#1. $(X, \mathcal{M}, \mu)$ $E_i \in \mathcal{M}$, $i \in \mathbb{N}$

$$\sum_{i \geq 0} \mu(E_i) < \infty$$

$$? \; \mu\left(\bigcap_{i \geq 0} E_i\right) = 0$$

$$\sum_{i \geq 0}^{\infty} \mu(E_i) = \lim_{n \to \infty} \sum_{i \geq 0}^{n} \mu(E_i)$$

$$= \lim_{n \to \infty} \sum_{i \geq 0}^{n-1} \mu(E_i) < \infty$$

$$\Rightarrow \lim_{n \to \infty} \left(\sum_{i \geq 0}^{n} \mu(E_i) - \sum_{i \geq 0}^{n-1} \mu(E_i) \right) = 0.$$

$$\Rightarrow \lim_{n \to \infty} \mu(E_n) = 0.$$

$$\mu\left(\bigcap_{i \geq 0}^{n} E_i\right) \leq \mu(E_n) \quad \forall n \in \mathbb{N}.$$

$$\mu\left(\bigcap_{i \geq 0} E_i\right) = \lim_{n \to \infty} \mu\left(\bigcap_{i \geq 0}^{n} E_i\right) \leq \lim_{n \to \infty} \mu(E_n) = 0.$$

#2. $A = \{x \in X \mid x \in E_i$ for infinitely many $i\}$.

! $\mu(A) = 0$.

$$\boxed{A = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E_i}$$

(For every $x \in A$, $x \in \bigcup_{i=n}^{\infty} E_i \quad \forall n \in \mathbb{N}$.

and for $y \in \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E_i$, y is contain in

infinitely many E_i)

Let $A_n = \{x \mid x \in E_i$ at least n E_i's$\}$

then $n\mu(A_n) \leq \sum_{n=1}^{\infty} \mu(E_i) < \infty$

because $\mu(A_n)$ is measured n-times

in $\sum_{n=1}^{\infty} \mu(E_i)$.

$\lim_{n\to\infty} A_n = A$ then.

$\mu(A) = \lim_{n\to\infty} \mu(A_n) \leq \frac{1}{n} \sum \mu(E_i) \quad \forall n \in \mathbb{N}$.

$\mu(A) = 0$

#3. Trendy number $x \in \mathbb{R}$

$$\left| x - \frac{p_i}{q_i} \right| < \frac{1}{q_i^3} \quad \forall i \in \mathbb{N}. \text{ for some } \left\{ \frac{p_i}{q_i} \right\} \subseteq \mathbb{Q}$$

Let $T = \{ \text{trendy numbers} \}$.

$$E_i = \left\{ x \in [0,1] \;\middle|\; \left| x - \frac{p_i}{q_i} \right| < \frac{1}{q_i^3} \right\}.$$

$\mu(E_i) \leq \frac{2}{q_i^3}$ then $\sum_{i=1}^{\infty} \mu(E_i) \leq \sum_{i=1}^{\infty} \frac{2}{q_i^3} < \infty$

Moreover, $T \cap [0,1] \subseteq T'$

$T' = \{ x \in [0,1] \mid x$ is in the infinitely many E_i's$\}$.

By the problem 2. $\mu(T') = 0$. then.

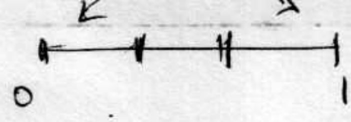
$\mu(T \cap [0,1]) = 0$.

$$\mu(T) = \sum_{n=0}^{\infty} \mu(T \cap [n, n+1])$$

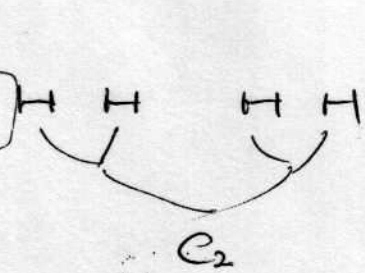
$$+ \sum_{n=0}^{\infty} \mu(T \cap [-n-1, -n])$$

is the countable sum of 0's. (μ is invariant under translation)

Then $\mu(T) = 0$.

#4. $C = \bigcap_{i \geq 1} C_i$. 

Middle third Cantor set

$\boxed{C_1 \supset C_2 \supset C_3 \supset \cdots \supset C_n \supset \cdots}$ 

1. Every closed and bounded interval is Borel measurable set.
2. The countable union ~~and~~ or the countable intersection of the closed interval is also Borel measurable.

1, 2 $\Rightarrow$ C is Borel measurable.

$$C_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]$$

$$C_2 = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1].$$

$\vdots$

$C_n = \cdots\cdots$

$$\boxed{\begin{array}{c} C_1 \supset C_2 \supset \cdots \supset C_n \supset \cdots \\ \Downarrow \\ \mu(C) = \lim_{n \to \infty} \mu(C_n) \end{array}}$$

$$\mu(C_n) = 2^n \cdot \left(\frac{1}{3}\right)^n$$

$$\lim_{n \to \infty} \mu(C_n) = \lim_{n \to \infty} \left(\frac{2}{3}\right)^n = 0. \quad \text{Hence } \mu(C) = 0.$$

#5 $p(x)$: polynomial. For <u>all</u> solutions of

$y' = p(y)$. $y(t)$ exists for all $t \in \mathbb{R}$.

$\boxed{\text{degree } p = ?}$

1. <u>Consider</u> $y' = y^2$ separable equation.

$$\int \frac{dy}{y^2} = \int dt$$

$$-\frac{1}{y} = t + C$$

$$y = -\frac{1}{t+C}$$

However, for each $C \in \mathbb{R}$, y does not exist at $t = -C$. Then $\deg p \neq 2$. satisfying hypothesis.

2. Assume that deg ≥ 3. and leading coefficient of y is positive.

$p(y) > y^2$ for all sufficiently larg y.

Consider $y' > y^2$

#5 $\int \frac{dy}{y^2} > \int dt.$

(*) $-\frac{1}{y} > t + C$ for all sufficiently large y.

Let y_0 be the value when $y > y_0$ satisfies (*). $y' > y^2$ then $y(t)$ is increasing function ($y_0 > 0$.). Then we can choose t_0 uniquely such that $y_0 = y(t_0)$.

and $-\frac{1}{y} > t + C$ is satisfied

whenever, $t > t_0$.

It's contradiction because we can choose

$t_1 \in \mathbb{R}$ such that $t_1 + C > 0$ and $t_1 > t_0$

Then. $-\frac{1}{y} \leq t_1 + C$.

$\lightning$.

Hence. $\deg P \leq 1$. ($\deg P = 0, 1$ is possible, skip the solution)

#6. 1. $f'' = -f$ on some interval I.

! $(f')^2 + f^2 = C$ on I

Check. $((f')^2 + f^2)' = 0$

$$((f')^2 + f^2)' = 2f' \cdot f'' + 2f \cdot f'$$

$$= 2f'(f'' + f) = 0.$$

2. $g'' = -e^g$, $g(0) = 0$ and $g'(0) = 10$.

Maximal interval of existence of g?

Let $h = g'$. $g''' = -e^g \cdot g' = -g'' g'$

then $h'' + h \cdot h' = 0$.

$$h' = \frac{dh}{dg} g' = \frac{dh}{dg} \cdot h = g'' = -e^{+g}.$$

$$\int h \, dh = \int -e^g \, dg.$$

$$\frac{1}{2} h^2 = -e^g + C$$

$$\frac{1}{2} \cdot 10^2 = -e^0 + C \Rightarrow C = 51.$$

$$h^2 = -2e^g + 102.$$

$$u = g' = \sqrt{-2e^g + 102}.$$

$$\int \frac{dg}{\sqrt{-2e^g + 102}} = \int dx$$

Let $u = \sqrt{-2e^g + 102}$ $\qquad u(0) = \sqrt{-2+102}$

$$u^2 = -2e^g + 102 \qquad = \sqrt{100}$$

$$= 10.$$

$$2u\,du = -2e^g\,dg$$

$$= (u^2 - 102)\,dg.$$

$$\int \frac{2\cancel{u}}{\cancel{u}(u^2 - 102)}\,du = \int dx$$

$$\int \frac{1}{\sqrt{102}}\left(\frac{1}{u - \sqrt{102}} - \frac{1}{u + \sqrt{102}}\right)du = \int dx$$

$$\frac{1}{\sqrt{102}} \ln\left|\frac{u \;\; \sqrt{102}}{u + \sqrt{102}}\right| = x + C_0$$

$$\frac{1}{\sqrt{102}} \ln\left|\frac{10 \;\; \sqrt{102}}{10 + \sqrt{102}}\right| = C_0.$$

Hence, $\dfrac{1}{\sqrt{102}} \ln\left|\dfrac{\sqrt{-2e^{9}+102}-\sqrt{102}}{\sqrt{-2e^{9}+102}+\sqrt{102}}\right|$

$$= x + \frac{1}{\sqrt{102}} \ln\left|\frac{10-\sqrt{102}}{10+\sqrt{102}}\right|$$

$$\cancel{u =} \quad \sqrt{102}\,x + \ln\left|\frac{10-\sqrt{102}}{10+\sqrt{102}}\right| = \ln\left|1-\frac{2\sqrt{102}}{u+\sqrt{102}}\right|$$

$$u^2 = \left(\frac{2\sqrt{102}}{-e^{\sqrt{102}x}\cdot\left(\frac{10-\sqrt{102}}{10+\sqrt{102}}\right)+1} - \sqrt{102}\right)^2$$

$$g(x) = \ln\left|\frac{1}{2}\left(u^2 - 102\right)\right|$$

is defined on the whole real #.

ii). Hölder's inequality

$$\sum_{i=1}^{n} a_i x_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} x_i^q\right)^{\frac{1}{q}}.$$

Lemma. (Young's inequality)

For $a, b > 0$ and $p, q \ge 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

Def. f is a (strictly) convex function if $f(tz + (1-t)y) < t f(z) + (1-t) f(y)$ for $0 < t < 1$.

example) $\boxed{\text{exp. is a convex function.}}$

$$ab = \exp(\ln ab) = \exp\left(\frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q\right)$$

$$< \frac{1}{p}\exp(\ln a^p) + \frac{1}{q}\exp(\ln b^q) = \frac{a^p}{p} + \frac{b^q}{q}$$

$0 < \frac{1}{p}, \frac{1}{q} < 1$ and exp is convex.

(check the case, p or $q = 0$, independently).

(skip the proof)

Let $\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} = \|a\|_p$ and $\left(\sum_{i=1}^{n} x_i^q\right)^{\frac{1}{q}} = \|x\|_q$

By the above lemma,

$$\frac{a_i x_i}{\|a\|_p \|x\|_q} \le \frac{1}{p} \frac{a_i^p}{\|a\|_p^p} + \frac{1}{q} \frac{x_i^q}{\|x\|_q^q}. \text{ Then.}$$

$$\sum_{i=1}^{n} \frac{a_i x_i}{\|a\|_p \|x\|_q} = \frac{\sum_{i=1}^{n} a_i x_i}{\|a\|_p \|x\|_q}$$

$$\le \sum_{i=1}^{n} \left(\frac{1}{p} \frac{a_i^p}{\|a\|_p^p} + \frac{1}{q} \frac{x_i^q}{\|x\|_q^q} \right) = \frac{1}{p} \frac{\sum_{i=1}^{n} a_i^p}{\|a\|_p^p} + \frac{1}{q} \frac{\sum_{i=1}^{n} x_i^q}{\|x\|_q^q}$$

$$= \frac{1}{p} + \frac{1}{q} = 1.$$

By definition of $\|a\|_p$ and $\|x\|_q$,

$\sum a_i^p = \|a\|_p^p$ and $\sum x_i^q = \|x\|_q^q$.

Hence, $\sum_{i=1}^{n} a_i x_i \le \|a\|_p \|x\|_q = \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} x_i^q\right)^{\frac{1}{q}}$

#8. Minkowski's inequality.

$$\left(\sum_{i=1}^{n} (x_i + y_i)^p \right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} y_i^p \right)^{\frac{1}{p}}.$$

$x_i, y_i \ge 0$, $\frac{1}{p} + \frac{1}{q} = 1$.

$$\sum_{i=1}^{n} (x_i + y_i)^p = \sum_{i=1}^{n} (x_i + y_i)(x_i + y_i)^{p-1}$$

$$= \sum_{i=1}^{n} x_i (x_i + y_i)^{p-1} + \sum_{i=1}^{n} y_i (x_i + y_i)^{p-1}$$

$$\le \left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \left((x_i + y_i)^{p-1} \right)^q \right)^{\frac{1}{q}} + \left(\sum_{i=1}^{n} y_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \left((x_i + y_i)^{p-1} \right)^q \right)^{\frac{1}{q}} \quad \left(\text{by Hölder's inequalty.} \right)$$

$$= \left[\left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} y_i^p \right)^{\frac{1}{p}} \right] \left[\sum_{i=1}^{n} (x_i + y_i)^{pq - q} \right]^{\frac{1}{q}}$$

$$\left(\begin{aligned} \frac{1}{p} + \frac{1}{q} = 1 &\iff p + q = pq \\ &\iff pq - q = p \end{aligned} \right)$$

$$= \left[\left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} y_i^p \right)^{\frac{1}{p}} \right] \cdot \left[\sum_{i=1}^{n} (x_i + y_i)^p \right]^{\frac{1}{q}}$$

Divide both sides by $\left[\sum_{i=1}^{n}(x_i+y_i)^p\right]^{\frac{1}{q}}$

$\uparrow$ positive #.

$$\left[\sum_{i=1}^{n}(x_i+y_i)^p\right]^{1-\frac{1}{q}}$$

$$= \left[\sum_{i=1}^{n}(x_i+y_i)^p\right]^{\frac{1}{p}} \le \left(\sum_{i=1}^{n}x_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}y_i^p\right)^{\frac{1}{p}}$$

//.