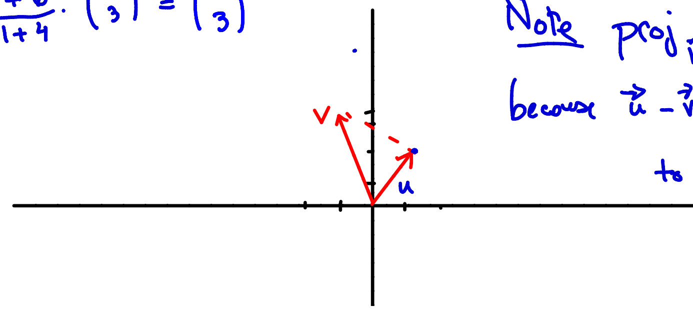
(1) Find the projection of the vector $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ onto the vector $\vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

$\text{proj}_{\vec{v}}(\vec{u}) = \dfrac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\cdot\vec{v} = \dfrac{-1+6}{1+4}\cdot\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$



Note $\text{proj}_{\vec{v}}(\vec{u}) = \vec{v}$ because $\vec{u} - \vec{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is perpendicular to $\vec{u}$.

(2) Find a single equation for the plane $\mathcal{P}$ in $\mathbb{R}^3$ which passes through the $P = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and has normal vector $\vec{n} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$.

The equation of a plane normal to $\vec{n}$ has the form $\vec{n}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c$ where c is a scalar.

Since P is in the plane then $c = \vec{n}\cdot P = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = -1+6 = 5$

Then the equation is $-x+3y+z=5$ is an equation for the plane $\mathcal{P}$.

(3) and (4) solved in class

(5) Find the vector form of the line in $\mathbb{R}^2$ passing through $P = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and perpendicular to the line with general equation $3x - 2y = 33$.

A line perpendicular to the line of equation $3x-2y=33$ has direction vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ (which is the normal vector to the given line)

Thus, the required equation can be written as $\begin{pmatrix} 2 \\ 2 \end{pmatrix} + t\cdot\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ where t in $\mathbb{R}$

(6) done in class

(7) Determine which of the equations below are linear. If any equation is not linear, explain why.

(a) $x + y + \sin z = 3$. Not linear because of the term $\sin z$

(b) $x + y^2 = 0$. Not linear because of the term y^2

(c) $2x + 2y = 2$. Linear.

(d) $x.y = 2$. Not linear because of the term $x.y$

(8) Determine geometrically whether each of the systems has a unique solution, infinitely many solutions, or no solutions. Then solve the systems algebraically to confirm your answer.

(a) $x + y = 1$
$x - 2y = 1$

These equations can be interpreted as two lines. Since the corresponding normal vectors are $\binom{1}{1}$ and $\binom{1}{-2}$ are linearly independent, the lines intersect at a unique point. Thus, the system has a unique solution.

(b) $x + y = 1$
$2x + 2y = 2$

Since the first equation multiplied by 2 gives the second, the two equation represent the same set of points, Thus set of points is the solution of the system. Thus the system has infinitely many solutions.

(c) $x + y = 1$
$2x + 2y = -2$

These are two parallel lines ($\binom{1}{1}$ is a normal vector to both directions)

They represent different lines because the first equation is equivalent to $2x+2y=2$. Thus, the lines are disjoint and the system has no solutions.

(9) in class

(10) Solve the given system of equations.

$$x+y+2z=0$$
$$x+z=-1$$
$$4x-2y+z=2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & -1 \\ 4 & -2 & 1 & 2 \end{array}\right) \xrightarrow{R_2=R_2-R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -1 & -1 \\ 4 & -2 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R_3=R_3-4R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & -6 & -7 & 2 \end{array}\right) \xrightarrow{R_1=R_1+R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & -1 & -1 & -1 \\ 0 & -6 & -7 & 2 \end{array}\right)$$

$$\xrightarrow{R_3=R_3-6R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & 8 \end{array}\right) \xrightarrow{R_2=-R_2} \left(\begin{array}{cccc} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 8 \end{array}\right) \xrightarrow{R_1=R_1+R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 8 \end{array}\right)$$

$$\xrightarrow{R_2=R_2+R_3} \left(\begin{array}{cccc} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & -1 & 8 \end{array}\right) \xrightarrow{R_3=-R_3} \left(\begin{array}{cccc} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & -8 \end{array}\right)$$

Thus the system has a unique solution, namely

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \\ -8 \end{pmatrix}$$

(11) Determine by inspection (that is, without performing any calculations) whether a linear system with each of the given augmented matrices has a unique solution, no solution or infinitely many solutions. Justify your answer.

(a) $\left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$ The red. row echelon form has 2 leading variables and a free one. Thus, the system has infinitely many sol'n.

(b) $\left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 2 \end{array}\right)$ → By applying elementary row operation (swapping rows) we obtain a red. row echelon form with a unique solution

(c) $\left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 2 \end{array}\right)$ → The last row of the matrix can be translated to $0 = 2$. Thus the system has no sol'n.

(d) $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right)$ → This is a "triangular system", that can be solved by back substitution and has a unique solution

(15) Given $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ find all matrices B such that $AB = BA$,

(16) Given the matrix $a = \begin{pmatrix} 1 & 1 \\ 2 & -4 \end{pmatrix}$.

(a) Find A^{-1}.

(b) Use A^{-1} found in (a). to solve the system $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(12) (a) Show that $\mathbb{R}^3 = \operatorname{span}(\vec{u}, \vec{v}, \vec{w})$, where $\vec{u} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$.

(b) Write $z = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{u}, \vec{v}$ and $\vec{w}$.

$$\begin{pmatrix} -1 & 0 & -2 & x \\ -2 & 2 & 3 & y \\ 1 & -1 & -1 & z \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} -1 & 0 & -2 & x \\ 0 & 2 & 7 & -2x+y \\ 1 & -1 & -1 & z \end{pmatrix}$$

$$\xrightarrow[R_3+R_1]{R_3} \begin{pmatrix} -1 & 0 & 2 & x \\ 0 & 2 & 7 & -2x+y \\ 0 & -1 & -3 & x+z \end{pmatrix} \xrightarrow{R_1 = -R_1} \begin{pmatrix} 1 & 0 & 2 & -x \\ 0 & 2 & 7 & -2x+y \\ 0 & -1 & -3 & x+z \end{pmatrix}$$

$$\xrightarrow{R_2/2} \begin{pmatrix} 1 & 0 & 2 & -x \\ 0 & 1 & 7/2 & -x+y/2 \\ 0 & -1 & -3 & x+z \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{pmatrix} 1 & 0 & 2 & -x \\ 0 & 1 & 7/2 & -x+y/2 \\ 0 & 0 & \frac{1}{2} & \frac{y}{2}+z \end{pmatrix}$$

$$\xrightarrow{R_3 = 2R_3} \begin{pmatrix} 1 & 0 & 2 & -x \\ 0 & 1 & 7/2 & -x+y/2 \\ 0 & 0 & 1 & y+2z \end{pmatrix} \xrightarrow{R_1 = R_1 - 2R_3} \begin{pmatrix} 1 & 0 & 0 & -x-2y-4z \\ 0 & 1 & 7/2 & -x+\frac{y}{2} \\ 0 & 0 & 1 & y+2z \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 - \frac{7}{2}R_3} \begin{pmatrix} 1 & 0 & 0 & -x-2y-4z \\ 0 & 1 & 0 & -x+\frac{y}{2}-\frac{7}{2}(y+2z) \\ 0 & 0 & 1 & y+2z \end{pmatrix} \begin{matrix} -x-2y+4z \\ -x-3y-7z \\ y+2z \end{matrix}$$

The system always has sol $\Rightarrow$ span $= \mathbb{R}^3$

(14) Determine whether the vectors $\vec{u} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$ are linearly independent.

If there exists x, y, z such that

$$x \cdot \vec{u} + y\vec{v} + z\vec{w} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

then $-1x + 0y + 0z = 0$

Thus $x = 0$.

$$\begin{cases} 2y + 3z = 0 \\ -y + 3z = 0 \end{cases}$$

Then $y = 0$ and $z = 0$.

Thus the vectors are linearly independent since the only way of obtaining $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ as a linear combination is with all the coefficients equal to 0.

(15) Given $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ find all matrices B such that $AB = BA$,

Set $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}$$

If $AB = BA$ then $c = 0$ and $a = d$

Thus $B = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$