easily for the differential operators (that is local operators) which will be met in the sequel:
i) the exterior derivative (1.119),
ii) the covariant derivative associated with a Riemannian metric (2.58)
iii) the Laplace operator on functions, and on differential forms (4.7, 4.29).

1.F.6 A characterization for tensors

The following test will be used frequently in the sequel (see for example 3.2 and 5.1).
1.114 **Proposition.** *Let P be a $\mathbf{R}$-linear map between two spaces of tensors $\Gamma(T^p_qM)$ and $\Gamma(T^r_sM)$. The following are equivalent:*
a) *P is $C^\infty(M)$-linear,*
b) *for s,s' in $\Gamma(T^p_qM)$, and m in M with $s_m = s'_m$, then $(Ps)_m = (Ps')_m$.*
Proof. It is clear that b) implies a).
Now if a) is satisfied, P is a local operator (use a test function as we did in lemma 1.111). Assume then that $s_m = s'_m$. Use a local trivialization of T^p_qM and lemma 1.111 to built an open subset U of M containing m, smooth functions $(f_i)_{(i=1,\cdots,N)}$ (where N is the dimension of the fiber), and sections $(\sigma_i)_{(i=1,\cdots,n)}$ of T^p_qU such that $(s-s')_{|U} = \sum f_i\sigma_i$. Hence

$$P(s-s')_{|U} = P\left(\sum f_i\sigma_i\right) = \sum f_iP(\sigma_i),$$

therefore $P(s-s')_m = \sum f_i(m)P(\sigma_i)_m = 0$.
Simpleminded example. Let $P : \Gamma(T^1_0M) \to C^\infty(M)$ be a $C^\infty(M)$-linear map. For any vector field X and any $m \in M$, the previous proposition ensures that the real $P(X)_m$ depends only on X_m in T_mM. It is clear that the map $X_m \to P(X)_m$ is a linear form on T_mM. Hence we got a section (with a priori no regularity) of the bundle T^*M. One can use local trivializations of T^1_0M and prove that this section is smooth. Conversely, it is clear that if ξ is a smooth section of T^*M, the map $X \to \xi(X)$ from $\Gamma(T^1_0M)$ to $C^\infty M$ is $C^\infty(M)$-linear.
This is a general phenomenon. One can use the isomorphisms $E^*\otimes F = \mathrm{Hom}(E,F)$ and $(E\otimes F)^* = E^*\otimes F^*$ (for finite dimensional vector spaces), and the proposition 1.114, to prove that it is equivalent to give a $C^\infty(M)$-linear map from $\Gamma(T^p_qM)$ to $\Gamma(T^r_sM)$, or a section of the bundle $\Gamma\big((T^p_qM)^*\otimes(T^r_sM)\big)$, that is a (q+r,p+s)-tensor. Do not be satisfied with these abstract considerations, but apply them to the curvature tensor (3.3).
Counter-example. The bracket, seen as a bilinear map from $\Gamma(TM)\times\Gamma(TM)$ to $\Gamma(TM)$, is not a tensor. For X, Y in $\Gamma(TM)$ and f, g in $C^\infty(M)$, we have indeed

$$[fX,gY] = f(X.g)Y - g(Y.f)X + fg[X,Y].$$

Another example. A q-covariant tensor can be seen as a q-linear form on the $C^\infty(M)$ module $\Gamma(TM)^{\otimes q}$. This point of view will be very useful in the following: we shall principally meet the bundle S^2M of bilinear symmetric forms, (which is clearly a subbundle of T^0_2M), the bundles Λ^kM of antisymmetric k-forms ($k = 1,\cdots,n = \dim M$), and the corresponding spaces of sections, that is the bilinear symmetric forms, and the exterior forms on M.
1.115 **Exercise.** Let $S \in \Gamma(T^0_qM)$, and $X,X_1,\dots X_q$ be (q+1) vector fields. Show that

$$(L_XS)(X_1,\cdots,X_q) = X.S(X_1,\cdots,X_q)$$
$$-\sum_{i=1}^{q} S(X_1,\cdots,X_{i-1},[X,X_i],\cdots,X_{i+1},X_q).$$

1.G Differential forms

The most important tensors are differential forms. The main reason for their importance is the fact that, under mild compactness assumptions, it is possible to define the integration of a form of degree k on a (sub)manifold of dimension k.

1.G.1 Definitions

1.116 **Definition.** A *differential form of degree* k on a manifold M is a smooth section of the bundle Λ^kM. We will set $\Gamma(\Lambda^kM) = \Omega^kM$.
1.117 **Algebraic recall:**
For a vector space E, there exists on $\otimes^kE$ an antisymmetrization operator, defined on the decomposed elements by:

$$\mathrm{Ant}(x_1\otimes\cdots\otimes x_k) = \sum_{s\in S_k}\mathrm{sign}(s)x_{s(1)}\otimes\cdots\otimes x_{s(k)}.$$

This formula is more suggestive for k-forms: for $f\in\otimes^kE^*$ we have

$$(\mathrm{Ant}f)(x_1,\cdots,x_k) = \sum_{s\in M}\mathrm{sign}(s)f(x_{s(1)},\cdots,x_{s(k)}).$$

We denote by $\bigwedge^k E^*$ the vector space of antisymmetric k-forms on E.
The *exterior product* of $f\in\Lambda^kE^*$ and $g\in\Lambda^lE^*$ is the (k+l) antisymmetric form defined by

$$f\wedge g = \frac{1}{k!l!}\mathrm{Ant}(f\otimes g).$$

Example. For f, g in E^*,

$$(f\wedge g)(x,y) = f(x)g(y) - f(y)g(x).$$

We will admit that the exterior product is associative and anticommutative, that is

$$f \wedge g = (-1)^{kl} g \wedge f, \quad \text{for } f \in \Lambda^k E^* \quad \text{and } g \in \Lambda^l E^*$$

(see [Sp] t.1 for a proof).
Recall finally that $\Lambda^{(\dim E)} E^* \simeq \mathbf{R}$ and that $\Lambda^k E^* = 0$ for $k > \dim E$.
It can also be useful to consider forms with complex values, and the corresponding bundles. The amateur of abstract nonsense can check easily that that $L_{\mathbf{R}}(E, \mathbf{C}) = E^* \otimes_{\mathbf{R}} \mathbf{C}$ and that, if $E_{\mathbf{C}} = E \otimes_{\mathbf{C}} \mathbf{R}$ is the complexified of the vector space E, then $\Lambda^k E \otimes_{\mathbf{R}} \mathbf{C}$ is $\mathbf{C}$-isomorphic to $(\Lambda^k E_{\mathbf{C}})$.
It is clear that all the previous definitions extend directly to the subbundle $\Lambda^k M$ of antisymmetric tensors of $T^0_k M$.
As a consequence of the properties of the action of ϕ^* and L_X on tensors (1.107 and 1.110), we have

$$\phi^*(\alpha \wedge \beta) = \phi^*\alpha \wedge \phi^*\beta, \quad \text{and } L_X(\alpha \wedge \beta) = L_X\alpha \wedge \beta + \alpha \wedge L_X\beta.$$

1.118 **Exercises.** a) Let $\omega \in \Omega^1(S^2)$ be a differential 1-form on S^2 such that for any $\phi \in SO(3)$, $\phi^*\omega = \omega$ holds. Show that $\omega = 0$. State and prove an analogous result for differential forms on S^n.
b) Let p be the canonical projection from $\mathbf{C}^{n+1} \setminus \{0\}$ to $P^n\mathbf{C}$. Show that there exists a 2-form ω on $P^n\mathbf{C}$ such that

$$p^*\omega = \left(\sum_{k=0}^{n} dz^k \wedge d\bar{z}^k\right) / \left(\sum_{k=0}^{n} \mid z_k \mid^2\right),$$

$$\text{where} \quad dz^k = dx^k + i dy^k \quad \text{and} \quad d\bar{z}^k = dx^k - i dy^k.$$

Show that the form ω is invariant under the action of $U(n+1)$ on $P^n\mathbf{C}$, and that for $k = 2, \cdots, n$ the $2k$-form ω^k is non zero and $U(n+1)$ invariant.
We will see that the forms ω^k generate the cohomology of $P^n\mathbf{C}$ (see 4.35).
Exterior forms are more interesting than tensors, for the following reason: we shall define on $\sum_{k=0}^{\dim M} \Omega^k M$ a "natural" differential operator (see 4.1) – that is depending only on the differential structure of M. This operator gives information on the topology of the manifold (cf. 1.125).

1.G.2 Exterior derivative

1.119 **Theorem and definition.** *Let M be a smooth manifold. For any $p \in \mathbf{N}$, there exists a unique local operator d from $\Omega^p M$ to $\Omega^{p+1} M$, called the exterior derivative and such that*
i) *for p=0, $d : C^\infty(M) \to \Omega^1 M$ is the differential on functions,*
ii) *for $f \in C^\infty(M)$, we have $d(df) = 0$,*
iii) *for $\alpha \in \Omega^p M$ and $\beta \in \Omega^q M$, we have*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta.$$

Short proof. (for more details, see [Sp]). Let us first treat the case where M is an open subset U of $\mathbf{R}^n$. Then $\alpha \in \Omega^p U$ can be decomposed in a unique way as:

$$\alpha = \sum \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

where the sum is understood on all the strictly increasing sequences $i_1 < \cdots < i_p$ of $[1, n]$, and the $\alpha_{i_1, \cdots i_p}$ being smooth. We must have

$$d\alpha = \sum d\alpha_{i_1 \cdots i_p} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

One checks directly that the operator just defined (which satisfies i) and ii) by construction) also satisfies iii). To prove that d is a local operator, just imitate the argument of 1.111, using iii) for the product of a test function and a p-form.
To extend this result to manifolds, we will need the following:

1.120 **Lemma.** *Let $\phi : U \to V$ be a diffeomorphism between two open subsets of $\mathbf{R}^n$. Then $\phi^* \circ d = d \circ \phi^*$, that is the diagram below commutes.*

$$\begin{array}{ccc} \Omega^p(V) & \xrightarrow{\phi^*} & \Omega^p(U) \\ d\downarrow & & \downarrow d \\ \Omega^{p+}(V) & \xrightarrow{\phi^*} & \Omega^{p+1}(U) \end{array}$$

Proof. It is the chain rule in the case $p = 0$. Use 1.119 b) and the behaviour of ϕ^* with respect to the exterior product (1.117), and proceed by induction on p. ∎

End of the proof of the theorem. If M is equipped with an atlas (U_i, ϕ_i), we can define $d\omega_i$ for the local expressions of a p-form ω on M (begining of the proof) and, from the lemma:

$$\left(\phi_j \circ \phi_i^{-1}\right)^* \left(d\omega_{j|\phi_j(U_i \cap U_j)}\right) = d\omega_{i|\phi_i(U_i \cap U_j)}.$$

This proves (see 1.108) that the $d\omega_i$ are local expressions for a (p+1)-form on M, which is the form $d\omega$ we are looking for: use 1.120 and check that $d\omega$ doesn't depend on the atlas. ∎

1.121 **Proposition.** *For any p-form ω on M, the following holds:*
i) *$d(d\omega) = 0$, that is $d \circ d = 0$;*
ii) *for a smooth map $\phi : M \to N$, $d(\phi^*\omega) = \phi^*(d\omega)$, that is $\phi^* \circ d = d \circ \phi^*$;*
iii) *for a vector field X on M, $L_X d\omega = d(L_X\omega)$, that is $L_X \circ d = d \circ L_X$, and $L_X\omega = d(i_X\omega) + i_X(d\omega)$, that is $L_X = d \circ i_X + i_X \circ d$.*
Proof. i), ii) and the first part of iii) are clear. Let

$$P_X = d \circ i_X + i_X \circ d.$$

One checks directly that

$$P_X(\alpha\wedge\beta) = P_X\alpha\wedge\beta + \alpha\wedge P_X\beta,$$

i.e. that P_X is, as L_X, a derivation of the $C^\infty(M)$-algebra $\Omega(M)$. Look carefully at the existence theorem for Lie derivative on tensors (1.110), and note that a derivation is determined by its values on functions and 1-forms: hence we only need to check that $L_X = P_X$ on functions and 1-forms (even only on 1-forms which can be written as dx^i in local coordinates), which is immediate. ■

1.122 **Corollary.** *For $\alpha\in\Omega^p(M)$ and $(X_0,\cdots,X_p)$ $p+1$ vector fields on M, we have*

$$d\alpha(X_0,\cdots,X_p) = \sum_{i=0}^{p}(-1)^i L_{X_i}\big(\alpha(X_0,\cdots,\hat{X}_i,\cdots,X_p)\big)$$
$$+\sum_{0\le i<j\le p}(-1)^{i+j}\alpha([X_i,X_j],X_0,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_p).$$

Proof. It is the definition of d for $p=0$. Proceed by induction on p (use 1.115). ■

Example. For $\alpha\in\Omega^1(M)$, we have

$$d\alpha(X,Y) = X.\alpha(Y) - Y.\alpha(X) - \alpha([X,Y]).$$

Compare these formulas with those of 2.61, where d is computed in terms of the covariant derivative associated with a Riemannian metric.

1.123 **Exercise.** Check that the right member of the equation in 1.122 is $C^\infty(M)$-linear with respect to the X_i.

Remark. We can take the expression of d obtained at 1.122 as a definition for the exterior derivative. This point of view is technically less simple, (convince yourself by trying and prove that $d\circ d=0$ using Jacobi identities!), but is coordinate-free: that is why it is useful in infinite dimension (see for example [Bt]).

Another application of the second formula on 1.121 iii) is the proof of the the following

1.124 **Poincaré lemma.** *Let U be an open star-shaped subset of $\mathbf{R}^n$, and α be a p-form on U such that $d\alpha=0$. Then there exists a $(p-1)$-form $\beta\in\Omega^{(p-1)}(U)$ such that $d\beta=\alpha$.*

Proof. See [Wa] or [Sp] ■

The analogous property is false on general manifolds. This leads to the definitions of important (smooth, *in fact topological*) of the manifold.

1.125 **Definitions.** Let M be a smooth manifold. The p^{th} *de Rham group* of M, denoted by $H^p_{DR}(M)$, is the quotient

$$\{\alpha\in\Omega^p(M), d\alpha=0\}/d\Omega^{p-1}(M).$$

The *p-the Betti number* $b_p(M)$ is the dimension of $H^p_{DR}(M)$.

* **Remark.** In Algebraic Toplogy, Betti numbers are defined for any coefficient field K: $b_k(M,K) = \dim H^k(M,K)$. Both definitions coincide for real numbers, in view of de Rham's theorem (cf. [Wa] or [B-T]).*

It is clear that $H^p_{DR}(M)=0$ when $p>\dim M$, and that $H^0_{DR}(M)=\mathbf{R}^k$ if M has k connected components. It can be proved (see [W]) that the vector spaces $H_{DR}(M)$ are indeed topological invariants, and that for M compact, connected, orientable and n-dimensional, $H^n_{DR}(M)=\mathbf{R}$. Finally, $H^k_{DR}(S^n)=0$ for $0<k<n$ (ibidem). We will come back to this subject in chapter IV, and we will see how to compute de Rham cohomology by using analytic methods.

1.G.3 Volume forms

1.126 **Definition.** A *volume form* on an n-dimensional manifold is a never zero exterior form of degree n.

The volume forms are interesting in view of the following theorem.

1.127 **Theorem.** *Let M be a countable at infinity and connected manifold. The following are equivalent:*
i) *there exists a volume form on M;*
ii) *the bundle Λ^nT^*M is trivial;*
iii) *M is orientable.*

Proof. i) and ii) are clearly equivalent: since Λ^nT^*M is a bundle of rank one, there exists a non zero section of this bundle (that is a volume form on M) if and only if it is trivial (1.35 a).

Let us show now that i) implies iii). Let ω be a volume form on M and (U_i,ϕ_i) be an atlas for M. The local expression of ω in a chart is:

$$\phi_i^{-1*}(\omega_{|U_i}) = a_i dx^1\wedge dx^2\wedge\cdots\wedge dx^n,$$

where the function $a_i\in C^\infty(\phi_i(U_i))$ is never zero. One can assume that the a_i are positive (compose ϕ_i with an orientation reversing symmetry if necessary). From the very definition of exterior forms, we know that on $\phi_i(U_i\cap U_j)$:

$$\left(\phi_i\circ\phi_j^{-1}\right)^*(a_i dx^1\wedge dx^2\wedge\cdots\wedge dx^n) = a_j dx^1\wedge dx^2\wedge\cdots\wedge dx^n,$$

and hence the jacobian $J\left(\phi_i\circ\phi_j^{-1}\right)=a_j/a_i$ is always positive.

That iii) implies i) is more technical, and uses partitions of unity (see 1.H). Assume that M is orientable, and let (U_i,ϕ_i) be an orientation atlas for M (the jacobians of all the transition functions are positive). From the hypothesis of the theorem, we can assume that the family (U_i) is locally finite. Hence there exists a subordinate partition of unity (ρ_i). Let $\omega_0 = dx^1\wedge dx^2\wedge\cdots\wedge dx^n$: since $\mathrm{supp}(\rho_i)\subset U_i$, the forms

$$\omega = \rho_i\big((\phi_i^{-1})^*\omega_0\big)$$

and $\omega=\sum_i\omega_i$ (finite sum at each point) are defined on the whole M. We must now check that ω is a volume form: work on the domain of a chart U_k, and note that

$$\phi_k^* \omega = \sum_i \rho_i \left(\phi_k \circ \phi_i^{-1}\right)^* \omega_0$$

(this sum is actually finite) is of the form $(\sum_i \mu_i)\omega_0$, where, from the hypothesis, the $\mu_i(x)$ are nonnegative and one of them at least is strictly positive at each point of U_k. ∎

Exercise. Using the previous theorem, prove that $P^n\mathbf{R}$ is orientable if and only if n is odd.

1.G.4 Integration on an oriented manifold

1.128 One says that two orientation atlases on an orientable manifold M are equivalent if their union is still an orientation atlas. An orientation for an orientable manifold is an equivalence class of orientation atlases. If M is connected, there are two distinct orientations.

It can be seen more easily using volume forms: from the proof of 1.127, a volume form defines an orientation. If ω and ω' are two volume forms, there exists a never zero function f such that $\omega = f\omega'$, and ω and ω' define the same orientation if and only f is positive.

Let now α be an exterior form of degree n, with compact support in M, and assume that M is oriented. We define the integral of α over M, denoted by $\int_M \alpha$ in the following way: let (U_i, ϕ_i) be an atlas for M such that the cover U_i is locally finite. If supp(α) is included in U_i, then

$$\left(\phi_i^{-1}\right)^* \alpha = f_i dx^1 \wedge \cdots \wedge dx^n,$$

where $f_i \in C^\infty\big(\phi_i(U_i)\big)$, and we set

$$\int_M \alpha = \int_{\mathbf{R}^n} f_i dx^1 \wedge \cdots \wedge dx^n.$$

Using the change of variables formula, and the fact that the jacobians are positive, we see that this integral does not depend on the chart (if supp$(\alpha) \subset U_i \cap U_j$).

In the general case, use a partition of unity (ρ_i) subordinate to the covering (U_i), and decompose α into a finite sum (since α has compact support) of n-forms with support in some U_i. The same use of the change of variables formula shows that the result does not depend on the orientation atlas. It can be checked that the result does not depend on the partition of unity either. But if we change of orientation, the integral is changed into its opposite. For more details, see [Wa], or the first volume of [Sp].

Now if ω is a volume form giving the orientation chosen for M, we associate to ω a positive measure μ on M by setting $\mu(f) = \int_M f\omega$. One checks easily that, if f is continuous and nonnegative, then $\mu(f) = 0$ if and only if $f \equiv 0$.

1.G.5 Haar measure on a Lie group

1.129 **Theorem.** *There exists on any Lie group G a non-trivial left-invariant measure (that is, if f is continuous with compact support, and if $g \in G$, $\mu(f \circ L_g) = \mu(f)$). This measure is unique up to a scalar factor.*

Proof. Let $n = \dim G$. For an n-exterior non zero form α on $\underline{G} = T_eG$, we define an L_g-invariant form on G by, for $g \in G$ and $x_i \in T_gG$:

$$\overline{\alpha}(g)(x_1, x_2, \cdots, x_n) = \alpha(e)(T_gL_{g^{-1}}x_1, \cdots, T_gL_{g^{-1}}x_n),$$

where $\alpha(e) = \alpha$ (compare to 2.90). Then, for $f \in C^0(G)$ with compact support, we have

$$\int_G f\overline{\alpha} = \int_G (f \circ L_g)L_g^*\overline{\alpha} = \int_G (f \circ L_g)\alpha.$$

The uniqueness, which will not be used in the sequel, is left to the reader. ∎

Remark. The previous theorem is true for any locally compact topological group, but the proof is more difficult.

Misusing the term, we will say that the measure defined above is "the" Haar measure on G.

1.130 **Exercises.** a) Explicit the Haar measure μ (that is the corresponding exterior form) in the case $G = Gl(n, \mathbf{R})$.

b) Let K be a compact Lie subgroup of $Gl(n, \mathbf{R})$ (actually, any closed subgroup of a Lie group is a Lie subgroup, see [G]). Show that there exists a K-invariant quadratic form q on $\mathbf{R}^n$, and deduce there exists $g \in Gl(n, \mathbf{R})$ such that $g.K.g^{-1} \subset O(n)$ *(this is the starting point for the proof that all the maximal compact subgroups of a Lie group are conjugate)*.

c) Check the uniqueness we claimed in 1.129. Hint: let μ be left-invariant and ν right-invariant, both non trivial. Set $\check{\nu}(f) = \nu(\check{f})$, where $\check{f}(g) = f(g^{-1})$. Then $\check{\nu}$ is left-invariant.

Pick a function f such that $\mu(f) \neq 0$, and set

$$D_f(s) = \frac{1}{\mu(f)} \int_G f(t^{-1}s)d\nu(t)$$

Show that, for any $\phi \in C^0(G)$, $\nu(\phi) = \mu(D_f\phi)$. Infer that D_f does not depend on f. The take $s = e$ and conclude.

Remark. We will introduce in 3.90 the notion of *density* on a manifold, and will then be able to compute the integral of functions (in place of maximal degree exterior forms) without any orientability assumption.

1.H Partitions of unity

Be given a smooth manifold M, we know what is a smooth "object" on the manifold (for example a function, a vector field, an exterior form...): we just