MAT 552. HOMEWORK 9
SPRING 2014
DUE TU APR 15

1. Define $\mathrm{Sp}(n)$ as a group of $\mathbb{H}$-linear automorphisms of $\mathbb{H}^n$, which preserve an $\mathbb{H}$-hermitian form $< \cdot, \cdot >$ on $\mathbb{H}^n$. The form is defined by the formula $< x, y >= \sum_{i=1}^{n} x_i \bar{y}_i$.

(1) Use this data to show that $\mathrm{Sp}(n) \cong \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(2n)$
(2) Show that $\mathrm{Sp}(n) \cong \mathrm{GL}(n, \mathbb{H}) \cap \mathrm{U}(2n)$
(3) Explain why $\rho(A) = jAj^{-1}$ is an anti-holomorphic involution on $\mathrm{Sp}(2n, \mathbb{C})$.
(4) Show that the image of $\mathrm{GL}(n, \mathbb{C}) \to \mathrm{Sp}(2n, \mathbb{C})$ defined in HW8 is invariant under ρ. Deduce from this that $\mathrm{U}(n)$ is a subgroup of $\mathrm{Sp}(n)$.
(5) Show that diagonal matrices in $\mathrm{U}(n) \subset \mathrm{Sp}(n)$ is a maximal torus in $\mathrm{Sp}(n)$. Find the roots of $\mathrm{Sp}(n)$.

We define $\mathrm{SO}(n, \mathbb{C})$ as a space of complex invertible matrices $\{A \in \mathrm{GL}(n, \mathbb{C}) | (Av, Au) = (v, u)\}$ where $(v, u) = \sum_{i=1}^{n} x_i y_i, x_i, y_i \in \mathbb{C}$.

2.

(1) Show that there is a real involution ρ on $\mathrm{SO}(n, \mathbb{C})$ whose fixed points coincide with $\mathrm{SO}(n, \mathbb{R})$.
(2) Follow the analogy with $\mathrm{Sp}(2n, \mathbb{C})$ to define an embedding of $\mathrm{GL}(n, \mathbb{C})$ into $\mathrm{SO}(2n, \mathbb{C})$. (Hint use a complex basis in which inner product has a form $\sum_{i=1}^{n} x_i y_{n+i} + y_i x_{n+i}$)
(3) Show that $\mathrm{GL}(n, \mathbb{C})$ is invariant under ρ and the fixed points coincide with $\mathrm{U}(n)$.
(4) Find a decomposition of $T_e(\mathrm{SO}(2n, \mathbb{C}))$ into irreducible representation under the action of diagonal subgroup $\mathbb{T}^{\mathbb{C}} \subset \mathrm{GL}(n, \mathbb{C})$.
(5) Show that the maximal torus $\mathbb{T} \subset \mathrm{U}(n)$ is a maximal torus in $\mathrm{SO}(2n, \mathbb{R})$. Find the roots.

3. Repeat the same for $\mathrm{SO}(2n + 1)$.(Hint:use a complex basis in which the inner product on $\mathbb{C}^{2n+1}$ has a form $x_{2n+1} y_{2n+1} + \sum_{i=1}^{n} x_i y_{n+i} + y_i x_{n+i}$)