MAT 552. HOMEWORK 4
SPRING 2014
DUE TH FEB 20

1. Suppose X is a topological space, R is equivalence relation. Show that

(1) if the quotient space X/R is Hausdorff, then R is closed in the product space $X \times X$.
(2) if the projection p of a space X onto the quotient space X/R is open, and R is closed in $X \times X$, then X/R is a Hausdorff space.
(3) Give an example of equivalence relation R on the set X such that $X \to X/R$ is not open.

Definition 1. (1) *A continuous function $f : X \to Y$ is called proper if f maps closed sets to closed sets and $f^{-1}(K)$ is compact for all compact $K \subset Y$.*
(2) *Let G be a topological group acting continuously on a topological space X. The action is called proper if the map $: G \times X \to X \times X$ given by $(g, x) \to (x, gx)$ is proper.*

2. Show that

(1) If G acts by homeomorphisms, then the quotient map $p : X \to X/G$ is always open (contrary to general quotient maps). This is a generalization of Problem 2 HW3.
(2) X/G is Hausdorff if and only if the orbit equivalence relation is a closed subset of $X \times X$.
(3) If G acts properly on X then X/G is Hausdorff. In particular, each orbit Gx is closed. The stabilizer G_x of each point is compact and the map $G/G_x \to Gx$ is a homeomorphism.
(4) If H is a closed subgroup then G/H is Hausdorff.
(5) Let G be a topological group and N the component of the identity in G. Then G/N is Hausdorff.

3.

(1) Let V be an inner product space with signature $(1, -1, \dots, -1)$. Show that if $(l_1, l_1) > 0, (l_2, l_2) > 0$ then $(l_1, l_2)^2 \geq (l_1, l_1)(l_2, l_2)$
(2) Let $\mathbb{C}^2$ be a two-dimensional complex space with a basis $\{e, e'\}$. The space $\mathbb{C}^2 \underset{\mathbb{C}}{\otimes} \overline{\mathbb{C}}^2$ has a real structure $j, j^2 = 1$ defined by the formula $j(e \otimes e') = e' \otimes e, j(e \otimes e) = e \otimes e, j(e' \otimes e') = e' \otimes e'$. Identify the space of real points of $\mathbb{C}^2 \underset{\mathbb{C}}{\otimes} \overline{\mathbb{C}}^2$ with the space of Hermitian matrices M. Compute the signature of the bilinear form $\langle A, B\rangle$ associated with the homogeneous quadratic function $\det A$ on M. Verify that the group $G_{\mathbb{C}} = \{g \in \mathrm{GL}(2, \mathbb{C}) || \det g| = 1\}$ acts on the space M by the formula

$gA = gA\bar{g}^t$. Compute the signature of $\langle A, B\rangle$. Identify the group $Aut(\langle ., .\rangle)$. Compute the image and the kernel of the homomorphism.

(3) Let $\mathbb{C}^2$ be a two-dimensional complex space with a basis $\{e, e'\}$. The space $\mathbb{C}^2 \underset{\mathbb{C}}{\otimes} \mathbb{C}^2$ has a real structure $j, j^2 = 1$ defined by the formula $j(e \otimes e') = e \otimes e', j(e' \otimes e) = e' \otimes e, j(e \otimes e) = e \otimes e, j(e' \otimes e') = e' \otimes e'$. j defines a real structure on the symmetric part $\mathrm{Sym}^2\mathbb{C}^2$ of $\mathbb{C}^2 \underset{\mathbb{C}}{\otimes} \mathbb{C}^2$. Let M be the space of real points in $\mathrm{Sym}^2\mathbb{C}^2$. Compute the signature of the bilinear form $\langle A, B\rangle$ associated with the homogeneous quadratic function $\det A$ on M. Verify that the group $G_{\mathbb{R}} = \{g \in \mathrm{GL}(2, \mathbb{R}) | \det g = \pm 1\}$ acts on the space M by the formula $gA = gAg^t$ and preserves $\langle A, B\rangle$. Identify the group $Aut(\langle ., .\rangle)$ compute the image of the homomorphism $G_{\mathbb{R}} \to Aut(\langle ., .\rangle)$.

(4) One dimensional quaternionic space $\mathbb{H}$ is the same as two-dimensional complex space $\mathbb{C}^2$ with a structure map $j, j^2 = -1$. The space $\mathbb{C}^4 = \mathbb{C}^2 + \mathbb{C}^2$ carries the diagonal structure map j. Let $\Lambda^2\mathbb{C}^4$ be the skew-symmetric part of $\mathbb{C}^4 \otimes \mathbb{C}^4$. $j \otimes j$ defines a real structure ($j^2 \otimes j^2 = \mathrm{id} \otimes \mathrm{id}$) on $\Lambda^2\mathbb{C}^4$. The Pfaffian function $\mathrm{Pf}(A)$ can be used to define a bilinear form $\langle A, B\rangle$ on $\Lambda^2\mathbb{C}^4$. Compute the signature of $\langle A, B\rangle$. Verify that the group $G_{\mathbb{H}} = \{g \in \mathrm{GL}(4, \mathbb{C}) | \det g = \pm 1, gj = jg\}$ acts on the space M by the formula $gA = gAg^t$ and preserves $\langle A, B\rangle$. Compute the image of the homomorphism $G_{\mathbb{H}} \to Aut(\langle ., .\rangle)$.