MAT 552. HOMEWORK 8
SPRING 2014
DUE TH APR 3

- A Hausdorff topological space X is a complex or holomorphic manifold if it has an holomorphic atlas, that is $X = \bigcup U_\alpha$ such that open U_α are such that exist local homeomorphisms $\psi_\alpha : \mathbb{R}^{2n} \cong \mathbb{C}^n \supset V_\alpha \to U_\alpha$ that have holomorphic compositions $\psi_\alpha^{-1} \circ \psi_\beta : V_\beta \to V_\alpha$ defined on the intersection $U_\alpha \cap U_\alpha$.
- A holomorphic map $f : X \to Y$ between two complex manifolds X, Y is a C^∞-map, which in local charts is defined by holomorphic functions $\phi_\beta^{-1} \circ f \circ \psi_\alpha$. $\{\psi_\alpha, U_\alpha\}$ is an atlas on X, $\{\phi_\alpha, W_\alpha\}$ is an atlas on Y.
- If X and Y are holomorphic manifolds then $X \times Y$ is a holomorphic manifold.
- A holomorphic Lie group is a Lie group, which is equipped with a holomorphic atlas such that multiplication map $\mu : G \times G \to G$ is holomorphic in this atlas. In addition the inverse map is also holomorphic.
- A C^∞ maps $\rho : X \to X$ is an antiholomorphic involution on a complex manifold X if $\rho^2 = \mathrm{id}$ and in local charts ρ is defined by anti-holomorphic functions.
- An antiholomorphic involution on a complex Lie group is a homomorphism $\rho : G \to G$, such that ρ is an antiholomorphic involution of the underlying manifold.

1.

(1) Give an example of a antiholomorphic involution on the Lie group $\mathbb{C}^\times$ whose fixed points is
  (a) S^1
  (b) $\mathbb{R}^\times$
  (c) $\emptyset$

(2) Give an example of a antiholomorphic involution on the complex manifold $\mathbb{C}^\times$ whose set of fixed points is empty.

(3) Give examples of antiholomorphic involutions on $\underbrace{\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times}_{n}$, whose fixed points are manifolds of dimension 0 or n

2. Let X be a complex n-dimensional manifold, equipped with an antiholomorphic involution ρ. Suppose is fixed point set $X^\rho = Y$ of ρ is a n-dimensional submanifold of the C^∞ $2n$-dimensional manifold underlying complex manifold X. Show that for any $y \in Y$ complexification $T_y(Y) \otimes \mathbb{C}$ is *canonically* isomorphic to $T_y(X)$. By definition an isomorphism is canonical if it commutes with maps induced by complex homeomorphisms that preserve y and commute with ρ.

3.

(1) Prove that $\mathrm{SL}(2,\mathbb{C}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) | a,b,c,d \in \mathbb{C}, ad - bc = 1\}$ is a complex analytic manifold.
(2) Show that the map $\rho : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C})$, $\rho(A) = (\overline{A}^t)^{-1}$ is an antiholomorphic involution on $\mathrm{SL}(2,\mathbb{C})$, whose fixed point set is $\mathrm{SU}(2)$
(3) Show that $\mathbb{G}^* = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\} \subset \mathrm{SL}(2,\mathbb{C})$ is invariant under ρ and the set of fixed points $(\mathbb{G}^*)^\rho$ as a group is isomorphic to S^1
(4) The group $\mathrm{SL}(2,\mathbb{C})$ acts on $T_e(\mathrm{SL}(2,\mathbb{C}))$; the group $\mathrm{SU}(2)$ acts on $T_e(\mathrm{SU}(2))$. In addition $\mathrm{SU}(2)$ being a subgroup of $\mathrm{SL}(2,\mathbb{C})$ acts on $T_e(\mathrm{SL}(2,\mathbb{C}))$. Show that there is an isomorphism of $\mathrm{SU}(2)$-representations $T_e(\mathrm{SU}(2)) \otimes \mathbb{C} \cong T_e(\mathrm{SL}(2,\mathbb{C}))$.
(5) Determine the structure of the restriction of $T_e(\mathrm{SL}(2,\mathbb{C}))$ from $\mathrm{SL}(2,\mathbb{C})$ to $\mathbb{G}^*$.
(6) Determine the structure of the restriction of $T_e(\mathrm{SU}(2))$ from $\mathrm{SU}(2)$ to S^1.

4. Generalize results of Problem 3 from $\mathrm{SL}(2,\mathbb{C})$ to $\mathrm{SL}(n,\mathbb{C})$. In particular find an analogue of ρ, of the maximal torus T of $\mathrm{SU}(n)$ and the complex-analytic subgroup $\mathbb{T} \subset \mathrm{SL}(n,\mathbb{C})$. Your goal should be to establish decomposition of $\mathfrak{sl}_n(\mathbb{C})$ under $\mathbb{T}$ and relate it to decomposition of $\mathfrak{su}_n$ under T.

Definition 1. Define $\mathrm{Sp}(2n,\mathbb{C})$ as a subgroup of $\mathbb{C}$-linear transformations of $\mathbb{C}^{2n}$ that preserve a skew-symmetric bilinear form

$$\Omega[x,y] = \sum_{i=1}^{n} x_{2i-1}y_{2i} - y_{2i-1}x_{2i}$$

5.

(1) Show that the linear space $\mathfrak{sp}(2n,\mathbb{C}) = \{A \in Mat_{2n}(\mathbb{C}) | \Omega[Ax,y] + \Omega[x,Ay] = 0 \forall x,y \in \mathbb{C}^{2n}\}$ is closed under commutator and is a Lie algebra .
(2) Show that the exponential map $exp : \mathfrak{sp}(2n,\mathbb{C}) \to \mathrm{Sp}(2,\mathbb{C})$ defines a chart around $e \in \mathrm{Sp}(2,\mathbb{C})$.
(3) Define a bilinear operation $\{f,g\}$ on $\mathbb{C}[x_1 \dots, x_{2n}]$ by the formula:
$$\{f,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_{2i-1}}\frac{\partial g}{\partial x_{2i}} - \frac{\partial g}{\partial x_{2i-1}}\frac{\partial f}{\partial x_{2i}}$$
Show that $\{f,g\}$ defines a Lie algebra structure on $\mathbb{C}[x_1 \dots, x_{2n}]$. This bracket is called a Poisson bracket.
(4) Show that $\mathrm{Sp}(2n,\mathbb{C})$ acts on $\mathbb{C}[x_1 \dots, x_{2n}]$ by automorphism of the Poisson bracket.
(5) Show that representation of $\mathrm{Sp}(2n,\mathbb{C})$ in polynomials of degree two is isomorphic to adjoint representation.
(6) Let V be a complex vector space. Show that $V + V^*$ has a canonical symplectic form.
(7) Identify $\mathbb{C}^{2n}$ with some $\mathbb{C}^n + (\mathbb{C}^n)^*$. Define an embedding $\mathrm{GL}(n,\mathbb{C}) \to \mathrm{Sp}(2n,\mathbb{C})$. Decompose $\mathfrak{sp}_{2n}(\mathbb{C})$ into $\mathrm{GL}(n,\mathbb{C})$-invariant components.
(8) Decompose $\mathfrak{sp}_{2n}$ into irreducible representation of the subgroup of diagonal matrices $\mathbb{T} \subset \mathrm{GL}(n,\mathbb{C})$.