MAT 552. HOMEWORK 1
SPRING 2014
DUE TH FEB 6

1.

(1) Prove that M is Hausdorff space $\Leftrightarrow$ the diagonal $\Delta \subset M \times M$ is closed.
(2) Show that a topological group G is Hausdorff $\Leftrightarrow$ one point set $\{e\} \subset G$ where e is the unit is closed.
(3) Recall that Lie group is a topological group , which is Hausdorff and which admits an atlas of neighborhoods homeomorphic to $\mathbb{R}^n$ together with smoothness condition. Show that the Hausdorff property is automatically satisfied.

2. Let S^3 be a round sphere

$$S^3 = \{x \in \mathbb{R}^4 | x \cdot x = 1\}$$

Identify the space of pairs

$$X = \{(x, v) \in \mathbb{R}^4 \times \mathbb{R}^4 | x \cdot x = 1, x \cdot v = 0\}$$

with the tangent bundle to S^3 as it was defined in class.

Definition 1. *Cayley transform is a map* $Mat_n \xrightarrow{\#} Mat_n$

$$A^\# = (\mathrm{id} - A)(\mathrm{id} + A)^{-1}$$

defined for all matrices such that $\det(\mathrm{id} + A) \neq 0$. *We denote the set of such matrices by* R_n

3.

(1) Prove that $\#\# = \mathrm{id}$ and $\#(R_n) \subset R_n$.
(2) Fix a standard inner product on $\mathbb{R}^n$. Let $\mathrm{O}(n) = \{A \in Mat_n | AA^t = \mathrm{id}\}$. Show that $\#(\mathrm{O}(n) \cap R_n) = \Lambda^2\mathbb{R}^n \cap R_n$, where $\Lambda^2\mathbb{R}^n$ is a linear space of skew-symmetric matrices.
(3) Prove that $\mathrm{O}(n)$ is a Lie group.