f) Let q be the symmetric bilinear form defined on $\mathbf{R}^{n+1}$ by

$$q(x,y) = \left(-x_0y_0 + \sum_{i=1}^{n} x_iy_i\right),$$

and set

$$O(1,n) = \{A \in M_n(\mathbf{R})/\forall x, y \in \mathbf{R}^{n+1},\ q(Ax,Ay) = q(x,y)\}$$
$$H^n = \{x \in \mathbf{R}^{n+1}/q(x,x) = -1 \quad \text{and} \quad x_0 > 0\}$$
$$O_0(1,n) = \{A \in O(1,n)/A(H^n) = H^n\}.$$

We want to show that $O_0(1,n)$ acts transitively on H^n. First we show that the orthogonal symmetries (for q) with respect to an hyperplane of $\mathbf{R}^{n+1}$ belong to $O(1,n)$: if D is a non isotropic straight line, and if P is the orthogonal hyperplane to D for q, any vector $u \in \mathbf{R}^{n+1}$ can be decomposed as the sum of two vectors $u_1 \in D$ and $u_2 \in P$. The map

$$s : u_1 + u_2 \to u_1 - u_2$$

belongs to $O(1,n)$ since

$$q\big(s(u_1+u_2), s(v_1+v_2)\big) = q(u_1,v_1) + q(u_2,v_2)$$
$$= q(u_1+u_2, v_1+v_2).$$

Let now $x, y \in H^n$. To show that there exists an hyperplane containing 0 and $\frac{x+y}{2}$, and orthogonal to $y-x$, it is sufficient to prove that $q(x+y, x-y) = 0$, which is the case since $q(x,x) = q(y,y) = 1$.
The symmetry s with respect to P sends x to y. We still have to prove that $s \in O_0(1,n)$. Let

$$\overline{H}^n = \{x \in \mathbf{R}^{n+1}/q(x,x) = -1\}.$$

$\overline{H}^n$ has two connected components ($x_0 > 0$ or $x_0 < 0$) and is preserved under s. Since s was built to send x to y for a pair of points of H^n, s preserves also H^n and hence $s \in O_0(1,n)$. One can prove similarly that $SO_0(1,n)$ (matrices of $O_0(1,n)$ with positive determinant) acts transitively on H^n.
Now the isotropy group of e_0 is the set of matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

with $A \in O(n)$. The action of $O_0(1,n)$ on H^n is free and proper and hence $H^n = O_0(1,n)/O(n)$ (similarly we could prove that $H^n = SO_0(1,n)/SO(n)$). This result shows that $O(1,n)$ has four connected components ($\det = \pm 1$, preserving H^n or not): each of these components being isomorphic to $SO_0(1,n)$,

$$\frac{\partial}{\partial u} = \big(r'(u)\cos\theta, r'(u)\sin\theta, z'(u)\big) \tag{2.1}$$

$$\frac{\partial}{\partial \theta} = \big(-r(u)\sin\theta, r(u)\cos\theta, 0\big). \tag{2.2}$$

Hence $\mid \frac{\partial}{\partial u} \mid = \sqrt{r'(u)^2 + z'(u)^2}$, $\mid \frac{\partial}{\partial \theta} \mid = r(u)$ and $\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial \theta} \rangle = 0$ that is, since we assumed that c is parametrized by arclength,

$$g = (du)^2 + r(u)^2(d\theta)^2.$$

If we want to get such manifolds as the sphere, we have to accept points on the curve c with $r(u) = 0$: to make sure that the revolution surface generated by c is regular, we must demand that in these points, $z'(u) = 0$. For details concerning conditions at the poles, see [B1]. In the case of the sphere, the curve c is given by $c(u) = (\sin u, \cos u)$ and $r(u) = \sin u$. Outside the poles, the induced metric is given in local coordinate (u, θ) by

$$g = du^2 + (\sin^2 u)d\theta^2.$$

2.10 **The hyperbolic space.**
We introduce first the *Minkowski space* of dimension $n+1$, that is $\mathbf{R}^{n+1}$ be equipped with the quadratic form

$$\langle x, x\rangle = -x_0^2 + x_1^2 + \cdots + x_n^2.$$

Let H^n be the submanifold of $\mathbf{R}^{n+1}$ defined by

$$H^n = \{x \in \mathbf{R}^{n+1} / \langle x, x\rangle = -1, x_0 > 0\}$$

(the second condition ensures that H^n is connected, see 1.1). The quadratic form

$$-dx_0^2 + dx_1^2 + \ldots + dx_n^2 \in \Gamma(S^2T^*\mathbf{R}^{n+1})$$

induces on H^n a positive definite symmectric 2-tensor g. If indeed $a \in H^n$, T_aH^n can be identified with the orthogonal of a for the quadratic form $\langle , \rangle$, and g_a with the restriction of $\langle , \rangle$ to this subspace. Since $\langle a, a\rangle < 0$, g_a is positive definite from the Sylvester theorem, or an easy computation. The hyperbolic space of dimension n is just H^n, equipped with this Riemannian metric. Using 1.101 f), we see that $O_o(1, n)$ acts *isometrically* on H^n. There exist other presentations for this space. They are conceptually easier, but technically more complicated. They are treated in the following exercise.

2.11 **Exercise: Poincaré models for** (H^n, g).
Let f be the "pseudo-inversion" with pole $s = (-1, 0, \cdots, 0)$ defined by

$$f(x) = s - \frac{2(x - s)}{\langle x - s, x - s\rangle},$$

where $\langle , \rangle$ is the quadratic form defined in 2.10. For $X = (0, X_1, .., X_n)$ in the hyperplane $x_0 = 0$, we note that

$$\langle X, X\rangle = \sum_{i=1}^{n} X_i^2 = \mid X \mid^2,$$

(square of the Euclidean norm in $\mathbf{R}^n$).
a) Show that f is a diffeomorphism from H^n onto the unit disk

$$\{x \in \mathbf{R}^n, \mid x \mid < 1\}, \quad \text{and that} \quad (f^{-1})^* g = 4 \sum_{i=1}^{n} \frac{dx_i^2}{(1 - \mid x \mid^2)^2}.$$

b) What happens if we replace H^n by the unit sphere of the Euclidean space $\mathbf{R}^{n+1}$, and f by the usual inversion of modulus 2 from the South pole?
c) Let f_1 be the inversion of $\mathbf{R}^n$ with pole $t = (-1, 0, \cdots, 0)$, given by

$$f_1(X) = t + \frac{2(x - t)}{\mid x - t \mid^2}.$$

Show that f_1 is a diffeomorphism from the unit disk onto the half space $x_1 > 0$. If $g_1 = \left(f^{-1}\right)^* g$ (see b)), show that

$$(f_1)^* g_1 = \sum_{i=1}^{n} \frac{dx_i^2}{x_1^2}.$$

In that way, we have obtained two Riemannian manifolds which are isometric to (H^n, g), namely the *Poincaré disk* in a), and the *Poincaré half-plane* in c).

2.11 bis **Definition** Two Riemannian metrics g_0 and g_1 on a manifold M are *(pointwise) conformal* if there is a nowhere vanishing smooth function f on M such that $g_1 = f g_0$.
Both Poincaré models are conformal to the Euclidean metric (on the unit ball or the half-plane). For two conformal metrics, the angles are the same.

Let us now give other examples to show that isometric Riemannian metrics may look quite different.

2.12 **Exercise.** Let $C \subset \mathbf{R}^3$ be a *catenoid:* C is the revolution surface generated by rotation around the z-axis of the curve of equation $x = \cosh z$. Let $H \subset \mathbf{R}^3$ be an *helicoid:* H is generated by the straight lines which are parallel to the xOy plane and meet both the z-axis and the helix $t \to (\cos t, \sin t, t)$.
a) Show that H and C are submanifolds of $\mathbf{R}^3$, and give a "natural" parametrization for both.
b) If g is the Euclidean metric $dx^2 + dy^2 + dz^2$ of $\mathbf{R}^3$, give the expressions of $g_{|C}$ and $g_{|H}$ in the parametrizations defined in a), and show that C and H are locally isometric. Are they globally isometric?
It is not possible to guess from the embeddings that C and H are locally the same from the Riemannian point of view. For example, C does not contain any straight line, even no segment: the local isometries between C and H do not come from isometries of the ambiant space.