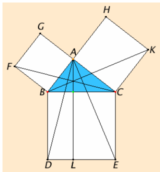
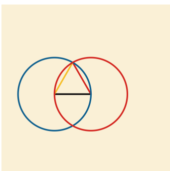
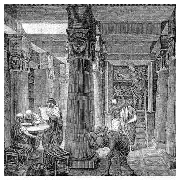


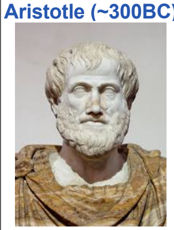
Euclid's Elements

Number theory in Euclid

Recall

- There are two distinct types of "quantities":
  - the continuous (magnitude)
    - *"A **magnitude** is that what which is divisible into divisible that are infinitely divisible."*
    - Example: lines, surfaces, bodies and time.
  - the discrete (number)
    - *A "**number**" quantities that is composed of distinct, separate units or parts that can be counted. (Examples 1, 2, 3, ...)*

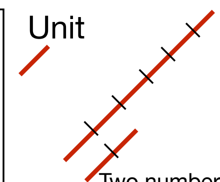
Aristotle (~300BC)



Book VII. Some Definitions

- A ***unit*** is (that) according to which each existing (thing) is said (to be) one.
- A ***number*** is a multitude composed of units.
- A number is ***a part* of a number**, the less of the greater, when **it measures the greater;**
- A ***prime number*** is that which is measured by a unit alone.
- Numbers ***relatively prime*** are those which are measured by a unit alone as a common measure.
- A ***composite number*** is that which is measured by some number.

Unit



Two numbers, 2 and 6.

2 measures 6, then 2 is a part of 6

Proposition IX.20: The prime numbers are more than any collection of prime numbers.

Which statement best captures Euclid's meaning? Choose the statement that matches Euclid's exact reasoning, not just the modern summary. Explain your choice.

a) There are infinitely many primes
b) For any finite list of primes, there exists a prime not on that list
c) No finite collection contains all primes
d) There are more primes than composites.

Euclid's Elements

Proposition IX.20: The prime numbers are more than any collection of prime numbers.

What "collection" meant to Euclid

Collection here means a finite group of numbers — Greek mathematicians did not conceive of completed infinite collections.

Euclid never uses the word "infinite"

Two ways of thinking infinity

*The infinite has a **potential** existence but not an **actual** one. It exists in the sense that one thing after another can be taken without end.* Aristotle

Potential infinity: The property that there is always something beyond any given quantity.

Actual infinity: The idea that an infinite collection exists as a completed whole.

Euclid never uses the word "infinite"

Consider the primes A=7, B=11 and C=13.

Define E =A.B.C+1.

G be the smallest prime that divides E.

What is G?

Proposition IX.20: The prime numbers are more than any collection of prime numbers.

(Modern) Proof: Assume that p_1, p_2, and p_3 are prime.

Suppose that the primes A, B, C are 2, 3, and 5. Let N be as above and let G be the largest prime that divides N. What is G?

Euclid's Elements

This proposition is not used in the rest of the Elements.

Table 3

P1	P2	P3	P1.P2.P3+1	FACTORS	FACTORS	FACTORS	FACTORS	FACTORS
2	3	5	31	31				
3	5	7	106	2	53			
5	7	11	386	2	193			
7	11	13	1002	2	3	167		
11	13	17	2432	2	19			
13	17	19	4200	2	3	5	7	
17	19	23	7430	2	5	743		
19	23	29	12674	2	6337			
23	29	31	20678	2	7	211		
29	31	37	33264	2	3	7	11	
31	37	41	47028	2	3	3919		
37	41	43	65232	2	3	151		
41	43	47	82862	2	13	3187		
43	47	53	107114	2	7	1093		
47	53	59	146970	2	3	5	23	71
53	59	61	190748	2	43	1109		

Proposition IX.20: The prime numbers are more than any collection of prime numbers.

- Let *A, B,* and *C* be the assigned prime numbers.
- I say that there are more prime numbers than *A, B,* and *C.*
- Take the least number *DE* measured by *A, B,* and *C.* Add the unit *DF* to *DE.*
- Then *EF* is either prime or not.
- Let it, first of all, be prime. Thus, the (set of) prime numbers A, B, C, EF, (which is) more numerous than A, B, C, has been found.
- And so let EF not be prime.
- Thus, it is measured by some prime number [Prop. VII.31]. Let it be measured by the prime (number) G.
- I say that G is not the same as any of A, B, C. For, if possible, let it be (the same).
- And A, B, C (all) measure DE.
- Thus, G will also measure DE.
- And it also measures EF.
- (So) G will also measure the remainder, unit DF, (despite) being a number [Prop. 7.28]. The very thing (is) absurd. Thus, G is not the same as one of A, B, C.
- And it was assumed (to be) prime. Thus, the (set of) prime numbers A, B, C, G, (which is) more numerous than the assigned multitude (of prime numbers), A, B, C, has been found. (Which is) the very thing it was required to show.

Euclid's Elements

1. Choose one (or more):
a) What surprised you most about how mathematics is organized in Euclid's Elements?
b) What is still confusing?
c) Name one idea from today that changes how you think about proofs or the Elements.

2. What is an axiomatic system?
Hint: chosen starting points (axioms), definitions, and proofs, logic.

3. Euclid never says "there are infinitely many primes." Instead, he writes that "the primes are more than any collection of primes." What does this wording tell you about how Euclid (and Greek mathematicians) thought about infinity?

4 Choose a or b (or both):
a) The "all triangles are isosceles" fallacy used a misleading diagram. What does that teach us about using figures in proofs?
b) Is 'all right angles are equal' a definition, postulate, or proposition? Why?

5 (Optional) Euclid vs. today:
Euclid treated postulates as self-evident truths; modern math treats axioms as chosen assumptions. What difference does this make for how we do mathematics?

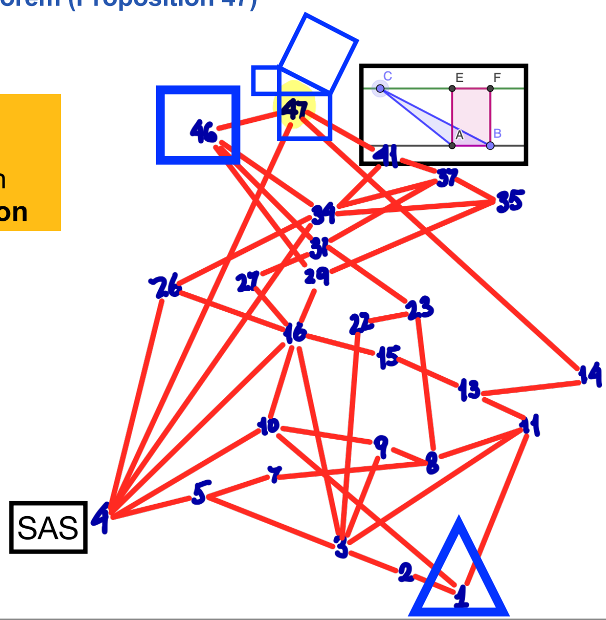
Proof of Pythagorean Theorem

What is the statement of the Pythagorean Theorem?

In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

The structure of the proportions in Euclid's Elements leading to the Pythagorean Theorem (Proposition 47)

The proposition in Euclid's Elements Book I, arranged in **logical progression**

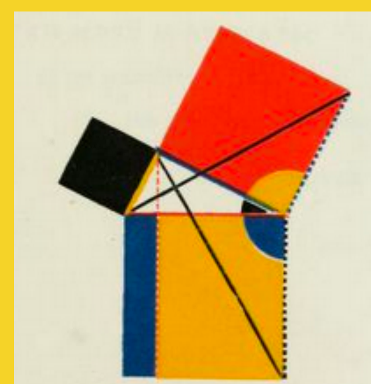


As you follow the next proof, pay attention to two key ideas:

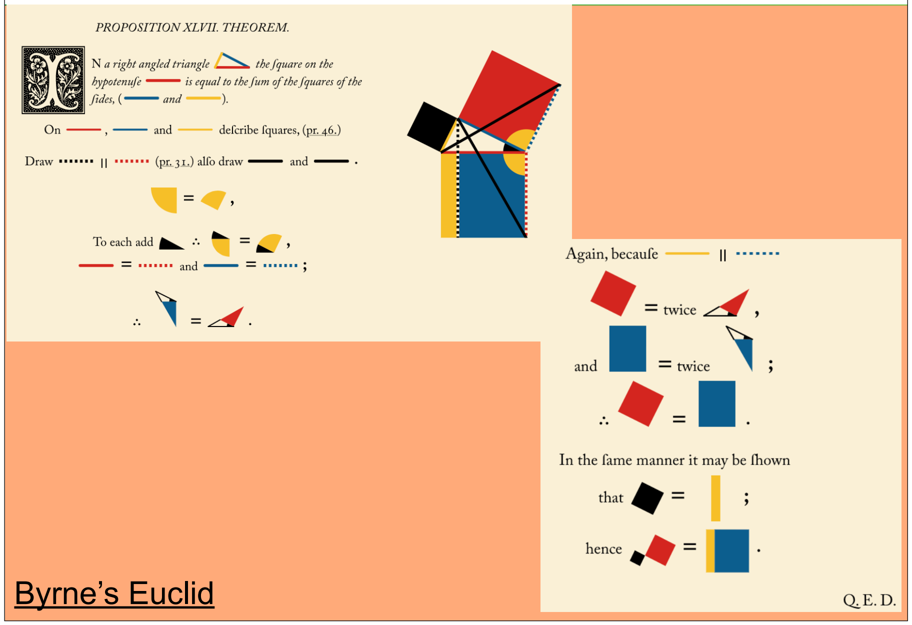
- How does the area of a triangle relate to the area of a rectangle?
- How are congruent triangles used to explain why certain areas are equal?

Try to connect what we saw in the GeoGebra activity to the steps in the proof. You'll be asked to explain this connection afterward.

Later

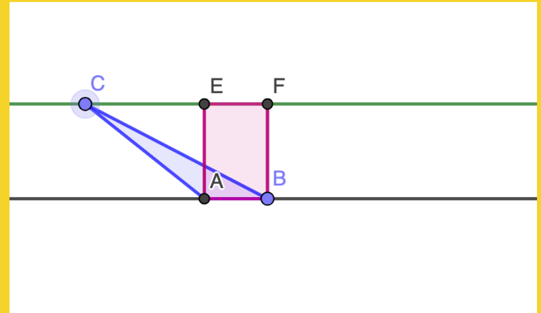


Euclid's Elements Proof of the Pythagorean Theorem



Byrne's Euclid

What is the ratio of the areas of triangle ABC and rectangle EFBA?
Does this ratio change when you drag point C along the green line? Explain why or why not.



One Step in the Proof of Pythagoras' Theorem

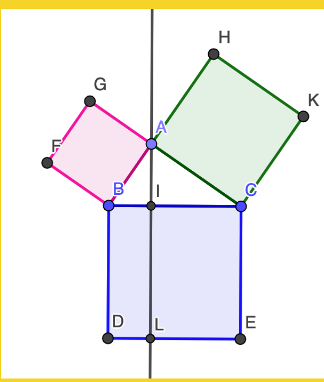
https://www.geogebra.org/m/xyfcawtm

Join the lesson at www.geogebra.org/classroom with the code:

FSTY 4V36

Or you can also share the following link with your students:

www.geogebra.org/classroom/fsty4v36



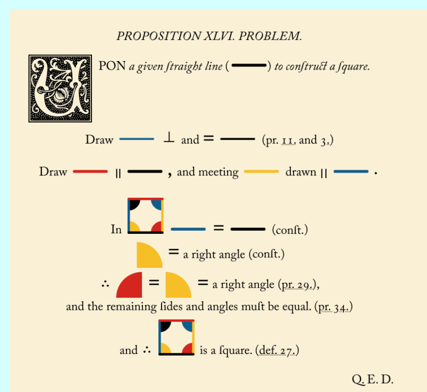
Geogebra (QR Code)
Wooclap What relation holds between the areas of these figures?

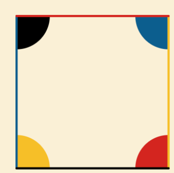
https://www.geogebra.org/classroom/fsty4v36

Euclid's Elements Proof of the Pythagorean Theorem

Prop 46: To describe a square on a given straight line.

Similar to construction of equilateral triangle.





Translation by R. Fitzpatrick

Postulates

1. Let it have been postulated† to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).‡

Common Notions

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.†
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part.

Euclid’s Elements Proof of the Pythagorean Theorem

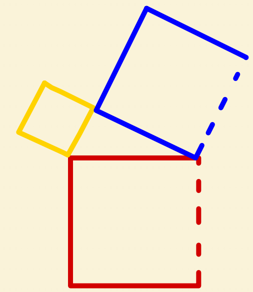
N a right angled triangle the *ſquare on the hypotenuſe* —— *is equal to the ſum of the ſquares of th ſides,* (—— *and* ——).

Check: http://aleph0.clarku.edu/~djoyce/java/elements/bookI/propI47.html

Euclid’s Elements Proof of the Pythagorean Theorem

N a right angled triangle the *ſquare on the hypotenuſe* —— *is equal to the ſum of the ſquares of th ſides,* (—— *and* ——).

On ——, —— and —— deſcribe ſquares, (pr. 46.)



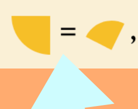
Byrne’s Euclid

Euclid’s Elements Proof of the Pythagorean Theorem

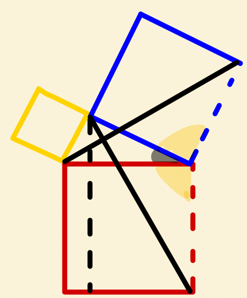
N a right angled triangle the *ſquare on the hypotenuſe* —— *is equal to the ſum of the ſquares of th ſides,* (—— *and* ——).

On ——, —— and —— deſcribe ſquares, (pr. 46.)

Draw ······· || ······· (pr. 31.) alſo draw —— and —— .



Postulate 4



Byrne’s Euclid

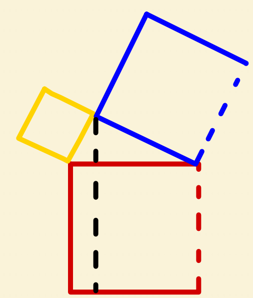
Euclid’s Elements Proof of the Pythagorean Theorem

N a right angled triangle the *ſquare on the hypotenuſe* —— *is equal to the ſum of the ſquares of th ſides,* (—— *and* ——).

On ——, —— and —— deſcribe ſquares, (pr. 46.)

Draw ······· || ······· (pr. 31.)

Postulate 5



Observe the doted black line.

Byrne’s Euclid

Euclid's Elements Proof of the Pythagorean Theorem

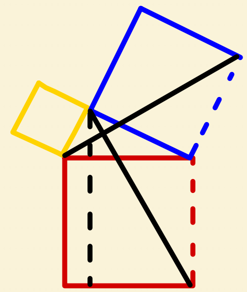
IN *a right angled triangle* *the fquare on the hypotenufe* *is equal to the fum of the fquares of the fides,* (*and*).

On , and defcribe fquares, (pr. 46.)

Draw || (pr. 31.) alfo draw and .

Postulate 1

Byrne's Euclid



Euclid's Elements Proof of the Pythagorean Theorem

IN *a right angled triangle* *the fquare on the hypotenufe* *is equal to the fum of the fquares of the fides,* (*and*).

On , and defcribe fquares, (pr. 46.)

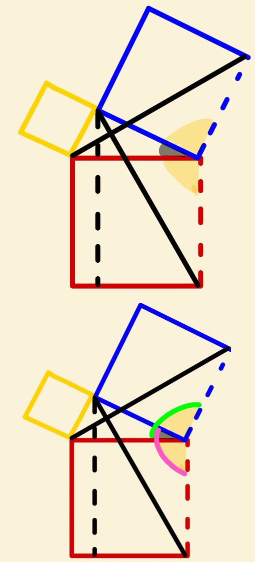
Draw || (pr. 31.) alfo draw and .

= ,

To each add ∴ = ,

Common Notion 2

Byrne's Euclid



Euclid's Elements Proof of the Pythagorean Theorem

IN *a right angled triangle* *the fquare on the hypotenufe* *is equal to the fum of the fquares of the fides,* (*and*).

On , and defcribe fquares, (pr. 46.)

Draw || (pr. 31.) alfo draw and .

= ,

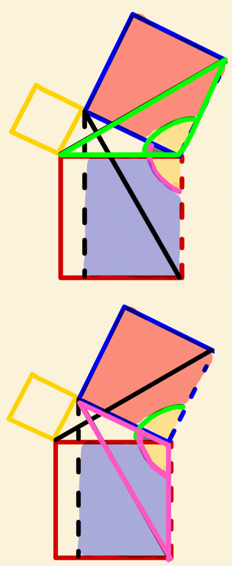
To each add ∴ = ,

= and = ;

∴ = .

Proposition 4

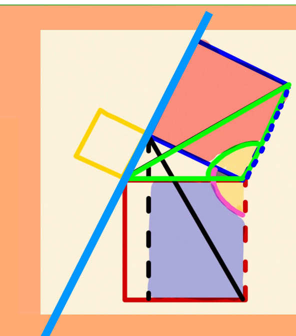
Byrne's Euclid



Euclid's Elements Proof of the Pythagorean Theorem

Again, becaufe ||

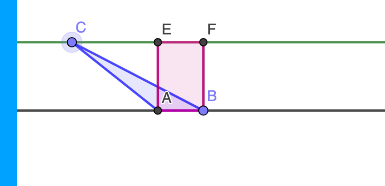
= twice ,

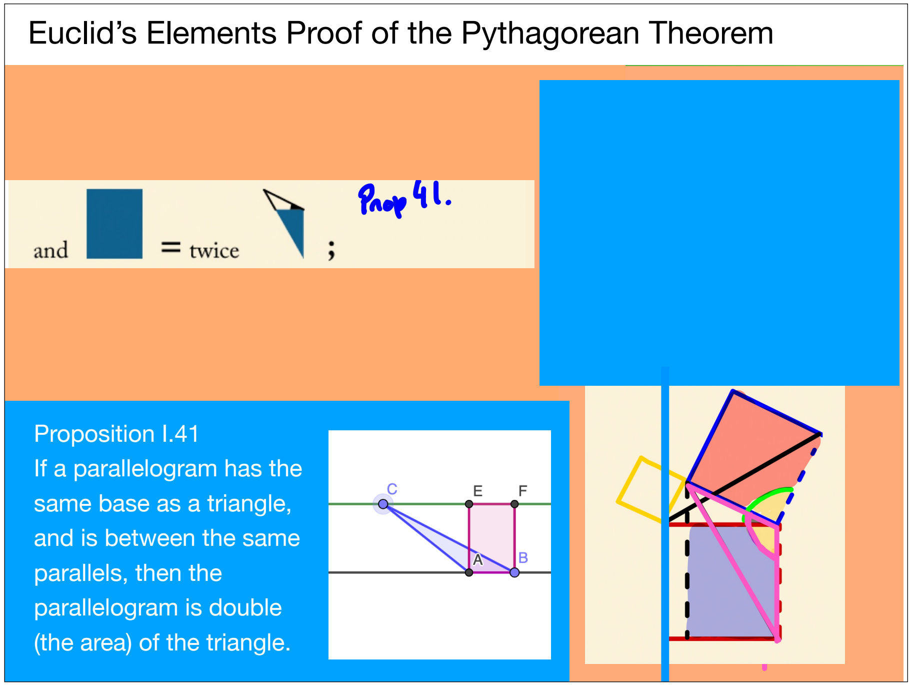


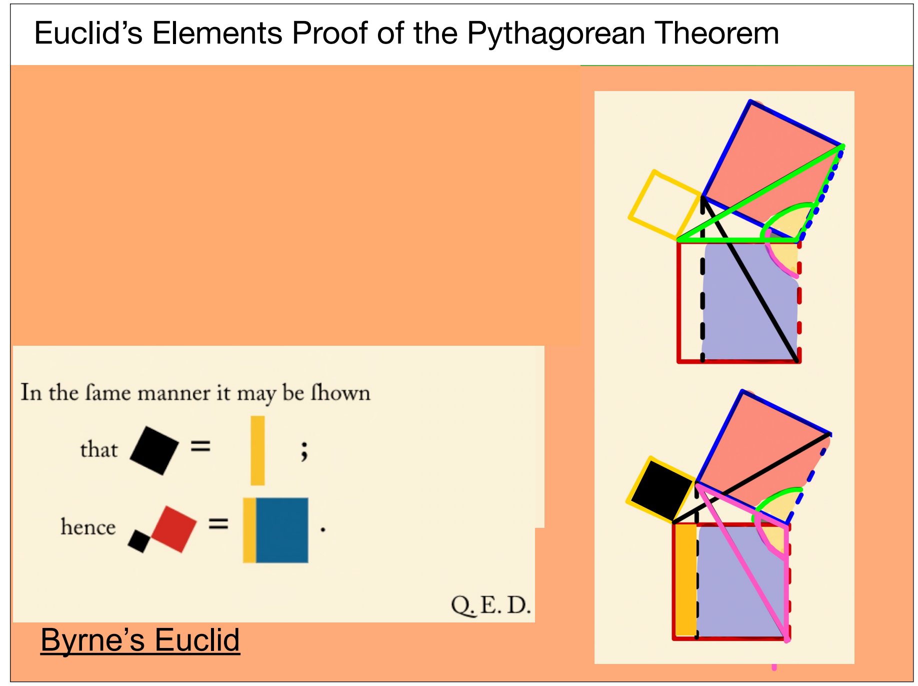
Proposition I.41

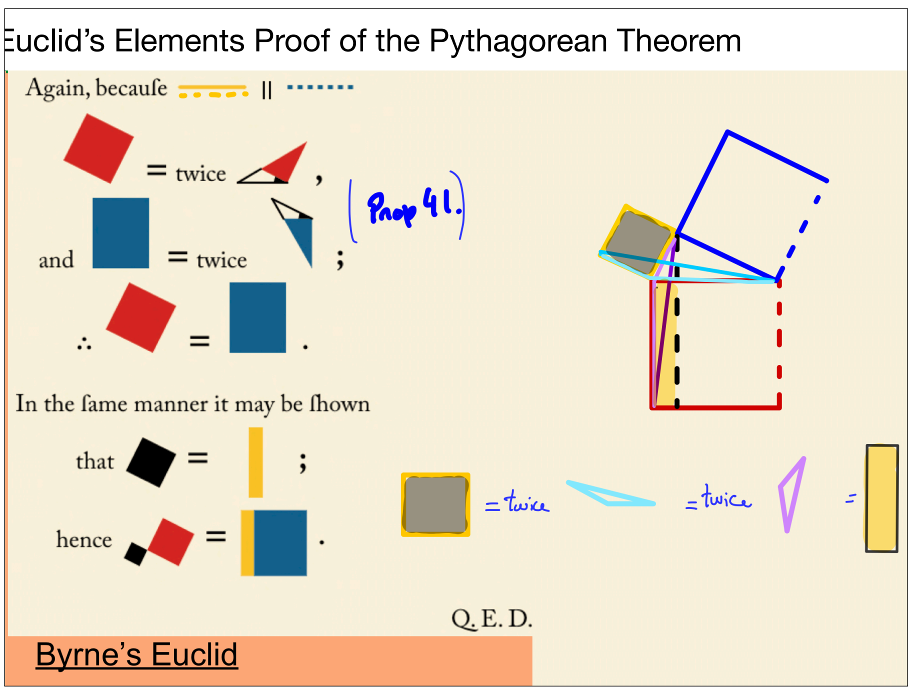
If a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle.

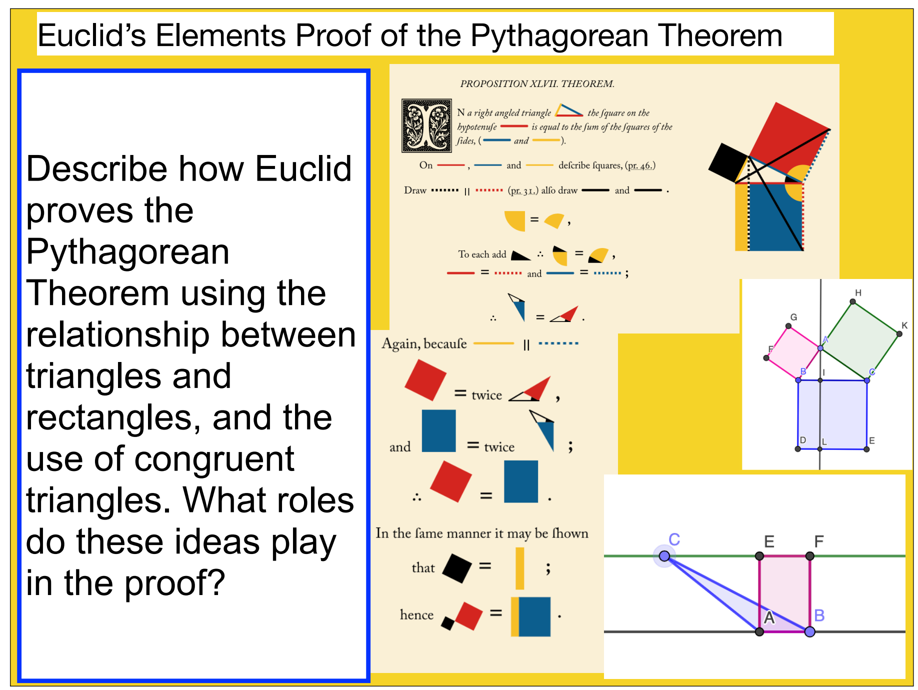
Recall




Euclid's Elements Proof of the Pythagorean Theorem
and = twice ;
Prop 41.
Proposition I.41
If a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle.
C
E
F
A
B


Euclid's Elements Proof of the Pythagorean Theorem
In the ſame manner it may be ſhown
that = ;
hence = .
Q. E. D.
Byrne's Euclid

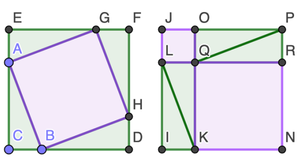

Euclid's Elements Proof of the Pythagorean Theorem
Again, becauſe ||
= twice ,
and = twice ;
(Prop 41.)
∴ = .
In the ſame manner it may be ſhown
that = ;
hence = .
Q. E. D.
Byrne's Euclid
= twice
= twice
=

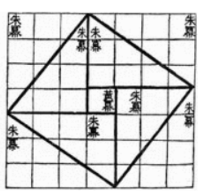

Euclid's Elements Proof of the Pythagorean Theorem
Describe how Euclid proves the Pythagorean Theorem using the relationship between triangles and rectangles, and the use of congruent triangles. What roles do these ideas play in the proof?
PROPOSITION XLVII. THEOREM.
Again, becauſe ||
= twice ,
and = twice ;
∴ = .
In the ſame manner it may be ſhown
that = ;
hence = .
C
E
F
A
B

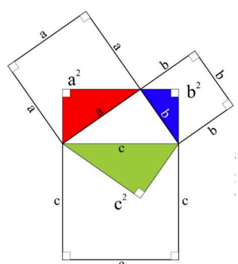
The Pythagorean Theorem Around the World

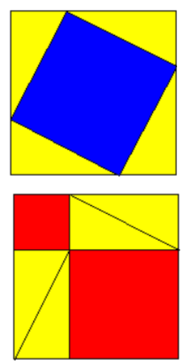
The Pythagorean Statement Across Time and Space

- **Mesopotamia** (c. 1900–1600 BCE): Computed specific right-triangle ratios and √2; no general statement or proof.
- **India** (c. 800–500 BCE) Śulba Sūtras: verbal general rule for rectangles; used in altar geometry.
- **China** (c. 500 BCE – 200 CE) Gou-gu theorem; geometric dissection proof by Zhao Shuang.
- **Greece** (c. 500–300 BCE) Euclid I.47: first axiomatic proof that areas on the legs equal that on the hypotenuse.
- **Islamic World** (8th–15th c.) Many geometric and algebraic proofs; extensions toward the law of cosines.
- **Europe** (12th–16th c.) Theorem transmitted via Arabic → Latin → printed Elements; used in surveying, art, and education.
- No verified versions in Egypt or the Americas.









The illustrations come from the site

https://www.cut-the-knot.org/pythagoras/

where you can find 122 proofs of the Pythagorean theorem.

Euclid's proof is axiomatic.

https://mathvoices.ams.org/mathmedia/tonys-take-september-2023/#two

https://www.tandfonline.com/doi/epdf/10.1080/00029890.2024.2370240?needAccess=true

More About Euclid's Elements

Which of the following statements about Euclid's Elements is true? Explain your choice.

A. All the logical consequences of Euclid's postulates and common notions were stated and proven in the Elements. Therefore, no new propositions were proven after Euclid.

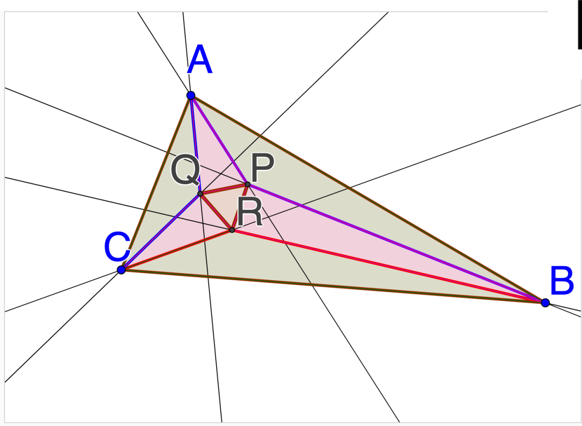
B. All the logical consequences of Euclid's postulates and common notions were either proven in the Elements or before the fifteenth century. Since then, no new propositions have been proven.

C. There are logical consequences of Euclid's postulates and common notions that have not yet been proven.

(Hint: Can a finite set of written propositions exhaust all the logical consequences of a system?)

Morley's trisector theorem, 1899

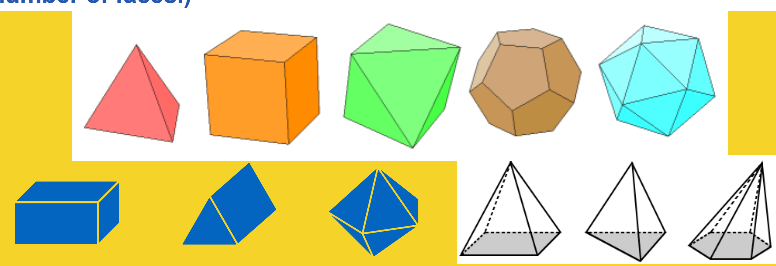
https://ggbm.at/FRZ9Nfec



Compute V, E, F and then

V – E + F

for two or more of the polyhedra below.
What value do you get?
Does the value depend on which polyhedron you chose?
Do you think this will always give the same value for *any* polyhedron?
(V is the number of vertices, E is the number of edges, and F is the number of faces.)



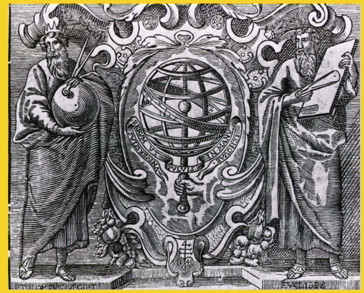
Euler's Polyhedral Formula

In a polyhedron, the following equation holds: **V – E + F = 2**, where V is the number of vertices, E is the number of edges, and F is the number of faces. Descartes (~1600), Euler (~1700)

If some of the propositions in Euclid's Elements are added as axioms, then…

a) A contradiction can be deduced from the new (larger) set of axioms.
b) All the propositions in Euclid's Elements can still be proven with the new (larger) set of axioms.
c) Some, but not all, of the propositions in Euclid's Elements can be proven with the new (larger) set of axioms.
Choose one option and explain your reasoning.

Hint: Think of axioms as tools in a toolbox. Propositions are things you build with those tools. If you put something you already built back into the toolbox, can you still build everything you built before? Could you accidentally "build something wrong" with the new set of tools?



Full length portraits of Euclid and Ptolemy, each holding a compass; a sphere on a staff is between them.
- Cardano, Girolamo, 1501-1576,

According to Proclus, Ptolemy (the pharaoh) once asked Euclid if there was not a shorter road to the knowledge of geometry than by the study of the Elements, and Euclid replied:

There is no royal road to geometry.

The Philosophers (Ptolemy and Euclid with Their Pupils) Pietro della Vecchia - 1600

What do you think Euclid meant by saying 'There is no royal road to geometry'?

Do you think there's a 'royal road' to geometry today that didn't exist in Euclid's time?

Tetradrachm of Ptolemy I, British Museum, London

All the classical theorems of geometry in the plane and geometry in the space, as well as many results in number theory are deduced from the axioms in Euclid's Elements. 😮!!!

Constructions

- On a given line, a right angle can be constructed. (I.11)
- From a point, a perpendicular can be dropped to a given line. (I.12)
- Through a point not on a line, a parallel to the line can be drawn. (I.31)
- A given angle can be bisected. (I.9)
- A regular pentagon can be inscribed in a circle. (IV.11)

Plane geometry (triangles & parallels)

- In an isosceles triangle, the base angles are equal. (I.5)
- The angles of a triangle are equal two right angles. (I.32)
- Opposite sides and opposite angles of a parallelogram are equal. (I.34–35)
- The Pythagorean theorem. (I.47)
- The converse of the Pythagorean Theorem: if the square on one side equals the sum on the other two, the angle is right. (I.48)

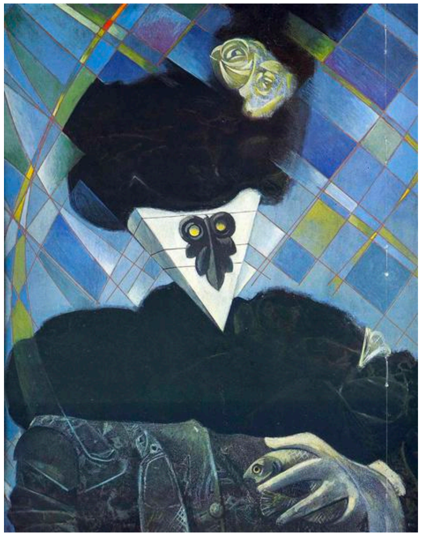
All the classical theorems of geometry in the plane and geometry in the space, as well as many results in number theory are deduced from the axioms in Euclid's Elements. 😮!!!

Circles

- A tangent is perpendicular to the radius at the point of contact. (III.18)
- The angle at the center is double the angle at the circumference standing on the same arc. (III.20)
- The angle in a semicircle is a right angle. (III.31)

Number Theory

- If a prime divides a product, it divides at least one factor (Euclid's Lemma). (VII.30)
- Any two numbers have a greatest common measure; the Euclidean algorithm finds it. (VII.1–2)
- Relatively prime numbers share no common measure except the unit. (VII.12)
- No finite list contains all primes; there is always another. (IX.20)
- Even perfect numbers arise from primes of the special form "two to a power minus one": the number is "two to one less than that power" times "two to the power minus one." (IX.36)



Max Ernst, 'Euclid'. 1945. - Fair Use