?

Short side s	Diagonal d	Row	r
45	75	11	2
$(45)_{60}$	$(1;15)_{60}$		$(2;0)_{60}$

The Reciprocal Pair Method: Row 11 of Plimpton 322

$a = (1;33,45)_{60} \rightarrow$ "area"
 $s = (0;45)_{60} \rightarrow$ "short side"
 $d = (1;15)_{60} \rightarrow$ "diagonal"
 $l \rightarrow$ "long side"



- We will discuss a method that could have been used to generate the table.
- A *regular number* is a number of the form $2^a \cdot 3^b \cdot 5^c$, where a, b, c are integers (possibly negative).
- Note: Earlier we had $s = (0;45)_{60}$; in the table, $s = (45)_{60}$. The tablets have no "decimal" point, so both are written the same way.

Short side s	Diagonal d	Row	r
45	75	11	2
$(45)_{60}$	$(1;15)_{60}$		$(2;0)_{60}$

The Reciprocal Pair Method: Row 11 of Plimpton 322

- Step 1:** Choose a regular number $\rightarrow r = 2$
Step 2: Find r 's reciprocal $\rightarrow 1/r = 1/2$
Step 3: Half-difference and half-sum $\rightarrow$
 $s/l = (r - 1/r)/2 = 3/4 = 45/60$
 $d/l = (r + 1/r)/2 = 5/4 = 75/60$
Step 4: Rescale (multiply by 60) $\rightarrow$
 $s = 45$
 $d = 75$
 Then $l = 60$.

Note: In our problems, l is a number that makes s and d "whole".

Short side s	Diagonal d	Row	r
119	169	1	12/5
$(1;59)_{60}$	$(2;49)_{60}$		$(2;24)_{60}$

The Reciprocal Pair Method: Row 1 of Plimpton 322

- Step 1:** Choose a regular number $\rightarrow r = 12/5 = (2;24)_{60}$
Step 2: Find r 's reciprocal $\rightarrow 1/r = 5/12 = (0;25)_{60}$
Step 3: Half-difference and half-sum $\rightarrow$
 $s/l = (r - 1/r)/2 = 119/120$
 $d/l = (r + 1/r)/2 = 169/120$
Step 4: Rescale (multiply by 120) $\rightarrow$
 $s = 119$
 $d = 169$
 Then $l = 120$.

Short side s	Diagonal d	Row	r
		5	9/4

The Reciprocal Pair Method: Row 5 of Plimpton 322

Find s and d on this case ($r = 9/4$), following the previous example (below)

Example from row 1

Step 1: Choose a regular number $\rightarrow r = 12/5 = (2;24)_{60}$

Step 2: Find r's reciprocal $\rightarrow 1/r = 5/12 = (0;25)_{60}$

Step 3: Half-difference and half-sum $\rightarrow$

$$s/l = (r - 1/r)/2 = 119/120$$

$$d/l = (r + 1/r)/2 = 169/120$$

Step 4: Rescale (multiply by 120) $\rightarrow$

$$s = 119$$

$$d = 169$$

Then $l = 120$.

Short side s	Diagonal d	Row	r
119	169	1	12/5
$(1;59)_{60}$	$(2;49)_{60}$		$(2;24)_{60}$

Short side s	Diagonal d	Row	r
65	97	5	9/4
$(1;59)_{60}$	$(2;49)_{60}$		$(2;15)_{60}$

The Reciprocal Pair Method: Row 5 of Plimpton 322

Step 1: Choose a regular number $\rightarrow 9/4 = (2;15)_{60}$

Step 2: Find r's reciprocal $\rightarrow 1/r = 4/9 = (0;26,40)_{60}$

Step 3: Half-difference and half-sum $\rightarrow$

$$s/l = (r - 1/r)/2 = (9/4 - 4/9)/2 = (65/36)/2 = 65/72$$

$$d/l = (r + 1/r)/2 = (9/4 + 4/9)/2 = (97/36)/2 = 97/72$$

Step 4: Rescale (multiply by 72) $\rightarrow$

$$s = 65$$

$$d = 97$$

Then, $l = 72$

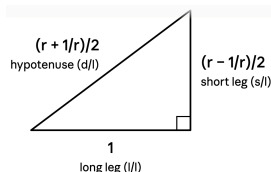
Justifying the Reciprocal Pair Method

Assume $r > 1$ is a positive rational,

Write $r = a/b$ in lowest terms (that is, $\gcd(a,b)=1$)

Then

- $(r - 1/r)/2 = (a^2 - b^2)/(2ab)$
- $(r + 1/r)/2 = (a^2 + b^2)/(2ab)$



Remark: $\gcd(a,b, a^2 + b^2) = 1$ and $\gcd(a,b, a^2 - b^2) = 1$
(because $\gcd(a, b)=1$ implies $\gcd(a, b^2)=1$ and $\gcd(b, a^2)=1$).

Then we obtain Pythagorean triples:

- $((a^2 - b^2)/2, (a^2 + b^2)/2, ab)$ if both a and b are odd
- $(a^2 - b^2, a^2 + b^2, 2ab)$ if one of them a or b is even.

Summary

Question	What the evidence suggests
What relationships do the numbers exhibit?	Pythagorean/right-triangle relationships, and trigonometric ratios
How were they probably generated?	Reciprocal-pair methods fit Old Babylonian mathematics
What was the tablet for?	Probably connected with scribal teaching/problem construction; not certain

Note: A correct modern mathematical description is not necessarily a correct historical interpretation.

Plimpton 322 interpretations - Summary

Plimpton 322: Evidence for a Teacher's Resource

- The choice of **regular numbers** follows typical scribal school practice.
- Rows can be generated from regular numbers using the **reciprocal pair method**, a standard computational procedure.
- **Structured** like other Old Babylonian school tables: the same setup repeated with varying numbers.
- The **error pattern** is consistent with copying and scaling reciprocals.

Most modern scholars agree that this is the most **historically consistent** interpretation.

Plimpton 322: Evidence against Pythagorean triples interpretation

- No similar table found anywhere in the Mesopotamian.
- The ordering of the missing columns would violate standard formatting rules
- Column I has no natural explanation.

Plimpton 322: Evidence against Trigonometry Interpretations

- No textual or material evidence for angle-based reasoning in Old Babylonian geometry.
- No evidence that scribes conceptualized “trigonometry” in any form.
- The tablet does not present ratios of sides in a way consistent with later trigonometric tables.
- The ordering of rows reflects numerical structure, not angular progression.

Even groups answer A (B if there is time) odd groups answer B (A if there is time)

A. Scholars have read Plimpton 322 as (i) a list of Pythagorean triples, (ii) a trigonometric table, or (iii) a teacher's tool for setting problems. Which do most historians favor today? Give one reason.

B. What did the triangle drawing activity reveal about cultural assumptions in mathematics?

Form groups of 3–4.

Choose one speaker.

Write one agreed-upon answer for each assigned question.

Do not write until all group members agree.

You have 6 minutes.

"Further, if we believe that **Plimpton 322 was intended to be a list of parameters to aid the setting of school mathematics problems** (and the typological evidence suggests that it was), the question '**how was the tablet calculated?**' does not have to have the same answer as the question 'what problems does the tablet set?' The first can be answered most satisfactorily **by reciprocal pairs**, as first suggested half a century ago, and the second by some sort of right-triangle problems."

E. Robson

Robson, Eleanor (August 2001), "Neither Sherlock Holmes nor Babylon: a reassessment of Plimpton 322", *Historia Math.*, 28 (3): 167–206, doi:10.1006/hmat.2001.2317,

Impact Review

Ancient mathematical texts and artefacts, if we are to understand them fully, must be viewed in the light of their mathematico-historical context, and not treated as artificial, self-contained creations in the style of detective stories.
Eleanor Robson

