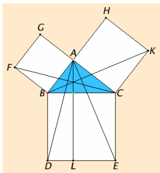
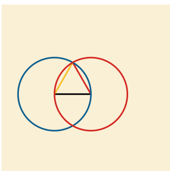
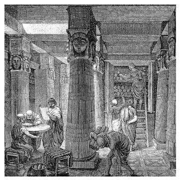


Euclid's Elements

Euclid

Alexandria
- Alexandria and the library

The elements
- Copies of copies. Remember books are manuscript.
- There were other Element's of Geometry. Only Euclid's arrived to us.
- Great impact around the world.
- Most copies printed after the Bible.
- Alteration by Theon.
- Fragments of people working out Euclid in broken pottery.
- Note that Euclid's element is the compilation of geometry that arrived to us. It was not created only by Euclid, and was not created overnight.
- Note that line (in the elements) means a curve, possible straight.
- It is really hard for students to understand what are axioms, and the logical structure of the elements.

- Many definitions are vague (point, line) and not used during the book. The circle is not an example of this, the definition is precise, and used

Common notions
- Logical statements, but rely on other terms, which are not defined in the Elements..

The postulates
- The first three are fine for students, although some fail to see the need of stating them. .
- The fourth is puzzling for student.
- The fifth is long has to be explained by making a figure, while reading it.

Logical structure
- Book 1: purpose is to prove Pythagorean theorem?
- Construction of equilateral triangle: Tell it is like a recipe for cooking.
- Gaps
- Are all triangles isosceles
- Number theory
- Axiom systems
- Morley theorem

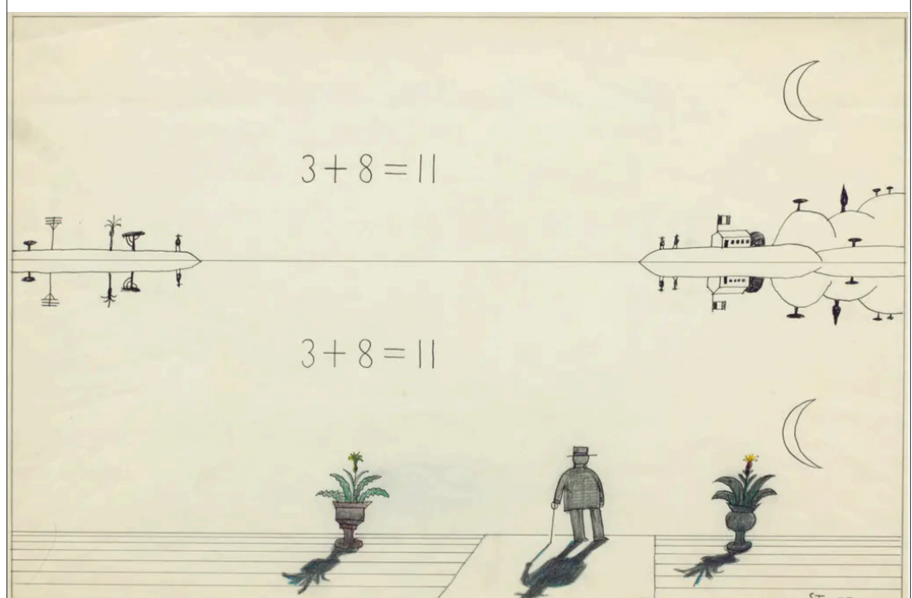
Euclid

L03
- Oct 15 The Two Earths of Eratosthenes — Takara
- Oct 20 Regular Polygons in Ancient Mathematics — Melody
- Oct 20 The Historical Development of the Platonic Solids — Keyue

L01
- Oct 15: The Two Earths of Eratosthenes
  - Mulik L01
  - Takara L03
- Oct 20: Regular Polygons in Ancient Mathematics
  - Ze Lect 1
  - Melody Lect 3
- Oct 20: The Platonic Solids
  - Yuna Lect 1
  - Keyue Lect 3 (THistorical Development of Plat.Sol.)
- Oct 22 The Parallel Postulate and its Role in Geometry
  - Lect 1 Isabella
  - Lect 3 Yutong
- Oct 22 The Story of the Binomial Theorem
  - Lect 1 Brian
  - Lect 3 Ellie

Math History Myth bursting
- Pythagoras
- Plimpton
- Ishango bone
- Equations.
- Euclid

Oct 15, 20 22.
Quiz 3 October 15



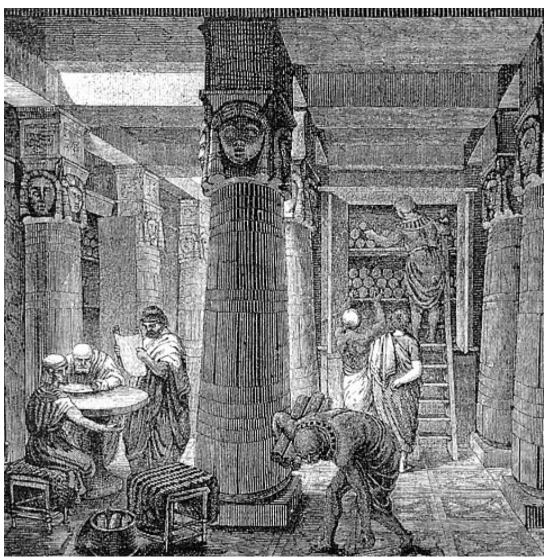
Steinberg

Alexandria

Map of Ancient Hellenic Mathematics Encyclopedia Britannica

It is likely that Euclid worked and taught there

About 500,000 volumes



Nineteenth-century artistic rendering of the Library of Alexandria by the German artist O. Von Corven, based partially on the archaeological evidence available at that time

Euclid's Elements: What are they

Put these four steps of the axiomatic method in order from start to finish. (If some overlap or seem simultaneous, choose any order)

- Use logical reasoning
- Derive a new proposition
- Establish definitions
- Justify each step using axioms or previously proved propositions
- Set up agreed principles (axioms)

Euclid's Elements

- Greek mathematical treatise
- Compiled around 300 BCE
- Attributed to Euclid
- Draws on centuries of accumulated mathematical work.
- Earliest extant work in **axiomatic method (**Propositions are arranged in a **logical progression**, where each step is justified by earlier results and axioms.)
- Its logical structure still underlies modern mathematics.

Covers

- **plane geometry** (shapes in the plane)
- **solid geometry** (shapes in space)
- **numbers & ratios** (divisibility, primes, proportions, irrationals)

Mathematics Is Built in The Elements: The Axiomatic Method

Axiomatic method:

- Starts with **definitions and agreed principles** (called **axioms** now**, postulates and common notions** in Euclid),
- New **propositions** are derived by **logical proof**;
- **Each step is justified** by
  - those principles or
  - by previously proved propositions.

Note: Nowadays, there are also undefined terms.

Euclid's Elements: Quick Overview

- **Definitions:** Terms like point, line, circle, angle, etc.
- **Postulates:** basic assumptions about geometry. (e.g., "*a straight line can be drawn between any two points*")
- **Common Notions**: intuitively obvious facts about magnitudes (e.g., "*things equal to the same thing are equal to each other*")
- **Propositions**:

  Theorems: In right-angled triangles, the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

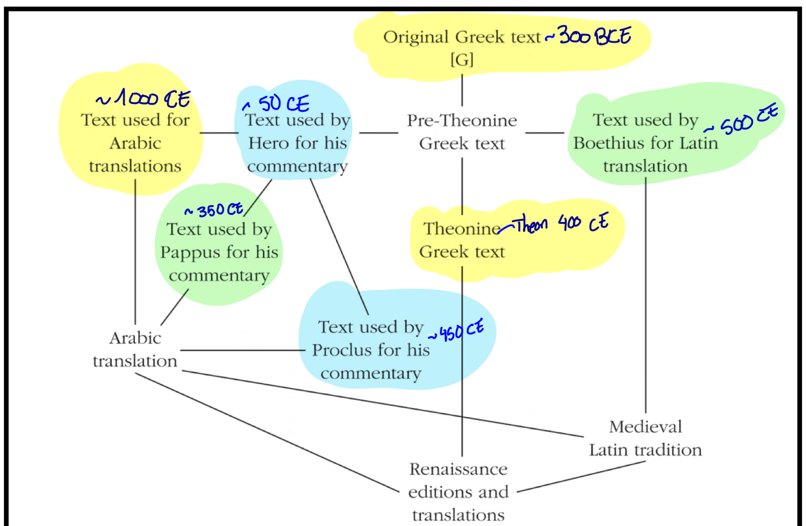
  Constructions: Construct an equilateral triangle, given a slide.

Axiomatic method: Starting from agreed principles (called **axioms**), new propositions are derived by logical proof; each step is justified by those principles or by previously proved propositions.

Euclid's elements	Today's math
Definitions	**Undefined** terms Definitions
Common notions Postulates **(self evident truths)**	Axioms **(assumptions wisely chosen)**
Propositions Constructions	Propositions Constructions

Axiomatic method: Starting from agreed principles (called **axioms**), new propositions are derived by logical proof; each step is justified by those principles or by previously proved propositions.

History and Impact. Variations



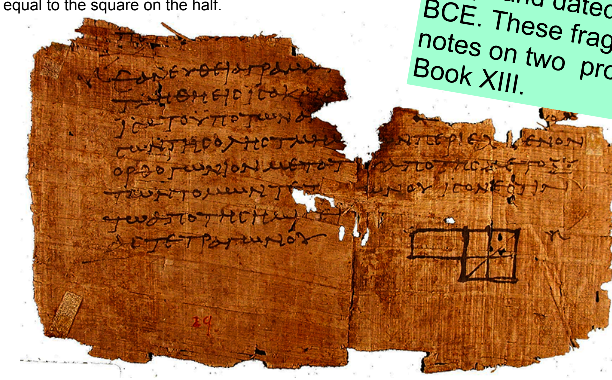
The medieval and Renaissance tradition of Euclid's *Elements* (adapted from Murdoch (1971))

Diagram from the book "Ancient Mathematics" by Serafina Cuomo

One of the Oldest Fragment of Euclid's Elements, dated from 1st century CE,

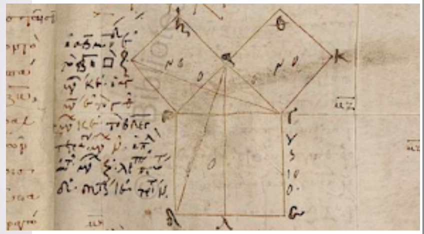
Proposition II.5: If a straight line is cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole, together with the square on the straight line between the points of the section, is equal to the square on the half.

Also there are fragments found in potsherds discovered in Egypt and dated from 225 BCE. These fragments contain notes on two propositions from Book XIII.



http://www.math.ubc.ca/~cass/Euclid/papyrus/

The 9th-century Vatican manuscript, Vat. gr. 190 Euclid's work possibly without major adulteration (in particular, without Theon's note).



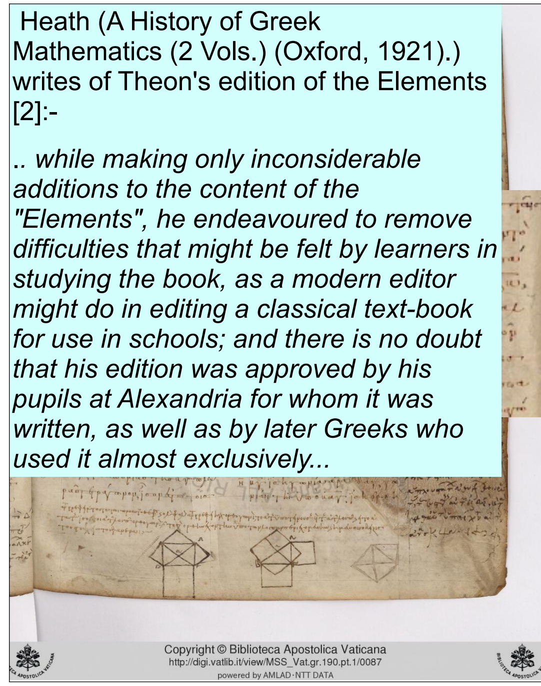
Book I proposition 47

The Pythagorean Theorem

https://digi.vatlib.it/view/MSS_Vat.gr.190.pt.1/0087

Heath (A History of Greek Mathematics (2 Vols.) (Oxford, 1921).) writes of Theon's edition of the Elements [2]:-

.. while making only inconsiderable additions to the content of the "Elements", he endeavoured to remove difficulties that might be felt by learners in studying the book, as a modern editor might do in editing a classical text-book for use in schools; and there is no doubt that his edition was approved by his pupils at Alexandria for whom it was written, as well as by later Greeks who used it almost exclusively...



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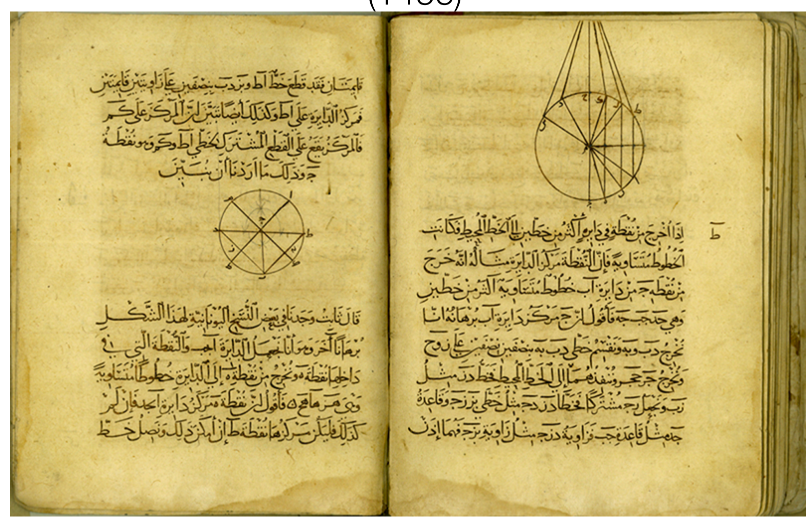
Book I proposition 47

The Pythagorean Theorem

https://digi.vatlib.it/view/MSS_Vat.gr.190.pt.1/0087

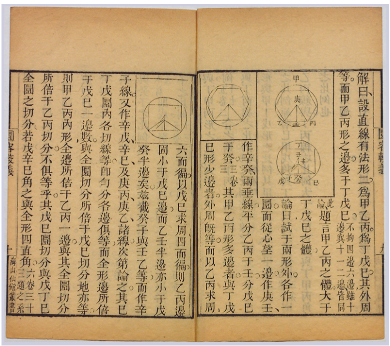
An illumination from a manuscript based on Adelard of Bath's translation of the *Elements*, *circa* 1309–1316; Adelard's is the oldest surviving translation of the *Elements* into Latin, done in the 12th-century work and translated from Arabic. (Wikipedia)

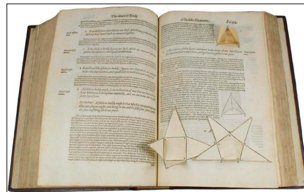
Early Translation of Euclid's Elements into Arabic (1466)

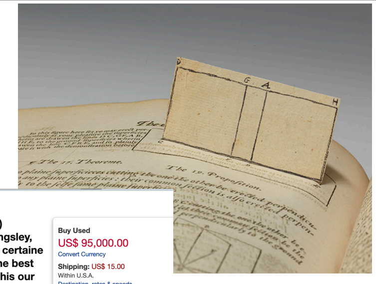


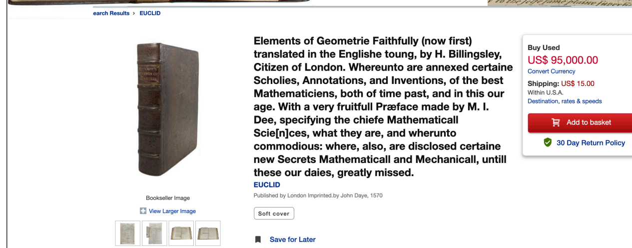
https://www.maa.org/press/periodicals/convergence/mathematical-treasure-early-translation-of-euclids-elements-into-arabic

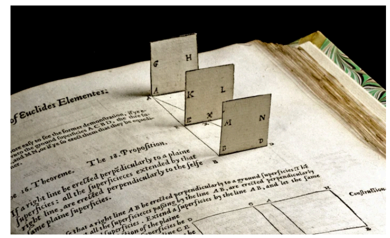
The Italian Jesuit Matteo Ricci (left) and the Chinese mathematician Xu Guangqi (right) published the Chinese edition of Euclid's Elements (幾何原本) in 1607. (Wikipedia)



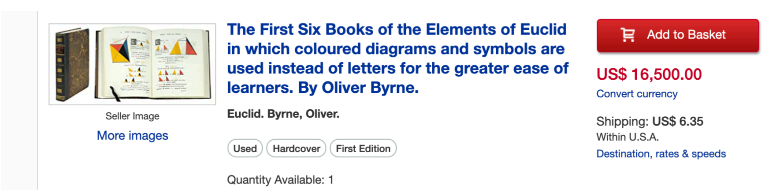




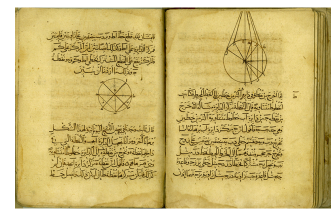


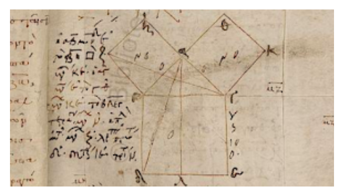


https://www.c82.net/euclid/



Euclid's element's

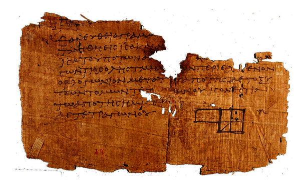




The 9th-century Vatican manuscript, Vat. gr. 190 Euclid's work possibly without major adulteration (in particular, without Theon's note).

were not created overnight

were not created by Euclid alone.



"Euclid alone has looked on Beauty bare."

BY EDNA ST. VINCENT MILLAY

Euclid alone has looked on Beauty bare.
Let all who prate of Beauty hold their peace,
And lay them prone upon the earth and cease
To ponder on themselves, the while they stare
At nothing, intricately drawn nowhere
In shapes of shifting lineage; let geese
Gabble and hiss, but heroes seek release
From dusty bondage into luminous air.
O blinding hour, O holy, terrible day,
When first the shaft into his vision shone
Of light anatomized! Euclid alone
Has looked on Beauty bare. Fortunate they
Who, though once only and then but far away,
Have heard her massive sandal set on stone.

https://www.poetryfoundation.org/poems/148566/euclid-alone-has-looked-on-beauty-bare

Definitions

In groups of 3:
Order these four Euclidean definitions from easiest to hardest to visualize and understand.
Then, explain the reasoning behind your order.
Hint: If you didn't already know the terms, could you understand what the object is from the definition alone?

Definition 1: A **point** is that which has no part.

Definition 2: A **line** is breadthless length.

Definition 10: When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is called **right**, and the straight line standing on the other is called a **perpendicular** to that on which it stands.

Definition 15: A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

Definition 1.
A ***point*** is that which has no part.

Definition 2.
A ***line*** is breadthless length.

Definition 3.
The ***ends*** of a line are points.

Definition 4.
A ***straight line*** is a line which lies evenly with the points on itself.

First four definitions of Book 1 of Euclid's Elements.

Axioms, postulates and common notions

Common notions

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part

The Greek text of J.L. Heiberg translated by Richard Fitzpatrick https://farside.ph.utexas.edu/Books/Euclid/Elements.pdf

Postulates

1. Let it have been postulated to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).

The Greek text of J.L. Heiberg translated by Richard Fitzpatrick https://farside.ph.utexas.edu/Books/Euclid/Elements.pdf

Euclid's five postulates

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines intersect each other on that side if extended far enough.

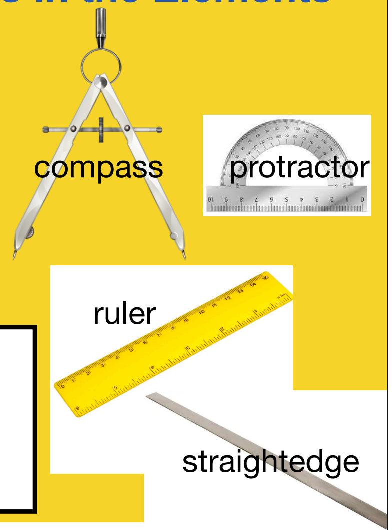
The first four postulates were straightforward; they were considered "truths". (Recall Aristotle's definition: "An axiom is a statement worthy of acceptance.")

The fifth postulate isn't simple or straightforward; it *feels* more like a proposition or theorem than an axiom.

The first three postulates in the Elements

- Postulate I asserts that it is possible to draw a segment through any two given points.
- Postulate II says that any segment can be extended to a longer segment.
- Postulate III states that it is possible to construct a circle with any given center and radius.

These first three postulates are associated with the tools that are used to implement them on a piece of paper.



Which tool is each of these postulates associated with? (Select the option with only the essential features)

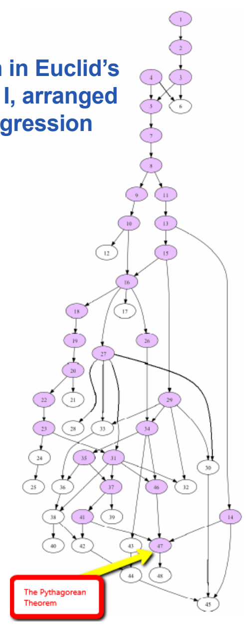
Postulate 4 states "*All right angles are equal one another.*" This seems self-evident. Why does Euclid need to write it down as a postulate?

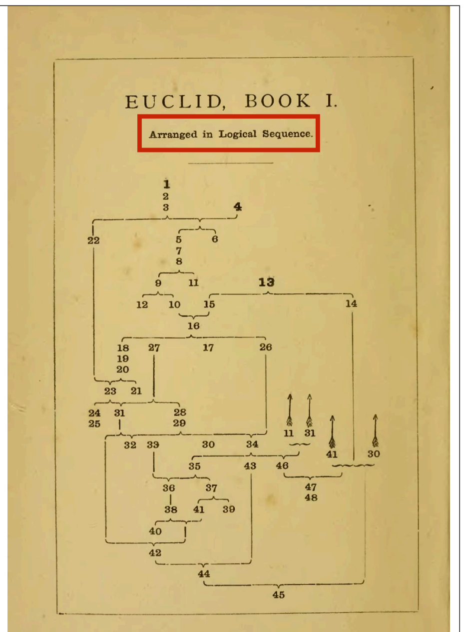
Euclid's Element Definition: When a straight line set up on another straight line makes the adjacent angles equal to one another, each of the equal angles is called a **right** angle.

Euclid's Element Definition in Modern Language: A **right** angle is an angle formed when one straight line stands on another straight line making the two adjacent angles equal to each other. Each of these equal angles is a right angle.

. Eyer's graph of Euclid's Book I
https://www.maa.org/book/export/html/590371

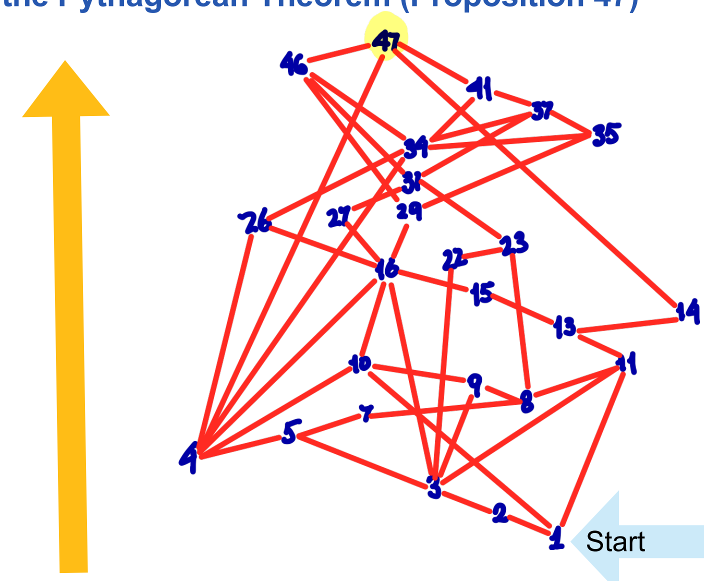
The proposition in Euclid's Elements Book I, arranged in logical progression

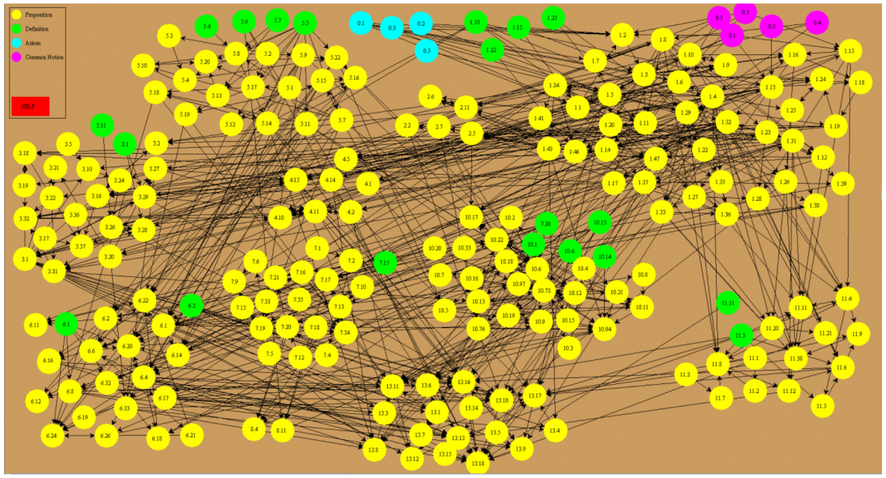




From Euclid and His Modern Rivals By Lewis Carroll

The structure of the proportions in Euclid's Elements leading to the Pythagorean Theorem (Proposition 47)





A graph of all 13 books of Euclid's *Elements*

https://www.maa.org/book/export/html/590371

Postulate 5'. (Playfair)
For any given point not on a given line, there is exactly one line through the point that does not meet the given line.

Byrne's version

If two ſtraight lines ( ) meet a third ſtraight line ( ) ſo as to make the two interior angles (and) on the ſame ſide leſs than two right angles, theſe two ſtraight lines will meet if they be produced on that ſide on which the angles are leſs than two right angles.

Postulate 5.
That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Proposition I.1 from Euclid's Elemen ts: Construction of an equilateral triangle

An equilateral triangle is a triangle that… (complete the sentence, you have one minute)

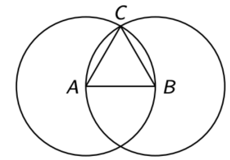
An equilateral triangle is a triangle that has three equal sides.

Proposition 1: Given any finite segment, it is possible to construct an equilateral triangle having that segment as a side. **(Modern Version)**

Construction (Part 1 of the proof):

- Draw a circle with center A and distance AB.(Post.3)
- Draw a circle with center B and distance AB.(Post.3)
- Let C be a point where the circles intersect.
- Draw segments AC and BC. (Post.1)

- ABC is the required triangle.



1. Post 1.Let it have been postulated to draw a straight-line from any point to any point.

3. Post. 3: to draw a circle with any center and radius.

Proposition 1 of Euclid's Elements. Construction of equilateral triangle

Join the lesson at www.geogebra.org/classroom with the code:

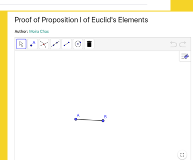
CWQM WCMD

Or you can also share the following link with your students:

www.geogebra.org/classroom/cwqmwcmd

Lecture 1
https://www.geogebra.org/classroom/cwqmwcmd

Go to https://www.geogebra.org/classroom
Code: CWQM WCMD



1. **Construction**: In the Geogebra window, construct an equilateral triangle.
2. **Now explain in Wooclap: Why does this construction guarantee an equilateral triangle? (Hint: recall the definition of equilateral triangle)** (Use complete sentences)
3. **Optional**: Justify each step of your explanation using the Postulates and Common Notions below.
4. **Super Optional:** Compare the instructions given in 1. with this one , which is a translation of the original Euclid's Element. What are the differences?)

Proposition 1 of Euclid's Elements. Construction of equilateral triangle

Join the lesson at www.geogebra.org/classroom with the code:

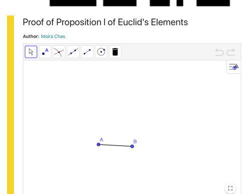
XSFW Q9BC

Or you can also share the following link with your students:

www.geogebra.org/classroom/xsfwq9bc

https://www.geogebra.org/classroom/xsfwq9bc

Go to https://www.geogebra.org/classroom
Code. XSFW Q9BC



1. **Construction**: In the Geogebra window, construct an equilateral triangle.
2. **Now explain in Wooclap: Why does this construction guarantee an equilateral triangle? (Hint: recall the definition of equilateral triangle)** (Use complete sentences)
3. **Optional**: Justify each step of your explanation using the Postulates and Common Notions below.
4. **Super Optional:** Compare the instructions given in 1. with this one , which is a translation of the original Euclid's Element. What are the differences?)

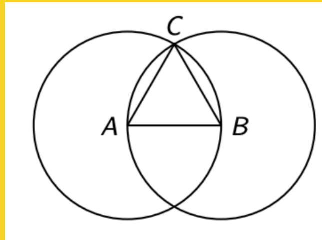
Why does this construction guarantee an equilateral triangle? (Hint: recall the definition of equilateral triangle)

Proposition 1: Given any finite segment, it is possible to construct an equilateral triangle having that segment as a side.

Construction (Part 1 of the proof):

- Draw a circle with center A and distance AB.(Post.3)
- Draw a circle with center B and distance AB.(Post.3)
- Let C be a point where the circles intersect.
- Draw segments AC and BC. (Post.1)

- ABC is the required triangle.



Modern version
Of Euclid's Elements

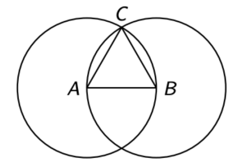
Proposition 1: Given any finite segment, it is possible to construct an equilateral triangle having that segment as a side. **(Modern Version)**

- Denote the segment by AB.
- Draw a circle with center A and distance AB.(Post.3)
- Draw a circle with center B and distance AB.(Post.3)
- Let C be a point where the circles intersect.
- Draw segments AC and BC. (Post 1)

- ABC is the required triangle.

- AB = AC (Definition 15 **AB and AC are radii of the same circle)**
- AB = BC (Definition 15) **AB and BC are radii of the same circle)**

- AC = BC (Common Notion 1)
- Hence, ABC is an equilateral triangle (Definition 20) and it is on the given line AB.



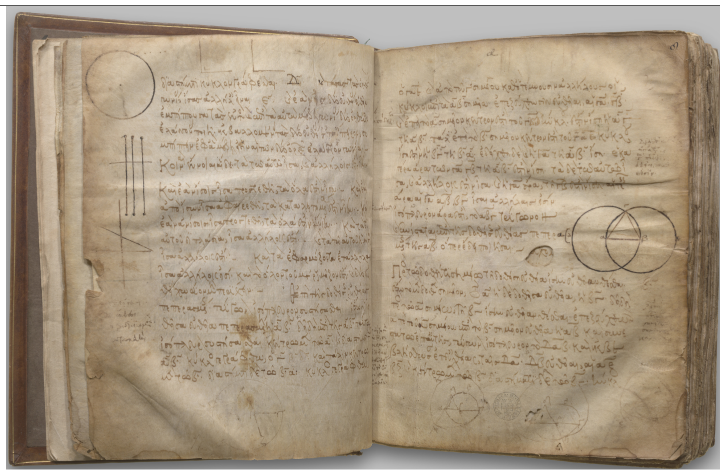
Post 1.Let it have been postulated to draw a straight-line from any point to any point.

Post.3: to draw a circle with any center and radius.

Definition 15. A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

Definition 20: If all three sides are of the same length, then a triangle is a **equilateral** /

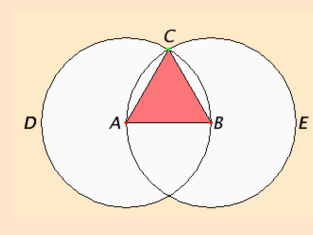
Common notion 1: Things that are equal to the same thing are also equal to one another.



Proposition I.1 – The thirteen books of Euclid's Elements MS D'Orville Manuscript, copied in 888 AD. (one of the two oldest extant complete manuscripts of the Elements.)
Original works of Euclid did not contain images embedded in the text, (we know this primarily through historical, textual, and manuscript evidence combined with knowledge of ancient Greek literary and mathematical practices)

Proposition 1: To construct an equilateral triangle on a given finite straight-line

Let *AB* be the given finite straight line.



It is required to construct an equilateral triangle on the straight line *AB*.

Describe the circle *BCD* with center *A* and radius *AB*. Again describe the circle *ACE* with center *B* and radius *BA*. Join the straight lines *CA* and *CB* from the point *C* at which the circles cut one another to the points *A* and *B*. I.Post.3 I.Post.1

Now, since the point *A* is the center of the circle *CDB*, therefore *AC* equals *AB*. Again, since the point *B* is the center of the circle *CAE*, therefore *BC* equals *BA*. I.Def.15

But *AC* was proved equal to *AB*, therefore each of the straight lines *AC* and *BC* equals *AB*.

And things which equal the same thing also equal one another, therefore *AC* also equals *BC*. C.N.1

Therefore the three straight lines *AC*, *AB*, and *BC* equal one another.

Therefore the triangle *ABC* is equilateral, and it has been constructed on the given finite straight line *AB*. I.Def.20

Q.E.F.

Text and figure by David Joyce https://mathcs.clarku.edu/~djoyce/elements/bookI/propI1.html

Postulates

1. Let it have been postulated† to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).‡

Common Notions

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.†
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part.

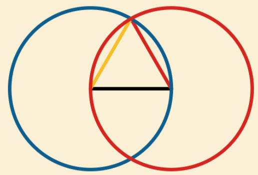
15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.

20. And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having three unequal sides.

Proposition 1: Given any finite segment, it is possible to construct an equilateral triangle having that segment as a side.

PROPOSITION I. PROBLEM.

ON *a given finite ſtraight line* ( ) *to deſcribe an equilateral triangle.*



Deſcribe and (poſtulate 3.); draw and (poſt. 1.). then will be equilateral.

For = (def. 15.);
and = (def. 15.),
∴ = (axiom. 1.);

and therefore is the equilateral triangle required.

Q. E. D. Byrne's Euclid

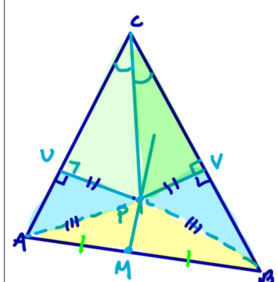
Is every step appropriately justified in Proposition I.1

Proposition 1: Given any finite segment, it is possible to construct an equilateral triangle having that segment as a side. **(Modern Version)**

- Denote the segment by AB.
- Draw a circle with center A and distance AB.(Post.3)
- Draw a circle with center B and distance AB.(Post.3)
- Let C be a point where the circles intersect.
- Draw segments AC and BC. (Post 1)
- ABC is the required triangle.

Is every step justified?

- AB = AC (Definition 15 **AB and AC are radii of the same circle)**
- AB = BC (Definition 15) **AB and BC are radii of the same circle)**
- AC = BC (Common Notion 1)
- Hence, ABC is an equilateral triangle (Definition 20) and it is on the given line AB.

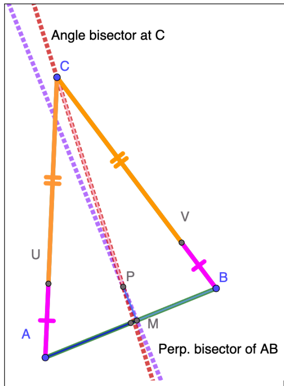


- Let **M** be the midpoint of **AB**
- Let **P** be the intersection of the perpendicular bisector of **AB** and the angle bisector at **C**
- Let **U** be the foot of the perpendicular from **P** to line **AC**
- Let **V** be the foot of the perpendicular from **P** to line **BC**

Congruent triangles:

- **△AMP ≅ △BMP**
- **△PCU ≅ △PCV**
- **△APU ≅ △BPV**

Find a statement proven by this argument



RH = Right angle-Hypotenuse-Side

If two right triangles have congruent hypotenuses and one pair of congruent legs, they're congruent.

https://www.geogebra.org/m/yyszz85d

- Let **M** be the midpoint of **AB**
- Let **P** be the intersection of the perpendicular bisector of **AB** and the angle bisector at **C**
- Let **U** be the foot of the perpendicular from **P** to line **AC**
- Let **V** be the foot of the perpendicular from **P** to line **BC**

Congruent triangles:

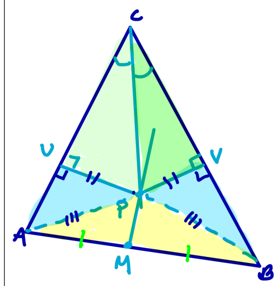
- △**AMP** ≅ △**BMP** (By SAS, since ∠**PMA** = ∠**PMB**, **AM** = **BM**, **MP** = **MP**). Then **AP** = **BP**
- △**PCU** ≅ △**PCV** (By AAS, since ∠**PUC** = ∠**PVC**, ∠**PCU** = ∠**PCV**, **PC** = **PC**). Then **PU** = **PV** and **CU** = **CV**
- △**APU** ≅ △**BPV** (By RH, since ∠**PUA** = ∠**PVB**, **AP** = **BP**, **PU** = **PV**). Then **AU** = **BV**

Conclusion:

Since **AU** = **BV** and **CU** = **CV**:

AC = **AU** + **CU** = **BV** + **CV** = **BC**

Therefore, **every triangle is isosceles!**



RH = Right angle-Hypotenuse-Side

If two right triangles have congruent hypotenuses and one pair of congruent legs, they're congruent.

https://www.geogebra.org/m/yyszz85d

- Let **M** be the midpoint of **AB**
- Let **P** be the intersection of the perpendicular bisector of **AB** and the angle bisector at **C**
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Congruent triangles:

- △**AMP** ≅ △**BMP** (By SAS, since ∠**PMA** = ∠**PMB**, **AM** = **BM**, **MP** = **MP**). Then **AP** = **BP**
- △**PCU** ≅ △**PCV** (By AAS, since ∠**PUC** = ∠**PVC**, ∠**PCU** = ∠**PCV**, **PC** = **PC**). Then **PU** = **PV** and **CU** = **CV**
- △**APU** ≅ △**BPV** (By RH, since ∠**PUA** = ∠**PVB**, **AP** = **BP**, **PU** = **PV**). Then **AU** = **BV**

Conclusion:

Since **AU** = **BV** and **CU** = **CV**:

AC = **AU** + **CU** = **BV** + **CV** = **BC**

Therefore, **every triangle is isosceles!**

Are all triangles isosceles?

Lect 3

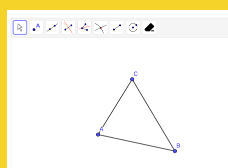
Join the lesson at www.geogebra.org/classroom with the code:

GMV5 JQQU

Or you can also share the following link with your students:

www.geogebra.org/classroom/gmv5jqqu

https://www.geogebra.org/classroom/gmv5jqqu



Go to https://www.geogebra.org/classroom

Code. GMV5 JQQU

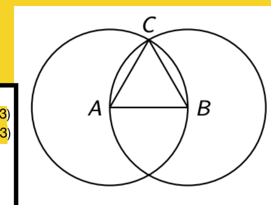
Follow the instructions in the app. Answer in Wooclap: Based on your GeoGebra construction, what is the flaw in this 'proof' that all triangles are isosceles?

What can you conclude from the "all triangles are isosceles" fallacy?

- Denote the segment by AB.
- Draw a circle with center A and distance AB.(Post.3)
- Draw a circle with center B and distance AB.(Post.3)
- **Let C be a point where the circles intersect.**
- Draw segments AC and BC. (Post 1)

ABC is the required triangle.

- AB = AC (Definition 15)
- AB = BC (Definition 15)
- AC = BC (Common Notion 1)
- Hence, ABC is an equilateral triangle (Definition 20) and it is on the given line AB.



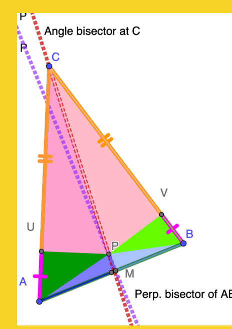
Post 1.Let it have been postulated to draw a straight-line from any point to any point.

Post.3: to draw a circle with any center and radius.

Common notion 1: Things that are equal to the same thing are also equal to one another.

Definition 15. A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

Definition 20: If all three sides are of the same length, then a triangle is a **equilateral** /

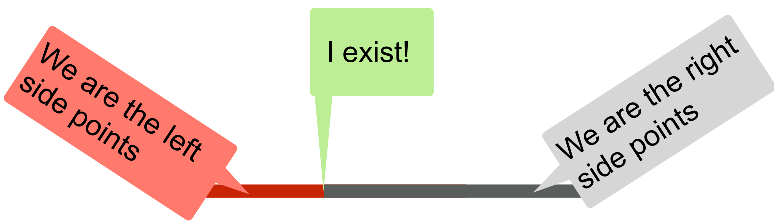


Proposition 1 of Euclid's Elements. Construction of equilateral triangle

Bad news: There is a gap in the proof of Proposition I.1: Euclid did not proved that the circles must intersect (remember that one can not "*prove by drawing*")

How do we deal with this issue nowadays:
Following **Hilbert**, modern geometry adds a **Completeness Axiom**, similar to an axiom you may know from Analysis.

Axiom of Completeness (for a line) which states.
If all points of a straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division.



Optional: More about the axioms

Hilbert's Axioms of Continuity (Group V)

1. **Archimedean Axiom**
   For any two segments a and b, there exists a natural number n such that n copies of a laid end to end exceed b.
   (No segment is infinitely small compared with another.)

2. **Completeness Axiom (Vollständigkeitsaxiom)**
   The geometric system that satisfies the previous groups of axioms (Incidence, Order, Congruence, Parallels, and the Archimedean Axiom) cannot be enlarged without contradiction. In other words, there is no larger system containing it as a proper part that still satisfies all those same axioms.
   (This ensures maximality and rules out "missing points" or gaps.)

From these, results such as line–circle and circle–circle intersection can be proved.

Optional: More about the axioms

Line–line continuity (crossing principle)
If a line has two points that lie in **different half-planes** determined by another line, then the two lines **meet** (the segment between those points crosses the other line).

Line–circle continuity
If a line has one point **inside** a circle and another point **outside** that circle, then the line **meets** the circle (in at least one point; generically two).

Circle–circle continuity
If one circle has a point **inside** and a point **outside** a second circle, then the two circles **meet** (in at least one point; generically two).

Notes: "Inside/outside" are with respect to the circle's center and radius; "half-planes" are the two regions determined by a line. These continuity principles are the modern fixes that justify the intersection steps Euclid sometimes assumed.

Review

Euclid's Elements

- Greek mathematical treatise
- Compiled around 300 BCE
- Attributed to Euclid
- Draws on centuries of accumulated mathematical work.
- Earliest extant work in **axiomatic method (**Propositions are arranged in a **logical progression**, where each step is justified by earlier results and axioms.)
- Its logical structure still underlies modern mathematics.

Covers
- **plane geometry** (shapes in the plane)
- **solid geometry** (shapes in space)
- **numbers & ratios** (divisibility, primes, proportions, irrationals)

Mathematics Is Built in The Elements: The Axiomatic Method

Axiomatic method:

- Starts with **definitions and agreed principles** (called **axioms** now, **postulates and common notions** in Euclid),
- New **propositions** are derived by **logical proof**;
- **Each step is justified** by
  - those principles or
  - by previously proved propositions.

Note: Nowadays, there are also undefined terms.

Euclid's Elements: Quick Overview

- **Definitions:** Terms like point, line, circle, angle, etc.
- **Postulates:** basic assumptions about geometry. (e.g., "*a straight line can be drawn between any two points*")
- **Common Notions**: intuitively obvious facts about magnitudes (e.g., "*things equal to the same thing are equal to each other*")
- **Propositions**:

  Theorems: In right-angled triangles, the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

  Constructions: Construct an equilateral triangle, given a slide.

Axiomatic method: Starting from agreed principles (called **axioms**), new propositions are derived by logical proof; each step is justified by those principles or by previously proved propositions.

Euclid's elements	Today's math
Definitions	**Undefined** terms Definitions
Common notions Postulates **(self evident truths)**	Axioms **(assumptions wisely chosen)**
Propositions Constructions	Propositions Constructions

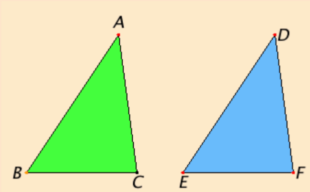
Axiomatic method: Starting from agreed principles (called **axioms**), new propositions are derived by logical proof; each step is justified by those principles or by previously proved propositions.

Congruence of triangles by SAS Proposition I.4

Proposition 4 (**SAS**) If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

Let *ABC* and *DEF* be two triangles having the two sides *AB* and *AC* equal to the two sides *DE* and *DF* respectively, namely *AB* equal to *DE* and *AC* equal to *DF,* and the angle *BAC* equal to the angle *EDF.*

I say that the base *BC* also equals the base *EF,* the triangle *ABC* equals the triangle *DEF,* and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides, that is, the angle *ABC* equals the angle *DEF,* and the angle *ACB* equals the angle *DFE.*



If the triangle *ABC* is superposed on the triangle *DEF,* and if the point *A* is placed on the point *D* and the straight line *AB* on *DE,* then the point *B* also coincides with *E,* because *AB* equals *DE.*

Again, *AB* coinciding with *DE,* the straight line *AC* also coincides with *DF,* because the angle *BAC* equals the angle *EDF.* Hence the point *C* also coincides with the point *F,* because *AC* again equals *DF.*

But *B* also coincides with *E,* hence the base *BC* coincides with the base *EF* and equals it. C.N.4

Thus the whole triangle *ABC* coincides with the whole triangle *DEF* and equals it. C.N.4

And the remaining angles also coincide with the remaining angles and equal them, the angle *ABC* equals the angle *DEF,* and the angle *ACB* equals the angle *DFE.*

Therefore *if two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.*

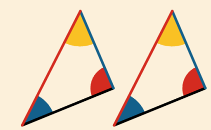
Q.E.D.

http://aleph0.clarku.edu/~djoyce/elements/bookI/propI4.html

Proposition 4: **SAS**

PROPOSITION IV. THEOREM.

IF two triangles have two ſides of the one reſpectively equal to two ſides of the other, (to and to) and the angles (and) contained by thoſe equal ſides alſo equal; then their baſes or their ſides (and) are alſo equal: and the remaining and their remaining angles oppoſite to equal ſides are reſpectively equal (= and =): and the triangles are equal in every reſpect.



Let the two triangles be conceived, to be ſo placed, that the vertex of one of the equal angles, or ; ſhall fall upon that of the other, and to coincide with , then will coincide with if applied: conſequently will coincide with , or two ſtraight lines will encloſe a ſpace, which is impoſſible (ax. 10), therefore = , = and = , and as the triangles and coincide, when applied, they are equal in every reſpect.

Q. E. D.

Byrne's Euclid

Solutions of Modern Mathematics for the gap in Euclid's proof of SAS

- Make SAS an axiom (Hilbert)
- Transformation geometry foundation
- Metric approach (Birkhoff)

Bertrand Russell about Euclid

Autobiography of Bertrand Russell (1872 1970)

At the age of eleven, I began Euclid, with my brother as my tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined that there was anything so delicious in the world. After I had learned the fifth proposition, my brother told me that it was generally considered difficult, but I had found no difficulty whatever. This was the first time it had dawned upon me that I might have some intelligence. From that moment until Whitehead and I finished Principia Mathematics when I was thirty-eight, mathematics was my chief interest, and my chief source of happiness. Like all happiness, however, it was not unalloyed. I had been told that Euclid proved things, and was much disappointed that he started with axioms. At first I refused to accept them unless my brother could offer me some reason for doing so, but he said: "If you don't accept them we cannot go on," and as I wished to go on, I reluctantly admitted them *pro tem*. The doubt as to the premisses of mathematics which I felt at that moment remained with me, and determined the course of my subsequent work.

pro tem = temporarily / for now

Unalloyed"= not mixed with anything else (like pure metal without other metals — no alloy).

Russell, Bertrand. The Autobiography of Bertrand Russell. London: George Allen & Unwin, 1975.

The Teaching of Euclid by Bertrand Russell

It has been customary when Euclid, considered as a text-book, is attacked for his verbosity or his obscurity or his pedantry, to defend him on the ground that his logical excellence is transcendent, and affords an invaluable training to the youthful powers of reasoning. This claim, however, vanishes on a close inspection. His definitions do not always define, his axioms are not always indemonstrable, his demonstrations require many axioms of which he is quite unconscious. A valid proof retains its demonstrative force when no figure is drawn, but very many of Euclid's earlier proofs fail before this test.

Russell, Bertrand. "The Teaching of Euclid." In The Mathematical Gazette, Vol. 2, No. 33 (1893), pp. 165–167.