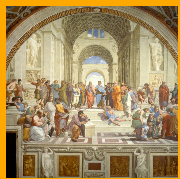
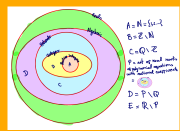
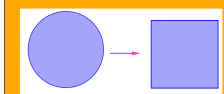
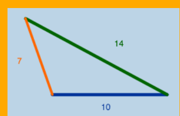
MAT 336 Hellenic Mathematics before Euclid.









- Overview and timeline .
- The School of Athens
- Impossible problems of Antiquity.
- Thales
- Pythagoras, the Pythagoreans and Pythagoras' theorem
- Zeno's paradoxes
- Hippocrates
- Plato
- Aristotle

- Introduction
- Rough timeline of Greek math
- Map, what when being Greek then
- Why the proofs were developed in Greece? Arguments
- The school of Athens
- Mesopotamian influence
- Attitudes toward math in the Ancient Greek World.

- Incommensurable magnitudes
- Impossible problems of antiquity.

- Pythagoras/Pythagoreans
- Plato and Aristotle

- Thales
- Parmenides, Zeno
- Hippocrates

Three lectures and two presentations
- Unit fractions in Egypt
- Oct 6 Thales and the Pyramid Ruixi(L1)
- Oct 6 Thales and the Pyramid Janey(L3)
- Oct 8 Quiz Egipt L1:18 students. Many finished in 10 minutes.
- Oct 8 Quiz Egipt L students.

Pre Euclid

- Quadrature Hippias
-

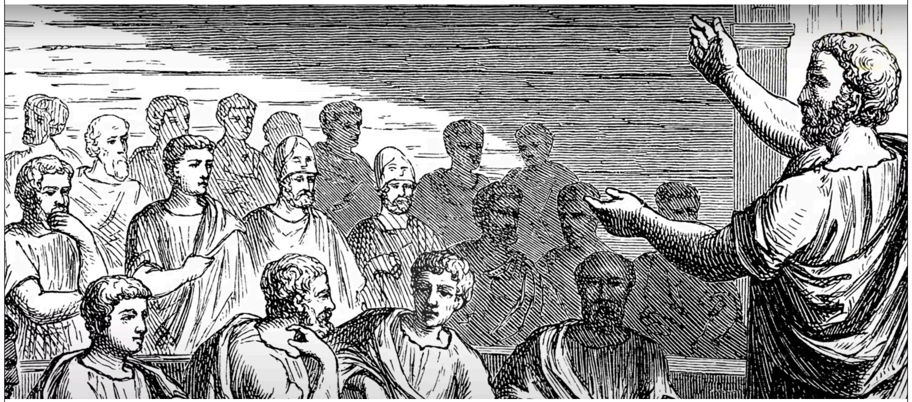
Math History Myth bursting
- Pythagoras
- Plimpton
- Ishango bone
- Equations.
- Euclid

Introduction

Ancient Greece 101 | National Geographic, https://youtu.be/6bDrYTXQLu8

1. Give one example (from the video) of Greek influence that still shows up today—in language, politics, sports, art, or architecture.
2. Name one way Athens' democracy differed from rule by a single monarch.

Sound arguments Public life



The Greek city-states fostered an environment of intellectual competition and debate. In the agora (public square), scholars engaged in **discussions and debates**, which encouraged the refinement of ideas and the need for rigorous argumentation.

The jurors were called upon to be not just witnesses, but also calculators – in principle, a *logistes* could tell with absolute certainty whether the account presented to him worked out or not, whether everything added up. It was this kind of persuasiveness, **mathematical persuasiveness,** that both speakers wished to claim for their arguments. (…) For now, let me observe that accounts – collective, public counting – were a pervasive practice in classical Athens. We find them not only in inscriptions, but also in various genres of literature. Along with its practical functions, **public counting was associated with political accountability**, and in fourth-century legal speeches it seems increasingly to symbolize the role itself of the Athenian citizen.

Cuomo, Serafina. Ancient mathematics. Routledge, 2005.

Greek or Hellenic Mathematics (Between ~600 BCE and 300 AD)

- In mathematics, the question

  "**why?**"

- was now asked together with the older question

  "**how?**"

Greek or Hellenic Mathematics Two important aspects

1. Emphasis on proofs
2. Involvement with specific, challenging problems.
   - need to decide which new arguments counted as genuine proof
   - in this way, the problems led to
     - new discoveries
     - increasingly sophisticated ideas about proofs

Greek or Hellenic Mathematics (Between ~600 BCE and 300 AD) Primary Sources

- No tablets or papyri or …
- Only copies of copies of copies of…
  - Restored versions of texts of Euclid, Archimedes, Apollonius and other.
- To study how this mathematics was developed we rely on these copies and remarks of writers of the time.

Early Greek mathematics

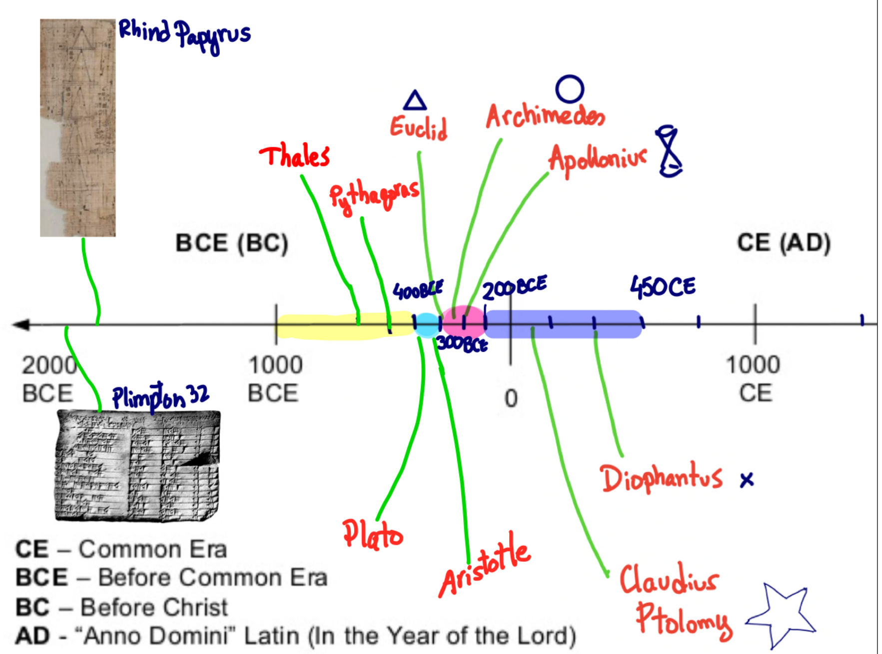
Early Greek mathematics was not one but many*; there were* ***various levels of practice,*** *from calculations on the abacus to indirect proofs concerning incommensurable lines, and varying attitudes, from laughing off attempts to square the circle to using attempts to square the circle as examples in a second-order discussion about the nature of demonstration. In sum,* ***different forms of mathematics were used for different purposes by different groups of people****. Perhaps one common feature is clearly distinguishable:* ***mathematics was a public activity,*** *it was played out in front of an audience, and it fulfilled functions that were significant at a communal level, be they counting revenues, measuring out land or exploring the limits of persuasive speech.*

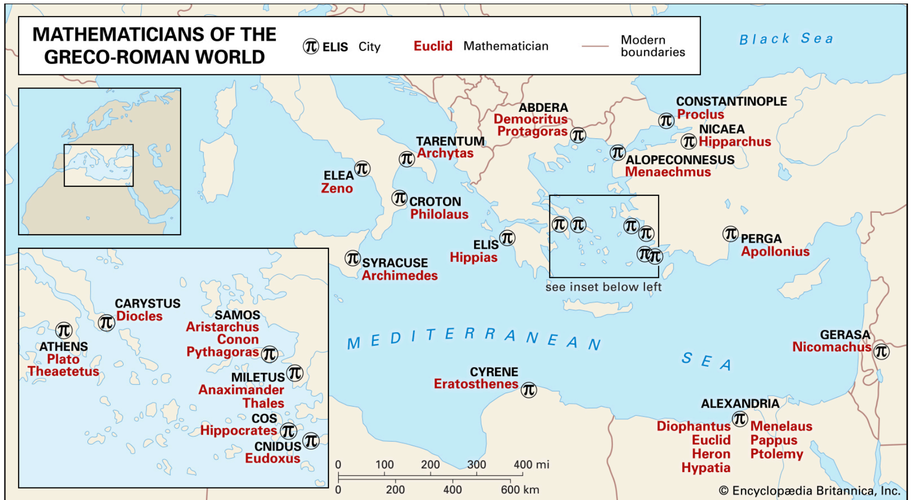
Cuomo, Serafina. Ancient mathematics. Routledge, 2005.

Mesopotamian Influence in Greek Astronomy

The superiority of Babylonian numerical methods has left traces still visible in modern times. The division of the circle into 360 degrees and the division of the hour into 60 minutes and 3600 seconds reflect the unbroken use of the sexagesimal system in their computations by medieval and ancient astronomers. But though the base 60 is the most conspicuous feature of the Babylonian number system, this was by no means essential for its success. The great number of divisors of 60 is certainly very useful in practice, but the **real advantage of its use in the mathematical and astronomical texts lies in the place-value notation,** which is consistently employed in all scientific computations. This gave the Babylonian number system the same advantage over all other ancient systems as our modern place-value notation holds over the Roman numerals. **The importance this invention can well be compared with that of the alphabet.** Just eliminates as the alphabet eliminates the concept of writing as an art to be acquired only after long years of training, notation eliminates mere computation as complex art in itself.

THE HISTORY OF ANCIENT ASTRONOMY : PROBLEMS AND METHODS Author(s): O. Neugebauer, Publications of the Astronomical Society of the Pacific, February 1946, Vol, 58.





[Map of Ancient Hellenic Mathematics](#)

Encyclopedia Britannica

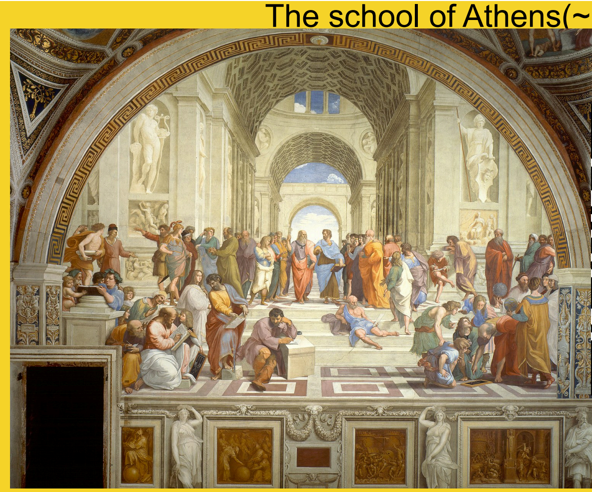
[Map of Ancient Hellenic Mathematics](#) Encyclopedia Britannica

[Mathigon timeline - Hellenic](#)

[mathigon.org](#)

Some of the characters in this painting (The school of Athens) are doing something mathematical. Write down their names. If you have time, write down also the name of mathematicians or mathematics-related characters in the painting who are not doing mathematics. (Google is allowed and encouraged for this question)

[The school of Athens(~1500), by Raffaele](#)



https://commons.wikimedia.org/wiki/File:%22The_School_of_Athens%22_by_Raffaello_Sanzio_da_Urbino.jpg

The school of Athens, by Raffaele

The school of Athens, by Raffaele

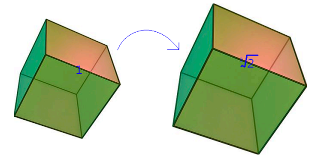
Impossible problems of the Antiquity

Greek geometers limited constructions to a straightedge and a compass.
Why only these two tools? What's special about them?

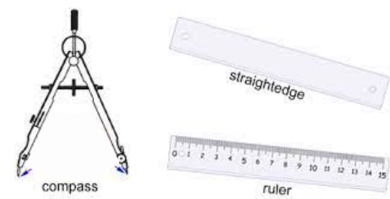
If cube C has side length 1 inch, what is the side length of a cube with twice the volume of C?

Three “impossible” problems

Doubling a cube: Given a cube, **construct** another cube with twice its volume.



“Construct” means “construct using only straightedge and compass



These problems originated around 400 BCE

What reason gave Plato to the Delians that for the oracle of the god?

In his work entitled *Platonicus* Eratosthenes says that, when the god announced to the Delians by oracle that to get rid of a plague they must construct an altar double of the existing one, their craftsmen fell into great perplexity in trying to find how a solid could be made double of another solid, and they went to ask Plato about it. He told them that the god had given this oracle, not because he wanted an altar of double the size, but because he wished, in setting this task before them, to reproach the Greeks for their neglect of mathematics and their contempt for geometry.

Theon of Smyrna

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Theon of Smyrna

The legend is doubtful but the **Delian problem was studied in Plato's Academy.**

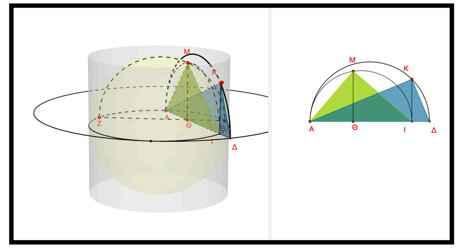
Plato insisted on an exact solution accomplished using only **ruler and compass.**

Suppose you can construct two mean proportionals between any lengths a and b. How could you construct $2^{1/3}$.

Hint: From the definition of double mean proportional, you get two equations. Combine them to eliminate x and find a relationship between y, a, and b. Then choose specific values for a and b.

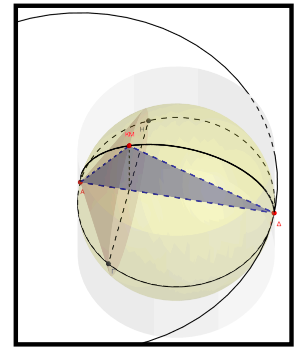
Double mean proportional: Given two lengths a and b, find x and y such that a/x = x/y=y/b

Reconstruction of Archytas method to find the double mean proportional I.



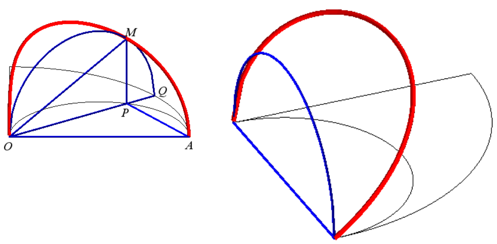
https://www.geogebra.org/m/tg4b5hmu#material/QgpK29uA

Reconstruction of Archytas method to find the double mean proportional I.



https://www.geogebra.org/m/dmr9daun

Double mean proportional: Given two lengths a and b, find x and y such that a/x = x/y=y/b



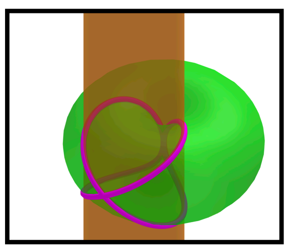
Source: https://mathcurve.com/courbes3d.gb/archytas/archytas.shtml

Construction of the curve:
The torus is produced by a circle (C_Q) of horizontal diameter $[OQ]$, the point Q having a movement circular around O, and the cylinder, by a vertical line (D) passing through P, the point P describing a fixed circle of diameter $[OA]$ $(OQ = OA)$.
Taking P on (OQ), the point M of intersection of (D) with (CQ) describes the Archytas curve.
Analytically:
$A(a,0,0), Q(a\cos t, a\sin t, 0), P(a\cos^2 t, a\cos t\sin t, 0), Q(a\cos^2 t, a\cos t\sin t, \pm a\sqrt{(1-\cos t)\cos t})$
Latitude $(\overrightarrow{OP}, \overrightarrow{OM}) = \lambda$

and longitude $(\overrightarrow{OA}, \overrightarrow{OP}) = \theta$

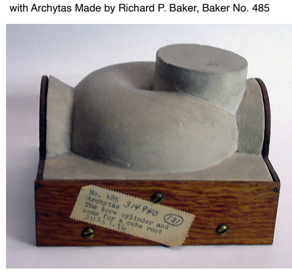
are connected by $\cos^2\lambda = \cos\theta$

Reconstruction of Archytas method to find the double mean proportional II.



https://www.geogebra.org/m/cq9szphj

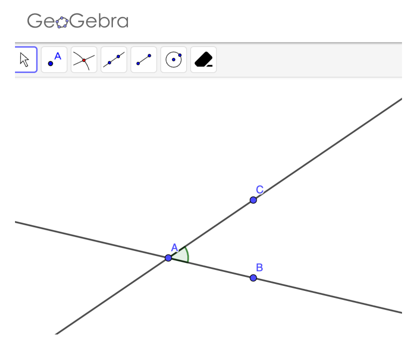
NMAH-NMAH2003-13018 Model of a Surface Associated with Archytas Made by Richard P. Baker, Baker No. 485



https://americanhistory.si.edu/it/collections/object/

Bisect the angle using only straightedge and compass. Hint: Use symmetry.

Lect 1



There is also a link in the course schedule

Join the lesson at www.geogebra.org/classroom with the code:

GXKA AERY

Or you can also share the following link with your students:

www.geogebra.org/classroom/gxkaaery

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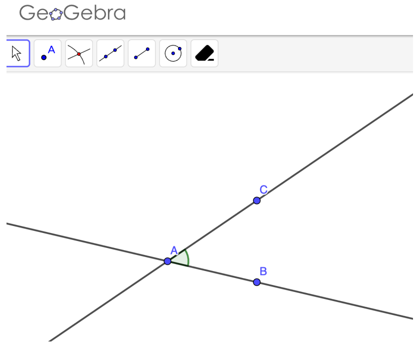
Lect 3



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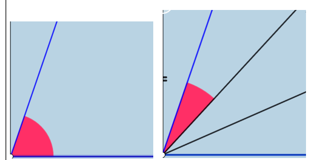
ZWAY YJKZ

Or you can also share the following link with your students:

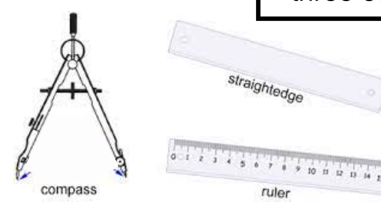
www.geogebra.org/classroom/zwayyjkz

Three "impossible" problems



Trisecting an angle: Given an angle between two straight lines, **construct** two lines that divide the angle into three equal parts.

"Construct" means "construct using only straightedge and compass



These problems originated around 400 BCE

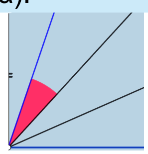
Trisecting an angle

Constructing an angle α from the angle 3α

equivalent

Constructing a segment of length cos(α) from a segment of length cos(3α).

Also, $\cos(3\alpha) = 4\cos^3(\alpha) - 3\cos(\alpha)$.
If we set $\cos(\alpha)=x$ and $\cos(3\alpha)=a$ then
$4x^3 - 3x - a = 0$



Thus, trisecting an angle is equivalent to solving certain cubic equation

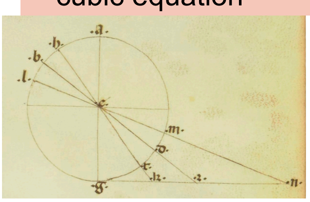


Diagram from 14th-century manuscript copy of Ptolemy's Almagest, folio 22 recto. Manuscript owned and digitized by gallica.bnf.fr / Bibliothèque nationale de France.

It is likely that the question of trisection of angles arose when trying to construct a table of chords for astronomical purposes. (A chord for the angle 3° can be constructed so it would have been natural to try to get a chord for the angle 1° from the chord for the angle 3°).

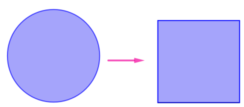
If a circle has radius 1 inch, what is the side length of a square that has the same area as the circle?

Three “impossible” problems

Squaring a circle is closely related to finding its area

“Construct” means “construct using only straightedge and compass

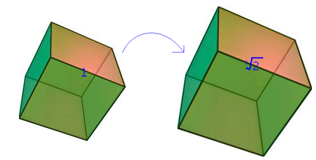
Squaring a circle: Given a circle, **construct** a square with the same area.

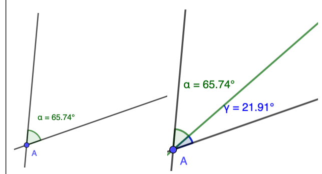


These problems originated around 400 BCE

Three “impossible” problems

Doubling a cube: Given a cube, **construct** another cube with twice its volume.

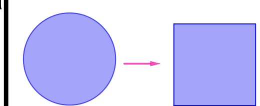




Trisecting an angle: Given an angle between two straight lines, **construct** two lines that divide the angle into three equal parts.

“Construct” means “construct using only straightedge and compass

Squaring a circle: Given a circle, **construct** a square with the same area.



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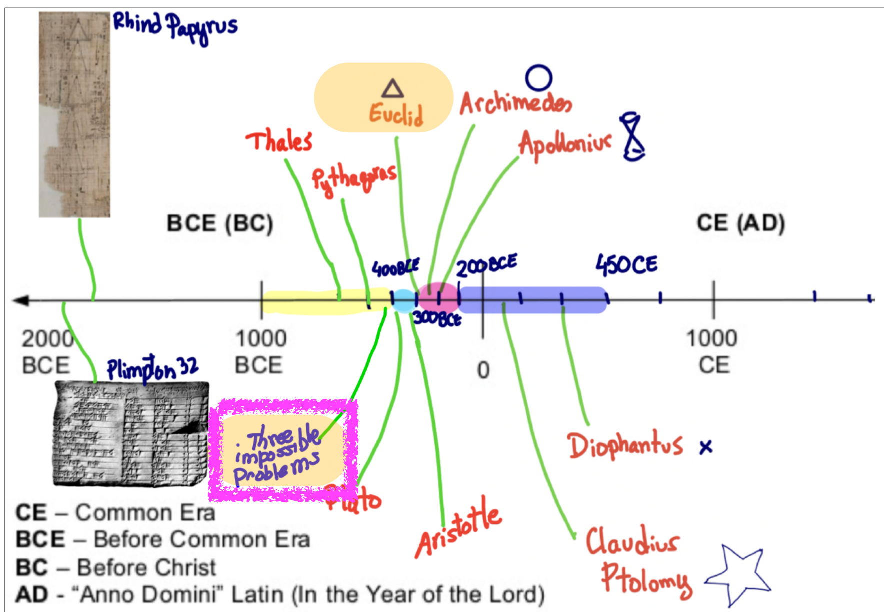
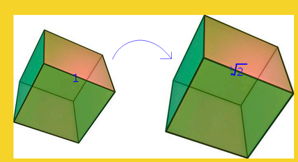
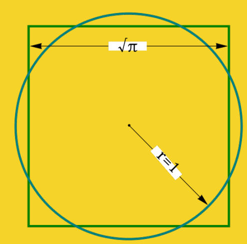


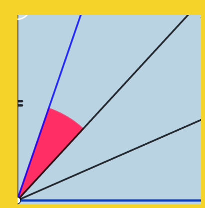
In your opinion, which of the three classical problems was the hardest to prove impossible — and why?

The three problems are :
- Squaring the circle
- Trisecting an angle
- Doubling the cube

Using only straightedge and compass.





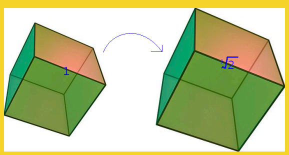


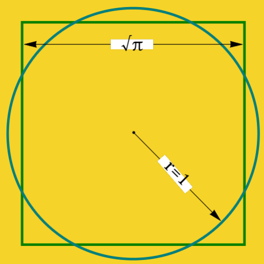
All three problems were eventually proven impossible.
Who proved each one impossible, and in what year?.
You may use Google and AI to answer this question.

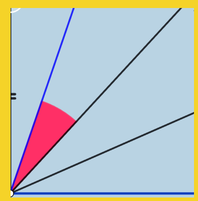
The three problems are :

- Squaring the circle
- Trisecting an angle
- Doubling the cube

Using only straightedge and compass.







The mathematical problems that the Greeks tackled, and indeed the whole geometrical bias of so much Greek mathematics, both **stimulated and was stimulated by construction methods**, and the question naturally arose as to which geometrical problems can be solved by the basic **line-and-circle** constructions, and which ones cannot.
The History of Mathematics: A Source-Based Approach, Vol. 1, (2014) June Barrow-Green, Jeremy Gray, Robin Wilson.

Takeaway: Problems shaped the tools, and the tools shaped the problems.

Consequence:
What looked like a purely "line-and-circle" obsession **seeded practical mathematics.**

Can you think of a practical technology that may trace back to line-and-circle constructions?

Hint: Think parabola, ellipse, triangles/triangulation.

Examples of practical applications

Constructions → conics (parabola / ellipse / hyperbola)

- Parabola → headlights, satellite dishes, solar ovens.
- Ellipse → planetary/satellite orbits; whispering galleries.
- **AI:** camera calibration and conic/ellipse detection in computer vision.

Proof culture → surveying / architecture / engineering

- Triangulation → land surveys, mapmaking, GPS logic.
- Geometry of structure → arches, trusses; constraints in CAD.
- **AI:** formal verification/specs for algorithms; constraint reasoning in planning and robotics.

Constructibility → algebra & number theory → real-world tech

- Public-key crypto (RSA, ECC) → HTTPS, banking, messaging (protects AI models/data).
- Error-correcting codes → QR codes, Wi-Fi, storage (reliable datasets/training).
- **AI:** hashing/modular arithmetic for indexing, sharding, and fast retrieval.

Obsession with Proofs and Abstract constraints (line & circle) → powerful ideas → tools embedded in today's communications, navigation, engineering, AI, …

1. What was "impossible" about doubling the cube using only straightedge and compass? *Hint: What kind of root do you need to find to double a cube's volume?*

2. Why might someone *in antiquity care about constructing mean proportionals?* *Hint: Think about how these problems were posed (recall the legend oracles, plague, altar) and what people hoped to achieve.*

3. What's so special about using only a straightedge and a compass? *Hint: Think about what shapes or operations those tools can make — and what they can't.*

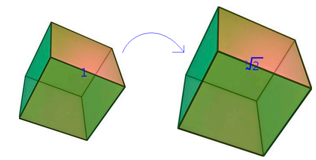
4. Can you think of a real-world technology that may have grown out of line-and-circle constructions? *Hint: Look back at today's applications: orbits, triangulation, QR codes...*

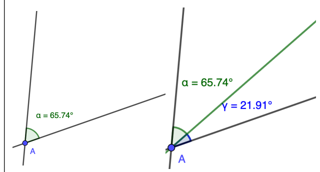
5. What surprised you most about how impractical Greek geometric work led to practical applications?*Hint: Think about how far downstream some of today's tools (like internet security or GPS) are from those early obsessions.*

Incommensurable magnitudes

Three “impossible” problems

Doubling a cube: Given a cube, **construct** another cube with twice its volume.

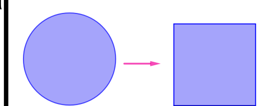




Trisecting an angle: Given an angle between two straight lines, **construct** two lines that divide the angle into three equal parts.

“Construct” means “construct using only straightedge and compass

Squaring a circle: Given a circle, **construct** a square with the same area.



These problems originated around 400 BCE

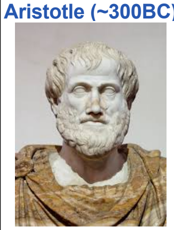
Do you think the Pythagoreans were right? Can every magnitude be expressed as a whole number or a ratio of whole numbers? Why or why not?

In the 5th century BCE, the Pythagoreans held that all magnitudes could be expressed by whole numbers or by ratios of whole numbers, reflecting their belief that “*all things are number*”.

Aristotle, Metaphysics

- There are two distinct types of “quantities”:
  - the continuous (magnitude)
    - *“A **magnitude** is that what which is divisible into divisible that are infinitely divisible.”*
    - Example: lines, surfaces, bodies and time.
  - the discrete (number)
    - *A “**number**” quantities that is composed of distinct, separate units or parts that can be counted. (Examples 1, 2, 3, ...)*

Aristotle (~300BC)



Is it possible to find integers p and q such that √2 = p/q? In other words, is √2 rational?

For all who effect an argument per impossibile infer syllogistically what is false, and prove the original conclusion hypothetically when something impossible results from the assumption of its contradictory; **e.g.** **that the diagonal of the square is incommensurate with the side, because odd numbers are equal to evens if it is supposed to be commensurate. One infers syllogistically that odd numbers come out equal to evens, and one proves hypothetically the incommensurability of the diagonal, since a falsehood results through contradicting this**.

Aristotle - The Organon ANALYTICA PRIORIA Book 1 Part 23- ~350 BCE

Proof of the irrationality of √2 discussed by Aristotle

It is clear then that the ostensive syllogisms are effected by means of the aforesaid figures; these considerations will show that reductiones ad also are effected in the same way. For all who effect an argument per impossibile infer syllogistically what is false, and prove the original conclusion hypothetically when something impossible results from the assumption of its contradictory; **e.g.** **that the diagonal of the square is incommensurate with the side, because odd numbers are equal to evens if it is supposed to be commensurate. One infers syllogistically that odd numbers come out equal to evens, and one proves hypothetically the incommensurability of the diagonal, since a falsehood results through contradicting this**. For this we found to be reasoning per impossibile, viz. proving something impossible by means of an hypothesis conceded at the beginning. Consequently, since the falsehood is established in reductions ad impossibile by an ostensive syllogism, and the original conclusion is proved hypothetically, and we have already stated that ostensive syllogisms are effected by means of these figures, it is evident that syllogisms per impossibile also will be made through these figures. Likewise all the other hypothetical syllogisms: for in every case the syllogism leads up to the proposition that is substituted for the original thesis; but the original thesis is reached by means of a concession or some other hypothesis. But if this is true, every demonstration and every syllogism must be formed by means of the three figures mentioned above. But when this has been shown it is clear that every syllogism is perfected by means of the first figure and is reducible to the universal syllogisms in this figure.

Aristotle - The Organon ANALYTICA PRIORIA Book 1 Part 23- ~350 BCE

Reasoning pattern: Proof by contradiction:

assume opposite →

derive impossibility →

conclude what we wanted to prove.

Some reasons why √2 irrationality matters

Proof by contradiction: A reasoning pattern

(Uses: diagnosis, debugging, risk, legal).

Frameworks have limits: find something not fitting → recognize limits → build better systems

(Uses: floating-point, measurement, model failure).

Paradigm shock: a core belief shatters → the field rebuilds.

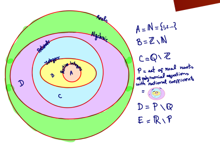
(Uses: Progress of science)

√2 is not rational

Does it have a "nice" expression in terms of whole numbers?

No pattern!

Commensurability



π is not rational

π is not even algebraic.

https://cosmosmagazine.com/mathematics/the-square-root-of-2/

Holy beard of Zeus! It's expanding irrationally and infinitely! Credit: Jeffrey Phillips

---

ℕ ⊂ ℤ ⊂ ℚ ⊂ **Algebraic** ⊂ ℝ

Positive integers (1,2,..)

⋂

Integers Z - (-2,-1,0, 1,2..)

⋂

Rationals Q (3/2, 1/2, 1, -1, 3/100,…)

⊂

Algebraic - roots of integer-coef polynomials - (1, -1/2, √3, ³√2…)

⋃

Real numbers (1, -1/2, √3, ³√2, π, e, …)

Positive integers

Integers

Rationals Q (3/2, 1/2, 1, -1, 3/100,…)

Algebraic - roots of polynomials with integer coefficients - (1, -1/2, √3, ³√2…)

Real numbers (1, -1/2, √3, ³√2, π, e, …)

Inner: ℕ Positive integers

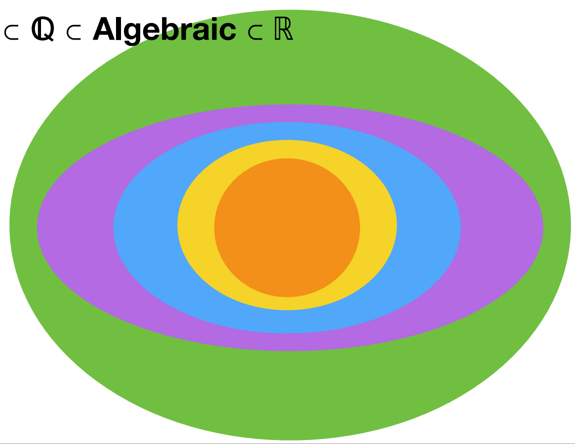
Next: **B** Integers \ A

Next: **C** Rationals \ ℤ

Next: **D** Algebraic irrationals (e.g., √2, ³√2)

Outer: **E** Transcendentals (π, e)

ℕ ⊂ ℤ ⊂ ℚ ⊂ **Algebraic** ⊂ ℝ



In which of the following sets is:

i. 11/5

ii. √2

iii. E

iv. -1000

v. $2^{1/3}$

vi. -11/5

A Positive integers

Integers

Rationals Q (3/2, 1/2, 1, -1, 3/100,…)

Algebraic - roots of polynomials with integer coefficients - (1, -1/2, √3, ³√2…)

Real numbers (1, -1/2, √3, ³√2, π, e, …)

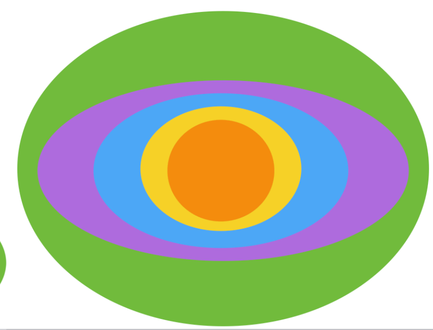
Inner: **A** Positive integers

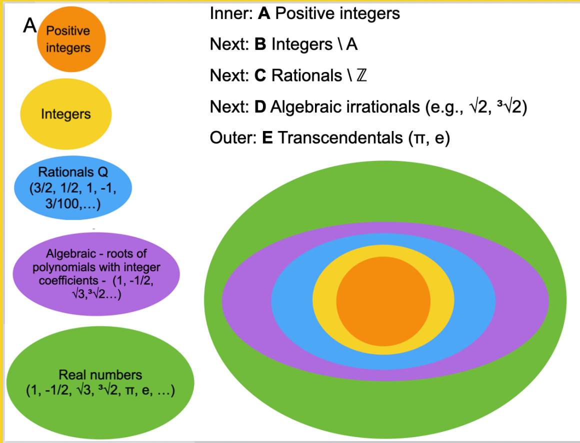
Next: **B** Integers \ A

Next: **C** Rationals \ ℤ

Next: **D** Algebraic irrationals (e.g., √2, ³√2)

Outer: **E** Transcendentals (π, e)



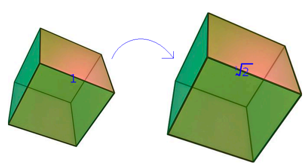


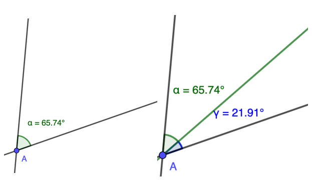
Write one number for each set ℕ, B, C, D, E. Write one number for each set N, B, C, D, E (no repeats). Add a one-line justification showing the evidence/ test you used. Example: Example: 7 — "integer, ≥1."

Commensurability

Three "impossible" problems

Doubling a cube: Given a cube, **construct** another cube with twice its volume.

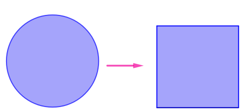




Trisecting an angle: Given an angle between two straight lines, **construct** two lines that divide the angle into three equal parts.

"Construct" means "construct using only straightedge and compass

Squaring a circle: Given a circle, **construct** a square with the same area.



These problems originated around 400 BCE

Zeno (~450BCE) and Paradoxes

The android is the one in grey, wearing pajamas-like clothes.
Why do you think the android self-destructs at the end of the clip?
(Optional) What does this suggest about how the android's reasoning differs from human reasoning?

A clip from Star Trek

https://youtu.be/EzVxsYzXI_Y?si=gfZ16Y6C5nQYfjcL

Paradox

Greek
- **para**: distinct from
- **doxa**: opinion, belief.

Colloquial

A statement contrary to common belief or expectation

Logical/Philosophical

A statement that starts from acceptable premises and follows sound reasoning, yet reaches a self-contradictory or illogical conclusion.

Which type of paradox destroyed the android?

I am a barber.
I shave anyone in my town who does not shave himself, and no one else.

Do I shave myself?

Barber paradox

Analogous to Russell's paradox

Analogous the paradoxes "*This sentence is false*" and the one in the Star Trek clip

Liar paradox

I am a liar

Is my previous sentence true of false?

Zeno's Dichotomy paradox

The first (argument) is the one which declares movement to be impossible because, however near the mobile is to any given point, it will always have to cover half, and then the half of that, and so on without limit before it gets there.

Aristotle - Physics

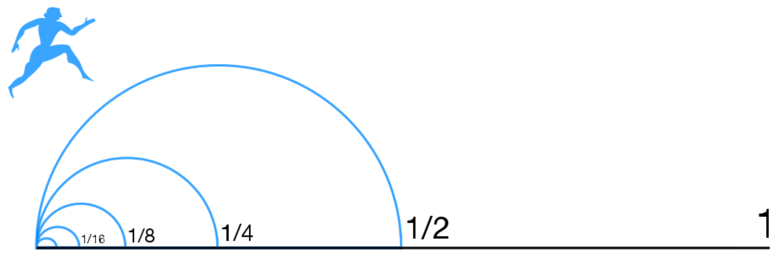
Read Aristotle's explanation. Then discuss:
What has to happen every time you try to reach a point?
Why does this create a logical problem?

Zeno's Dichotomy paradox

To go from A to B, you have to

- go midway to $A_{1/2}$,
- midway from A to $A_{1/2}$, $A_{1/4}$,
- and midway from A to $A_{1/4}$, $A_{1/8}$,
- and so on.

So you have to pass through infinite points in finite time, which is impossible.



How many Zenos does it take to change a light bulb?

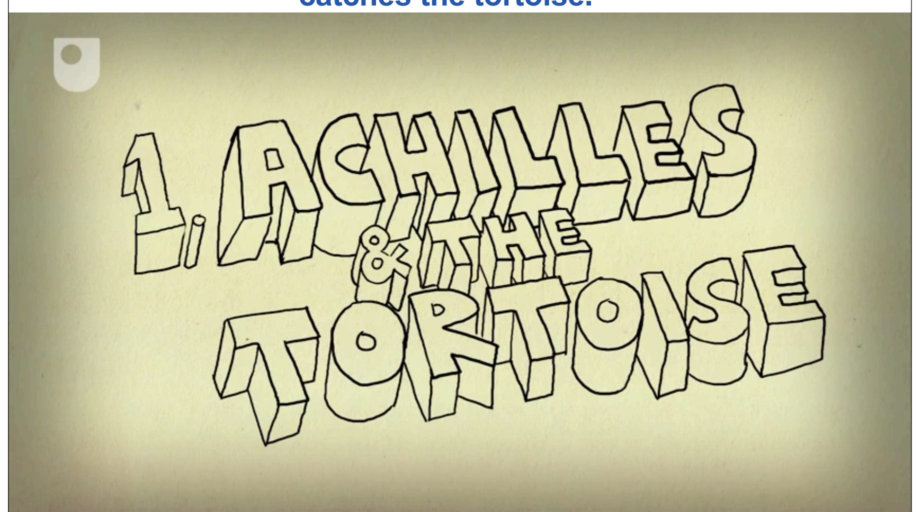
Infinite. One to screw it in halfway, one to screw it in half of what's left, another to screw it in half of what remains, and so forth...

Zeno showing the Doors to Truth and False

Youths the Doors to Truth and False (Veritas et Falsitas) by Pellegrino Tibaldi

Zeno's Achilles and the Tortoise paradox

Zeno's logic seems sound, but we know Achilles actually catches the tortoise.



Video from Open University https://www.youtube.com/watch?v=skM37PcZmWE

Zeno's logic seems sound, but we know Achilles actually catches the tortoise.

1: What assumption about space and time makes Zeno's argument work? (Hint: What does he keep dividing?)

2 How is the Achilles paradox similar to the Dichotomy paradox we discussed earlier?

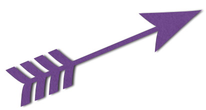
3: Zeno assumes infinite steps = infinite time. Why would ancient Greeks find this assumption convincing? Do you agree with Zeno? Why or why not?

Zeno's Achilles and the Tortoise paradox

"In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead."

Aristotle, Physics, VI:9

Zeno's Arrow Paradox



- At any instant in time, an arrow in flight occupies a precise position in space.
- At that instant, it's not moving (motion requires time to pass).
- If it's motionless at every instant, when does it move?
- The paradox: If time is composed of instants, and the arrow is frozen at each instant, motion is impossible.

Is movement impossible?

- *Nude descending a staircase*, by Marcel Duchamp 1912
- Provoked an scandal when first shown.
- The object at each moment is at a fixed position (according to Zeno at rest) how can it then move?

Why are paradoxes important in mathematics?
(Hint: What happens when mathematicians try to resolve a paradox? What might they discover or create in the process?)

Pythagoras(~600BC?)
The Pythagoreans

Pythagoras (~600BC?) - The Pythagoreans

- We know almost nothing about Pythagoras.
- Some scholars even doubt he existed. Most believed he as a Holy Man.
- The Pythagoreans form a very secretive sect.
- The mathematical and the mystical were merged.

Doll by UneekDollDesigns

All things are number

No primary sources ☹

Pythagoras (~600BC?) - The Pythagoreans

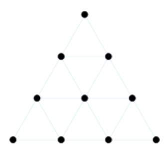
They [the Pythagoreans]were the first to advance this study [of mathematics], and having been brought up in it they thought its principles were the principles of all things. Since of these principles numbers are by nature the first, and in numbers they seemed to see many resemblances to the things that exist and come into being—… —and similarly almost all other things being numerically expressible; since, again, they saw that the attributes and the ratios of the musical scales were expressible in numbers; since, then, **all other things seemed in their whole nature to be modeled after numbers, and numbers seemed to be the first things in the whole of nature**, they supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number.

Aristotle - Metaphysics

Educated guess: Do Pythagorean ideas still shape today's science? Why or why not?
Hint: think measurement, models, data, music/acoustics.

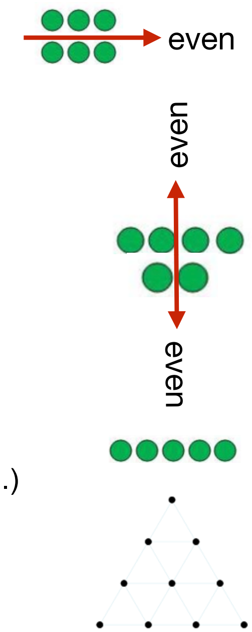
Pythagorean "mystical" views of number

- Number rules the universe.
- **1** = source of all things; neither even nor odd
- Odd = masculine;
- even = feminine;
- **5** = marriage (**2 + 3**).
- **7** considered sacred: "seven planets," seven strings on a lyre; Apollo's day = the 7th.
- **10** called "perfect": never meet in groups larger than 10; tetractys (1 + 2 + 3 + 4 = 10).



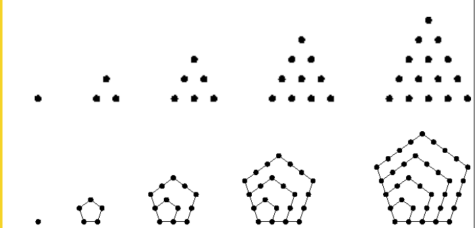
The Pythagoreans

- Recall Dichotomy between odd and even.
- Pythagoreans probably represented numbers with pebbles.
  - **Example**: A number is **even** if it can be represented by a configuration of pebbles that than be divided into two equal parts. Otherwise is odd. (Well, 1 was not considered odd, nor even)
    - Proof: 4+2 is even.
    - Proof: An even sum of odd numbers is even. (Similar to the proof of 4+2 is even.)



The Pythagoreans - Figurate numbers

The *gnomon* is the piece of the figure that needs to be added to a given figurate numbers in order to get the next greater figurate number.



What is the 6th triangular number
What is the 6th pentagonal number
If you finish early: Can you find a formula for the n-th triangular number? What about the n-th pentagonal number?

The Pythagoreans

Number rules the universe. ⟷ Great influence on scientists from then on

The Pythagoreans believed Pythagoras discovered that musical notes could be translated into mathematical equations when heard the sound of blacksmiths' hammers clanging against the anvils.

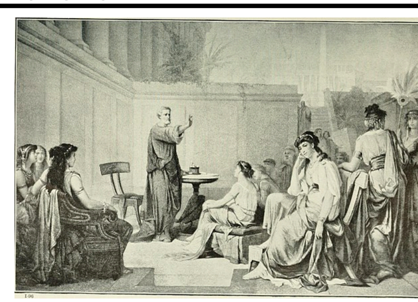


Illustration from 1913 showing Pythagoras teaching a class of women.

Western history of the Pythagorean theorem and how it was attributed to Pythagoras

- **c. 1800–1600 BCE (Mesopotamia):** Precise approximation of √2. Plimpton 322? New research discards Pythagorean links.
- **6th–5th c. BCE (Pythagoreans):** All things are number.
- **4th c. BCE — Plato:** *Meno* (diagonal doubles a square); *Republic/Timaeus* elevate math, discusses Pythagorean links.
- **4th c. BCE — Aristotle:** *Metaphysics* A5: Pythagoreans first to advance mathematics.
- **c. 300 BCE — Euclid's Elements** First extant general proof.
- **5th c. CE — Proclus:** Attributes to Pythagoras; legend spreads.
- **Medieval → modern West:** Greek historiography + Euclid's prestige fix the "Pythagorean" name despite earlier/independent proof.

1) What does this change about how we view mathematical discoveries, names, and credit? 2) (optional) Does it matter who first discovered an idea? Why or why not?

Most scholars will agree that there was a Pythagorean school of philosophy from the sixth until probably the fourth century BC, that they were involved in politics and that they had certain beliefs about life and the universe, including perhaps the tenet that 'everything is number', or that number holds the key to understanding reality. But most scholars today also think, for instance, that Pythagoras never discovered the theorem that bears his name.

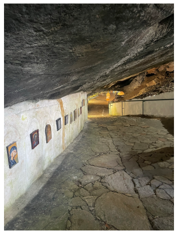
Cuomo, S. 2001. Ancient Mathematics, Science of Antiquity, Routledge.

The Discovery of √2: Drama Among the Pythagoreans?

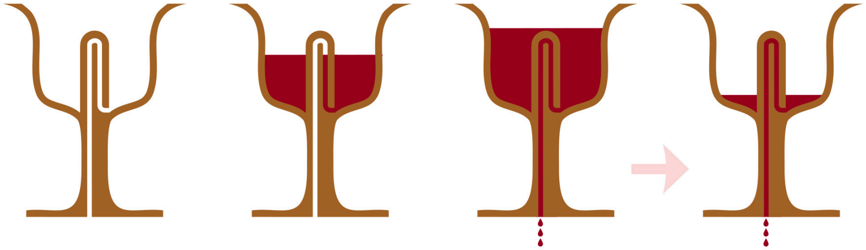
- The legend: **Hippasus**'s discovery that the square's diagonal (√2) is irrational clashed with the Pythagorean worldview.
- Some later accounts say he **revealed the secret** and was punished—**drowned** in a shipwreck.
- **Historically unverified**: sources are late and contradictory, but the story signals tension in the "all things are number" philosophy.

Hippasus of Metapontum

Hippasus, engraving by Girolamo Olgiati, 1580

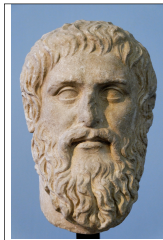


Cross section of a Pythagorean cup being filled: at B, it is possible to drink all the liquid in the cup; but at C, the siphon effect causes the cup to drain



https://commons.wikimedia.org/wiki/File:Physagorian_Pythagoras_Greedy_Tantalus_cup_05.svg

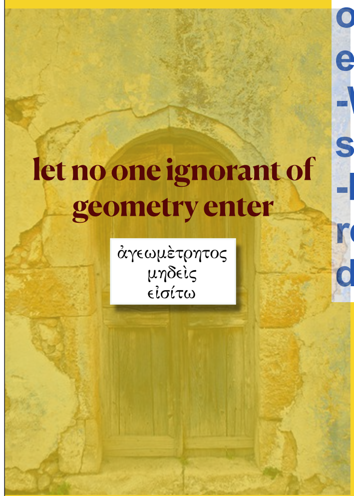
Plato ~ 400 BC



Plato (~400 BC)

In the philosophical school of thought of Plato's (429–348) academy, **mathematics was shaped as ideal case of purely deductive science**, which has influenced the development of this science enormously up to the present day. Plato argued that **mathematics had an intermediate position between the realm of mere ideas and the world of empirical objects.**

5000 Years of Geometry Mathematics in History and Culture
By Christoph J. Scriba, Peter Schreiber · 2015



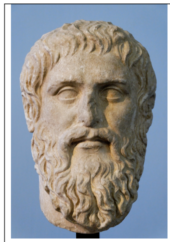
Prompt (at least one per group):
-Why would Plato post 'Let no one ignorant of geometry enter'?
-What qualities was he actually seeking in students?
-Elitist gatekeeping or reasonable prerequisite—defend your position.

There is no direct historical or archaeological evidence of such inscription but it reflects Plato's authentic emphasis on the importance of geometric reasoning.

Plato's Academy

Plato (~400 BC)

- Distinction between pure math and "merely useful"
- Math education is essential for the mind (of an elite)



Plato (~400 BC)

- Plato 'greatly advanced mathematics in general, and geometry in particular, because of his zeal for these studies'
- was an **important influence** upon the mathematicians of his time, by inspiration and direction.
- the works of Plato are some of our fullest and **best source of information about the mathematical developments at that time.**
- many scholars agree that defined the complex of general ideas forming the imperishable origin of Western thought

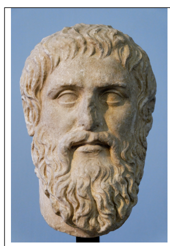
Why might Plato have seen mathematics as an 'ideal case of purely deductive science'? Choose one option and explain your reasoning.

A) Because mathematical truths, once proved from fixed axioms, don't rely on sense experience..

B) Because mathematics starts from basic definitions and axioms, then derives new truths through logical reasoning alone

C) Because mathematics only deals with abstract numbers, not physical objects

D) Because mathematicians in Athens were considered the wisest scholars



Plato (~400 BC)

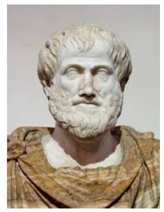
In sum, what information we can get from Plato is fascinating and rich, and fraught with problems. Beyond issues of detailed reconstructions and precise attributions, which may never be solved with any certainty, **he definitely is testimony to the vitality of mathematics at the time, and to the interest that mathematical issues aroused in the educated public at large.**

Cuomo, Serafina. Ancient mathematics. Routledge, 2005.

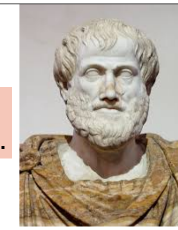
Aristotle ~ 400 BC

Aristotle (~300BC)

- An axiom is a statement worthy of acceptance
- By applying logic, one deduces new propositions.



Aristotle (~300BC)



Posterior Analytics: Two sorts of starting points for demonstration, **axioms** and **posits**.

An **axiom** (axiôma) is a statement worthy of acceptance and is needed prior to learning anything. (for instance: when equals taken from equals the remainders are equal.)

Example of **syllogism**
All men are mortal,
Socrates is a man,
therefore Socrates is mortal.

Posits are either:
Hypothesis (in math) the stipulation of objects at the beginning of a typical proof.
Definition expressions which are equivalent in some way to the defined term.

True proofs should be built out of **syllogisms** ("*a discourse in which , certain things being stated, something other than what is stated follows of necessity from their being so*." *Prior Analytics*)

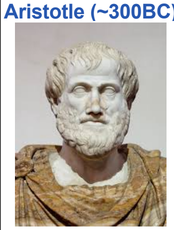
https://plato.stanford.edu/entries/aristotle-mathematics/

Is it possible to find integers p and q such that $\sqrt{2} = p/q$? In other words, is $\sqrt{2}$ rational?

For all who effect an argument per impossibile infer syllogistically what is false, and prove the original conclusion hypothetically when something impossible results from the assumption of its contradictory; **e.g. that the diagonal of the square is incommensurate with the side, because odd numbers are equal to evens if it is supposed to be commensurate. One infers syllogistically that odd numbers come out equal to evens, and one proves hypothetically the incommensurability of the diagonal, since a falsehood results through contradicting this**.

Aristotle - The Organon ANALYTICA PRIORIA Book 1 Part 23- ~350 BCE

Aristotle (~300BC)



- There are two distinct types of "quantities":
  - the continuous (magnitude)
    - *"A **magnitude** is that what which is divisible into divisible that are infinitely divisible."*
    - Example: lines, surfaces, bodies and time.
  - the discrete (number)
    - *A "**number**" quantities that is composed of distinct, separate units or parts that can be counted. (Examples 1, 2, 3, ...)*

Ideas of Math in the Ancient Greek World

300BCE - Scene depicted on a vase, likely with a counting board.

300BCE - Scene depicted on a vase, likely with a counting board.

Cuomo, Serafina. Ancient mathematics. Routledge, 2005.

Prometheus at Rockefeller Center, sculpture by Paul Manship, 1933

Paul Manship, Public domain, via Wikimedia Commons

Prometheus Bound By Aeschylus

PROMETHEUS: ..hear what wretched lives people used to lead, how babyish they were – until I gave them intelligence, **I made them masters of their own thought**. [...] they knew nothing of making brick-knitted houses the sun warms, nor how to work in wood. They swarmed like bitty ants in dugouts in sunless caves. They hadn't any sure signs of winter, nor spring flowering, nor late summer when the crops come in. All their work was work without thought, until I taught them to see what had been hard to see: where and when the stars rise and set. **What's more, I gave them the numbers, chief of all the stratagems.** And the painstaking, putting together of letters: to be their memory of everything, to be their Muses' mother, their handmaid! [...] In a word: listen! All the arts are from Prometheus.

Codex-Style Vessel CE 700–750.
Two teaching scenes with Itzam—an elderly creator/atlas-like deity—instructing four students

https://kimbellart.org/collection/ap-200404

Itzam ID cues: Aged face; netted headdress with a brush tucked in.

Scene 1: Itzam uses a pointer and a folded codex; a speech thread with **bar-and-dot numerals suggests arithmetical/calendrical calculation**.

Scene 2: Itzam taps the ground while addressing two students; spoken glyphs on a speech thread.

Takeaway: Elite scribal education in mathematics/ calendar lore framed as instruction from a creation deity.

Birds, by Aristophanes performed in 414 BC

A new city has to be founded from scratch. The main character, Peisthetaerus, is visited by various people who offer their services.

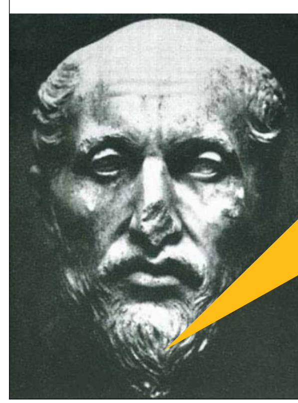
METON : With the straight rod I measure out, that **so the circle may be squared**; and in the centre a market-place; and streets be leading to it straight to the very centre; just as from a star, though circular, straight rays flash out in all directions.
PEISTHETAERUS : **Why, the man's a Thales!**

Translation Cuomo, Serafina. Ancient mathematics. Routledge, 2005.

Thales ~ 600 BC

No primary sources

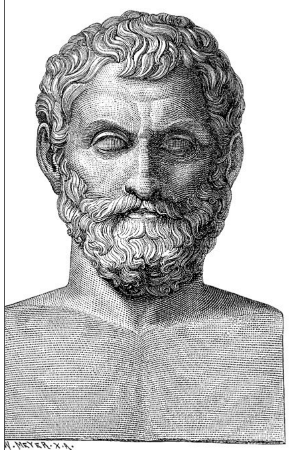
About Thales de Miletus ~ 600 BC



Hey, I am Proclus and I wrote:
"Thales was the first to go to Egypt and this study [geometry] bring back to Greece
He himself discovered many propositions and disclosed the underlying principles of many others to his successors, in some case his method being more general, in others more empirical..."

- *Proclus's Summary*
- ***written around 450 CE***
- *of Eudemus' History of Geometry*
- ***written around 320 BC***

Thales de Miletus ~ 600 BC



- We know very little.
- Earliest Greek mathematical investigations (and first proof!) that we know of
- According to Proclus
  - "*Thales travelled to Egypt and was the first to introduce mathematics into Greece".*
  - was the first to demonstrate that
    - *"the circle is bisected its diameter".*
    - *"A triangle inscribed on a circle, with a diameter as one of its sides is a right triangle."*

No primary sources ☹

ANSWER TWO OR MORE

1. **State the Pythagorean theorem.**
2. **Why the name stuck: The first extant proof is Euclid (~300 BCE), and the personal attribution to Pythagoras appears only with Proclus (5th c. CE). Explain why the label "Pythagorean theorem" nevertheless persisted.**
3. **Plato: Give one reason the Academy valued geometry and one trait Plato sought in students.**
4. **Aristotle: Define an axiom in plain language and state what it is used for**

Platonic solids

A *regular polygon* is a plane figure bounded by segments with equal sides and equal interior angles, such as the square.
Find two other examples of regular polygons and find how many regular polygons there are.
If you have time, discuss the analog to the regular polygons in dimension 3 (what they are, how many there are).

Polygons and Platonic solids

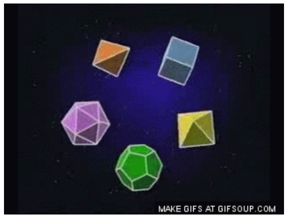
- A ***regular polygon*** is a plane figure bounded by straight lines, with equal sides and equal interior angles. There is an **infinite number** of such figures.

- In three dimensions the analog of the regular polygon is the regular polyhedron: **a solid bounded by regular polygons, with congruent faces and congruent interior angles at its corners**. One might suppose that these forms are also infinite, but in fact they are, as Lewis Carroll once expressed it, 'provokingly few in number.' There are only five regular convex solids. They are called the Platonic solids, and have been known since antiquity. Each is bounded by identical regular polygons.

Origami, Eleusis, and the Soma Cube: Martin Gardner's Mathematical Diversions" (Cambridge University Press).Origami, Eleusis, and the Soma Cube: Martin Gardner's Mathematical Diversions" (Cambridge University Press).

Platonic solids

Thaetus, a friend of Plato, may have been the first to recognize there are only 5 regular solids. Plato wrote about these solids which are now called "Platonic solids"



Some outliers notwithstanding, almost all Roman dodecahedrons were found in Britain, Gaul, and Roman Germany. IMPERIUM ROMANA - https://www.atlasobscura.com/articles/dodecahedrons-roman-empire

Two ancient Roman bronze dodecahedrons and an icosahedron from around 300.

The dodecahedrons excavated in Bonn.

The icosahedron in Arloff.

Almost all Roman dodecahedrons were found in Britain, Gaul, and Roman Germany.

wo Roman dodecahedrons (left and top) are displayed with the only known example of an icosahedron (20 sides, right) at the Rheinisches Landesmuseum in Bonn, Germany. KLEON3/WIKIMEDIA/CC BY-SA 4.0

Roman bronze dodecahedron from around 300.

Twenty-sided die (icosahedron) with faces inscribed with Greek letters

- Ptolemaic Period–Roman Period
- 2nd century B.C.–4th century A.D.
- From Egypt
- Medium: Serpentinite

https://www.metmuseum.org/art/collection/search/551072

Polygons and Platonic solids

The first systematic study of the five regular solids appears to have been made by the ancient Pythagoreans. They believed that the tetrahedron, cube, octahedron and icosahedron respectively underlay the structure of the tradition four elements: fire, earth, air and water. The dodecahedron was obscurely identified with the entire universe. Because these notions were elaborated in Plato's Timaeus, the regular polyhedrons came to be known as the Platonic solids. The beauty and fascinating mathematical properties of these five forms haunted scholars from the time of Plato through the Renaissance. The analysis of the Platonic solids provides the climactic final book of Euclid's Elements. Johannes Kepler believed throughout his life that the orbits of the six planets known in his day could be obtained by nesting the five solids in a certain order within the orbit of Saturn

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