$\Leftrightarrow$ bounded and ~~∀~~ for every $x \in M$ and $\varepsilon > 0$.

$\exists\, n \geq 0$ s.t. $\sum_{i \geq n} x_i^2 < \varepsilon$.

---

pf) pre-compact (Wikipedia).

$\Leftrightarrow$ every seq. has convergent subseq.
(may or may not convergent in M)

$\Leftrightarrow$ <u>totally bounded</u> (~~finite~~ complete metric space)

$\Leftrightarrow$ $\bar{M}$ is compact in l_2.

① $\Leftarrow$ Consider the ε-net (totally bounded)

$$\sum x_i^2 = \sum_{i<n} x_i^2 + \sum_{i \geq n} x_i^2$$

$$\leq \sum_{i<n} x_i^2 + \varepsilon. \leq \sup_{i<n} x_i^2 \cdot n + \varepsilon$$

~~The~~

$$< \varepsilon M. \quad M(x_i^2, n).$$

for some $M > 0$.

$\Rightarrow$ pre-compact $\Rightarrow$ totally bounded

Consider the seq. as a function. Then

$\boxed{\text{Arzela-Ascoli thm}}$ $\Rightarrow$ $\exists$ convergent subsequence.

$\rho(f(x), f(y)) < \rho(x, y) \quad x \neq y.$

$\boxed{\text{Existence}}$ of the fixed pt.

Let $x_n = f^n(x_0)$. K is compact $\Rightarrow$ There exists a subseq. of $\{x_n\}$, say $\{x_{n_i}\}$ such that $\lim_{i \to \infty} x_{n_i} = x \in K$

Then $f^{n_i - n_{i-1}}(x) = f^{n_i - n_{i-1}}(\lim_{i \to \infty} x_{n_{i-1}}) = \lim_{i \to \infty} f^{n_i - n_{i-1}}(x_{n_{i-1}})$

$= \lim_{i \to \infty} x_{n_i} = x.$

Then x is a periodic pt. Let p be the period of x.

If $\rho(x, f(x)) = \rho(f^p(x), f^{p+1}(x))$ then it's a contradiction

If $\_\_ > \_\_$

$\Rightarrow x = f(x)$ Hence f is a fixed pt.

$\boxed{\text{Uniqueness}}$ Let x and y be fixed points.

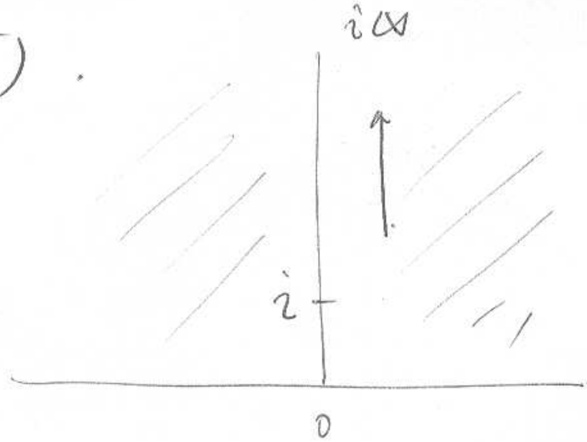
$\rho(f(x), f(y)) = \rho(x, y)$ then $x = y$

Hence, fixed point is unique.

If $\_\_ < \_\_$ then it's a contradiction.

(There exist many examples)

$\overline{\mathbb{H}}$: closure of the half plane with the hyperbolic metric. (which is isomorphic to $\overline{\mathbb{D}}$).



upper-half complex plane.

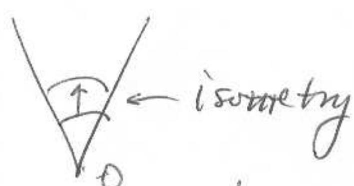
$$f(x) = x + i$$

then $\rho(f(x), f(y)) \le C\rho(x, y)$ $x \ne y$.

$C = C(p)$ $p \in \overline{\mathbb{H}}$

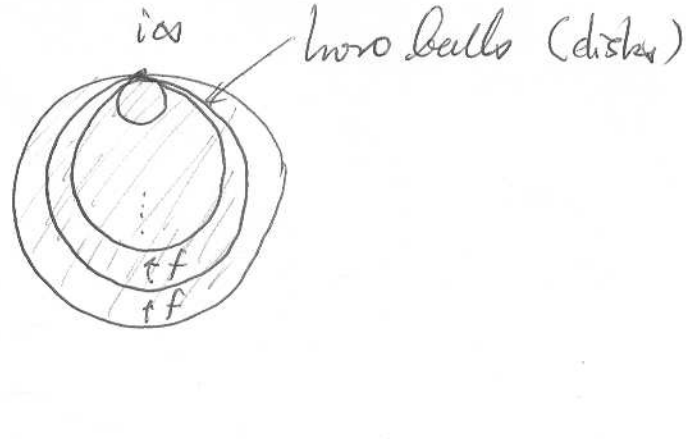
— $\lim_{p \to i\infty} C(p) = 1$.

(I skip the proof).



(see the plane hyperbolic geometry)

Disk model



thm $C(K)$ is separable. K: cpt in $\mathbb{R}$.

pf) $\Leftrightarrow$ It has countable basis

$\boxed{\text{Stone-Weirstrass thm.}}$

$C([a,b])$ is (uniformly) approximated by the polynomials. It can be extended to the I^n, the box $\underbrace{I \times I \times I \times \cdots \times I}_{n\text{-times}}$. Furthermore. It can be extended to any compact set K. (Skip the proof).

the Polynomials (with multivariables) has ~~to~~ the finite number coefficients, or a ~~the~~ basis of polynomials is a set of monomials $\{1, x_1, x_1^2, x_1^3, \cdots, x_2^0, x_2^2, \cdots, x_3, x_3^2, \cdots$

$\cdots x_n, x_n^2, \cdots\}$

and it's a countable basis.

Thus polynomials on K is separable and it's dense in $C(K)$. Therefore, $C(K)$ is seperable.

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

$$y' = f(t) + f(y), \quad y(3) = 5.$$

Global existence and uniqueness of the solution.

Rk. $\boxed{f \text{ is differentiable at } 0 \text{ (on } \mathbb{R}) \text{ and } \lim_{x \to \infty} \frac{f(x)}{x} = 1. \text{ and } \lim_{x \to -\infty} \frac{f(x)}{x} = -1.}$

(See Geller's book Chaper 2)

Thm. 2.2.2 Fundamental <u>local</u> existence thm for ODE. Thm 2.2.8 Uniqueness thm for ODE.

and the <u>global</u> existence can be proved when the given function $|F(t_1, Y_1) - F(t_1, Y_2)| \leq K|Y_1 - Y_2|$ for all $Y_1, Y_2 \in \mathbb{R}^n$ for all $t_1 \in (t_0 - h, t_0 + h)$. by extension of the <u>local</u> <u>unique</u> solution ~~to~~ on the whole domain.

Let $F(t, y) = f(t) + f(y)$

$$= t^2 \sin\left(\frac{1}{t}\right) + y^2 \sin\left(\frac{1}{y}\right)$$

Take $t_1 \in (3-\varepsilon, 3+\varepsilon)$ for some $\varepsilon > 0$.

$$|F(t_1, y_1) - F(t_1, y_2)| = \left| y_1^2 \sin\left(\frac{1}{y_1}\right) - y_2^2 \sin\left(\frac{1}{y_2}\right) \right|$$

✓ $f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$

is continuous except $x = 0$.

✓ Take $u = \frac{1}{x}$ and take $|u| \leq \frac{\pi}{2}$. Then.

$f'(u) = \frac{2 \sin u}{u} + (-\cos u)$ (even function)

is increasing on $(0, \frac{\pi}{2}]$

because $\frac{\sin u}{u}$ and $-\cos u$ are increasing.

and decreasing on $[-\frac{\pi}{2}, 0)$.

Then if $|y| \geq \frac{2}{\pi}$, then

$|F(t_1, y_1) - F(t_1, y_2)| \leq K |y_1 - y_2|$ where $K = \left| f'\left(\frac{\pi}{2}\right) \right|$

Consider

$$\sup_{y_1, y_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{y_1^2 \sin\left(\frac{1}{y_1}\right) - y_2^2 \sin\left(\frac{1}{y_2}\right)}{y_1 - y_2}$$

and take $y_1 = \dfrac{1}{(2n+1)\frac{\pi}{2}}$ and

$$y_2 = \frac{1}{(2n-1)\frac{\pi}{2}}$$

Then $\sin\left(\frac{1}{y_1}\right) = 1$ and $\sin\left(\frac{1}{y_2}\right) = -1$

for all $n \in \mathbb{N}$. but $y_1, y_2 \to 0$ when $n \to +\infty$

Then $\cancel{\sup_{y_1, y_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]}} \dfrac{y_1^2 \sin\left(\frac{1}{y_1}\right) - y_2^2 \sin\left(\frac{1}{y_2}\right)}{y_1 - y_2}$

$$= \frac{y_1^2 + y_2^2}{y_1 - y_2}.$$ is unbounded

Hence, f is not Liptschitz near 0.

Therefore there doesn't exist the ~~unique~~ global solution on $\mathbb{R}$.

Claim.

Function $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$

satisfies.

$$\left| f(x_1) - f(x_2) \right| \leq K |x_1 - x_2|.$$

Proof. It suffice to prove for $x_1 x_2 \geq 0$ because for $-x_2 < 0$

$$\left| f(x_1) - f(-x_2) \right| = \left| f(x_1) + f(x_2) \right| \leq |f(x_1)| + |f(x_2)| \leq$$
$$\leq K\left(|x_1| + |x_2| \right) = K \left| x_1 - (-x_2) \right|.$$

Suppose $x_2 = 0$. $x_1 > 0$ then inequality $|f(x_1)| \leq K|x_1|$ is equivalent to boundness of $\left| \frac{f(x)}{x} \right|$ in $\mathbb{R} \setminus \{0\}$. The later is equal to $\left| x \sin\frac{1}{x} \right|$.

Observe that $\sin x \leq x \quad x \geq 0$. (Lagrange thm.)

$\Rightarrow \quad \sin\frac{1}{x} \leq \frac{1}{x} \quad x \geq 0.$

$\Rightarrow \left| x \sin\frac{1}{x} \right| \leq \left(x \cdot \frac{1}{x} \right) \leq 1.$

The derivatime of $f(x)$ on $\mathbb{R} \setminus \{0\}$ is $2x \sin\frac{1}{x} - \cos\frac{1}{x}$.
From this we conclude that $|f'(x)| \leq 2\left| x \sin\frac{1}{x} \right| + \left| \cos\frac{1}{x} \right| \leq 3.$

By Lagrange theorem $x_1, x_2 > 0 \quad f(x_1) - f(x_2) = f'(\theta)(x_1 - x_2)$
then $\left| f(x_1) - f(x_2) \right| \leq 3 \, |x_1 - x_2|.$

It follows that f satisfies

$\left| f(x_1) - f(x_2) \right| \leq 3 \, |x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}$.

We prove that.

$\dot{y} = f(t) + f(y)$ for any initial condition t_0 y_0 has a global solution.

As usual we define $A : C[a,b] \to C[ab]$ by the formula

$$(A\varphi)t = y_0 + \int_{t_0}^{t} f(\varphi(\tau))\,d\tau + \int_{t_0}^{t} f(\tau)\,d\tau.$$

Then we claim that for arbitrary $\varphi_0 \in C[ab]$ and $\varphi_i = A^i\varphi_0$ the sequence of functions satisfies

$$|\varphi_{i+1}(t) - \varphi_i(t)| \leq C\,\frac{[K\,(b-a)]^i}{i!} = C\,\frac{[K(b-a)]^i}{i!}$$

For some constants $K, C > 0$

Proof

$$\left|\varphi_2(t) - \varphi_1(t)\right| \doteq \left|\int_{t_0}^{t} [f(\varphi_1(\tau)) - f(\varphi_0(\tau))]\,d\tau\right| \leq$$

$$\leq \int_{t_0}^{t} |f(\varphi_1(\tau)) - f(\varphi_0(\tau))|\,d\tau \leq 3 \int_{t_0}^{t} |\varphi_1(\tau) - \varphi_0(\tau)|\,d\tau$$

$C = \max_{a \leq t \leq b} |\varphi_1(\tau) - \varphi_0|$

$$\leq 3\,C \int_{t_0}^{t} d\tau = 3\,C(t - t_0) \leq 3C(b-a). \qquad \boxed{K = 3}$$

$$|\varphi_3(t) - \varphi_2(\tau)| \leq 3 \int_{t_0}^{t} |\varphi_2(\tau) - \varphi_1(\tau)|\,d\tau \leq 3 \int_{t_0}^{t} 3C(t - t_0)\,d\tau$$

$$= 3^2 C\,\frac{(t-t_0)^2}{2!}$$

$$\ldots$$

$$|\varphi_{n+1}(t) - \varphi_n(t)| \leq 3 \int_{t_0}^{t} |\varphi_n(\tau) - \varphi_{n-1}(\tau)|\,d\tau \leq 3 \int_{t_0}^{t} 3^n C\,\frac{(\tau - t_0)^n}{n!}$$

$$= 3^{n+1}\frac{(t-t_0)^{n+1}}{(n+1)!}$$

The sum $\sum_{n \geq 0} 3^n \frac{(t-t_0)^{n}}{n!}$ converges uniformly. $\{\varphi_n\}$ is a Cauchy sequence in $C[ab]$. Pass to the limit in $\varphi_n = A\varphi_{n-1}$ we get a unique solution $\varphi = A\varphi$. a, b - are arbitrary solution exist on $\mathbb{R}$.