MAT544 Fall 2009

Homework 8

Problem 1 Let $l^2(\mathbb{Z})$ denote the complex Hilbert space of sequences $f(n)(\ n \in \mathbb{Z})$ with the norm

$$\left(\sum_{n=-\infty}^{\infty} |f(n)|^2 \right)^{\frac{1}{2}}$$

Let $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ denote the operator

$$(Tf)(n) = f(n+1)$$

Let I denote the identity map. Show that for all $\lambda \in \mathbb{C}$, $T - \lambda I$ is injective and has a dense range.

Problem 2 Let V denote finite dimensional real vector space equipped with the positive-definite inner-product; and let $||.||$ denote the norm which comes from this inner product. Let $T : V \to V$ denote a linear transformation which satisfies $\|T(v)\| \leq \|T^*(v)\|$ for all $v \in V$. The map T^* is the adjoint, defined by the formula $(T^*(v_1), v_2) = (v_1, T(v_2))$. Show that T is normal, i.e. that $TT^* = T^*T$. (Hint $\mathrm{tr}(TT^* - T^*T) = 0$).

Problem 3 Let X be a real vector space with positive-definite inner product $< .,. >$. Show that ball about the origin are strictly convex, that is , to show that $x \neq y$ belong to a ball $\overline{B}_r$ of radius r then any point on segment between x and y is in a ball of smaller radius.

Is this true for any vector space with a norm?

Problem 4 Let $L : X \to Y$ be a continuous map of one Banach space to another. Assume that there exists a positive constant k such that $\|L(v)\| \geq k\|v\|$. Prove that the range of L in Y is closed.

Recall that a topological space is called separable if it contains a countable dense set.

Problem 5 Prove that the Banach space $l^1(\mathbb{N})$, formed by sequences $x = \{x_n\}$ with the norm $\|x\| = \sup |x_n|$ is not separable.

A linear map $L : X \to Y$ between two normed spaces is called bounded if there is a nonnegative constant k such that $\|L(x)\| \leq k\|x\|$.

Problem 6 Let $L : X \to Y$ denote a map between two normed spaces. Prove or construct a conterexample

1. If L is bounded then Ker$L$ is closed.
2. If Ker$L$ is closed then L is bounded.

A sequence $\{e_i\}$, $e_i \in H, i \geq 1$ form an orthonormal basis of a separable Hilbert space if

$$(e_i, e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and the span of $\{e_i\}$ is dense in H.

Problem 7 Let A be a bounded operator on a separable Hilbert space H and A^* be its adjoint operator, defined by $(A^* x, y) = (x, Ay)$ for all $x, y \in H$.

1. For an orthogonal basis $\{e_i\}$ for H, set $a_{ij} = (Ae_i, e_j)$. Show that

   $$\sum_{ij\geq 0} |a_{ij}| = \sum_{i\geq 0} \|Ae_i\|^2 = \sum_{i\geq 0} \|A^* e_i\|^2$$

   understood as an equality in $[0, \infty]$

2. Show that if for some orthonormal basis $\{e_i\}$

   $$\sum_{ij\geq 0} |a_{ij}| < \infty$$

   then the series is convergent for all orthonormal bases and the sum is independent on the choice of the basis.

Problem 8 Show that there does not exist a measure μ on $[0, 1]$ such that $\mu([0, 1]) < \infty$ and $\mu(\{x\}) > 0$ for any $x \in [0, 1]$.