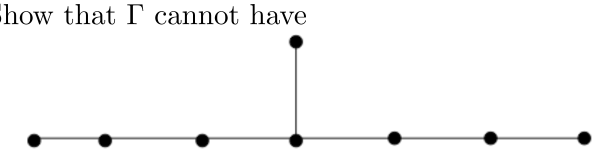
MAT 552. HOMEWORK 11
SPRING 2014
DUE TH MAY 1

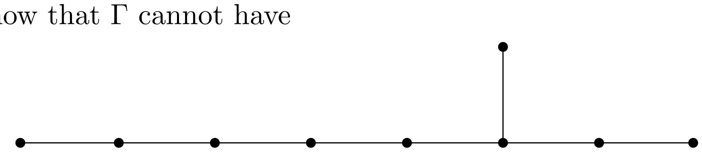
- Suppose that A is a symmetric $n \times n$ matrix with $A_{i,i} = 2$ for all i and $A_{i,j} \in \{-1, 0\}$ for $i \neq j$. To this matrix we can draw a non-oriented graph Γ by connecting i and j with an edge iff $A_{i,j} = -1$. The graph Γ completely determines A and we write A_Γ. The number of vertices of this graph $v(\Gamma) = n$.

1. Suppose A_Γ is positive definite, i.e. $x^t A x > 0$ when $x \neq 0 \in \mathbb{R}^{v(\Gamma)}$.

(1) Show that Γ has no cycles.
(2) Show that Γ cannot have a vertex with ≥ 4 edges.
(3) Show that a connected component of Γ cannot have two distinct vertices with each ≥ 3 edges.
(4) Show that Γ cannot have a connected subtree with only one three-valent vertex, having all its adjacent vertices valence two.
(5) Show that Γ cannot have

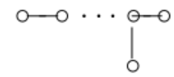
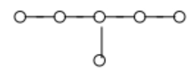
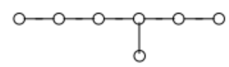
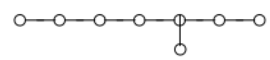


as a subgraph.
(6) Show that Γ cannot have



as a subgraph.

- Γ is called a Dynkin graph if A_Γ is positive definite
- A-D-E graphs are

- A_n :
- D_n: 
- E_6 : 
- E_7 : 
- E_8 : 

2.

(1) Show that if Γ_1 and Γ_2 are Dynkin graphs then their disjoint union $\Gamma_1 \coprod \Gamma_2$ is also a Dynkin graph.
(2) Compute det A_{A_n}.
(3) Compute det A_{D_n}.
(4) Compute det A_{E_n}.
(5) Verify that any connected Dynkin graph belongs to A-D-E family.
(Hint: use Sylvesters Criterion)

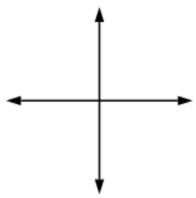
- A finite subset R of Euclidean space $\mathbb{R}^n$ is a root system if
  (1) $0 \notin R$ and R spanes $\mathbb{R}^n$.
  (2) $\alpha \in R \Rightarrow -\alpha \in R$, but $k \cdot \alpha$ is not in R if k is any real number other than ± 1.
  (3) For $\alpha \in R$, the reflection w_α in the hyperplane $\alpha^\perp$ maps R to itself,
  (4) For $\alpha, \beta \in R$, the real number

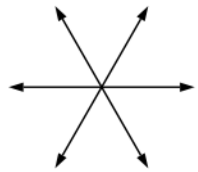
$$n_{\beta\alpha} = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}$$

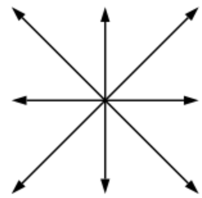
Is an integer.

3.

(1) Verify that the following subsets in $\mathbb{R}^2$ are root systems.
(2) Identify generators $\{w_\alpha\}$ of the Weyl group.

$A_1 \times A_1$: 

A_2: 

B_2: 

G_2: