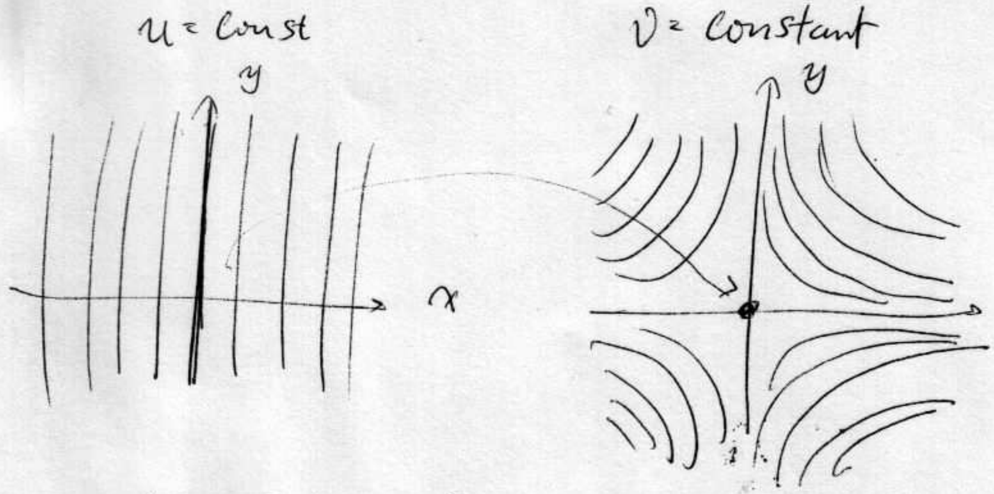
$f: (x,y) \mapsto (u,v) = (x, xy).$

HW 7



$x = 0$ $\quad (0,0) \longmapsto (0,0)$

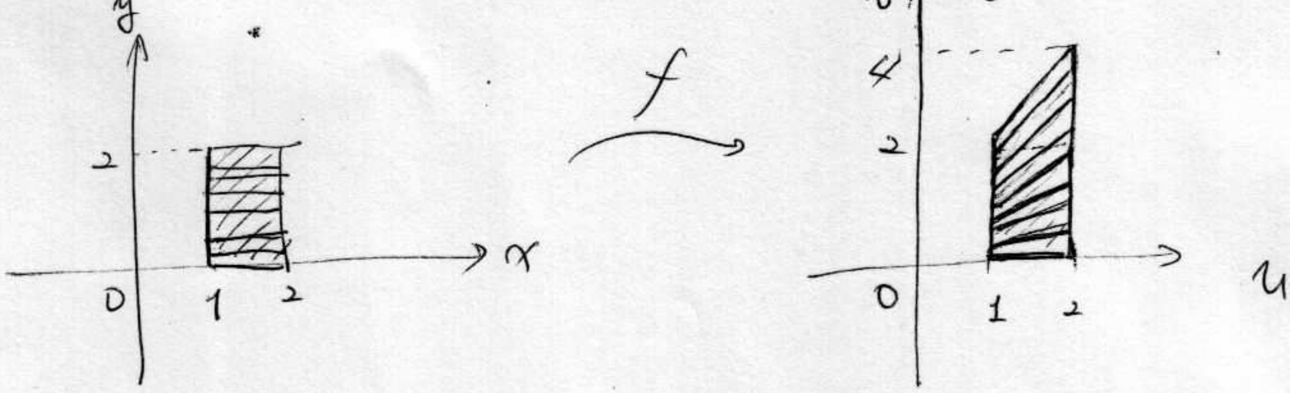
$\quad (0,1) \longmapsto (0,0)$ not 1-1

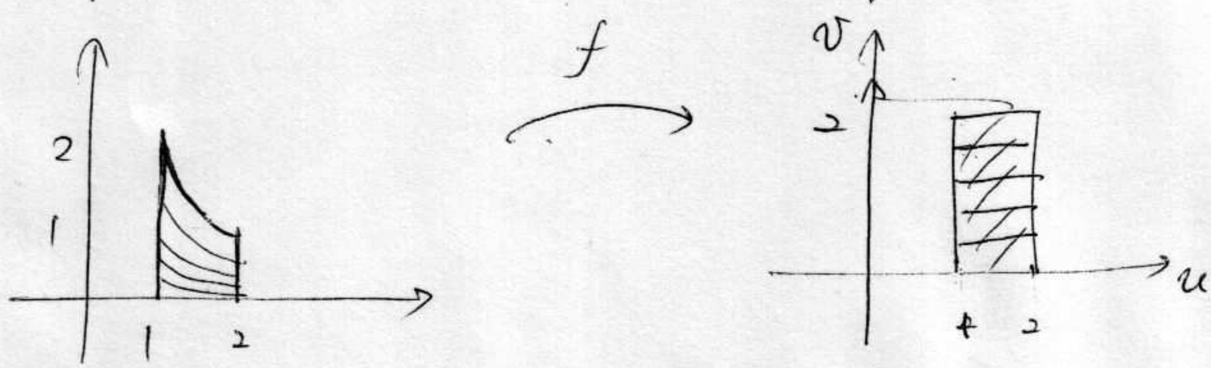
$x \neq 0$

$$Df = \begin{pmatrix} 1 & 0 \\ y & x \end{pmatrix} \qquad \det Df = x \neq 0.$$

Inverse function theorem.

$\exists$ local inverse near (x_0, y_0) when $x_0 \neq 0$





#2 $f(x) = \begin{cases} x + 2x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$

$$Df(0) = \lim_{h \to 0} \frac{1}{h}\left(f(h) - f(0)\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(h + 2h^2 \sin\left(\frac{1}{h}\right)\right)$$

$$= 1 + \lim_{h \to 0} 2h \sin\left(\frac{1}{h}\right)$$

$$= 1.$$

f cannot be injective near 0.

$k \mapsto Df(0)k = k$ one-to-one.

Take a nbhd of 0, $(-\varepsilon, \varepsilon)$.

$$f'(x) = 1 + 4x \sin\left(\frac{1}{x}\right) - 2\cos\left(\frac{1}{x}\right).$$

$f'(x)$ is continuous $\mathbb{R}\setminus\{0\}$, so is continuous at $(-\varepsilon, 0), (0, \varepsilon)$.
and for any (small) fixed $\varepsilon > 0$
We can choose large $n\pi \geq N$ such that $\frac{1}{N} < \varepsilon$.
natural number.

$f'\left(\frac{1}{2n\pi}\right) = -1$ and

$f'\left(\frac{1}{(2n+1)\pi}\right) = 1$, By the intermediate value thm.

$\exists\ x_0 \in \left(\frac{1}{(2n+1)\pi}, \frac{1}{2n\pi}\right)$ such that $f'(x_0) = 0 \quad \forall n \geq N$.

$\Rightarrow$ local max or min

#3. L_0 : one-to-one linear from $\mathbb{R}^p$ to $\mathbb{R}^p$,

$|L(z) - L_0(z)| < a|z|$ for $z \in \mathbb{R}^p$ for some $a > 0$.

/ L is one-one.

There exist $r > 0$ such that $|L_0(z)| > r|z|$

because L_0 is one-to-one and onto ~~to~~

(it's finite dimensional) (See open mapping thm).

Then $L_0(z) - a|z| < |L(z)| < L_0(z) + a|z|$

Choose sufficiently small a, $a < r$.

then. for some $a_0 > 0$, $a_0|z| < |L(z)|$

Then $L(z)$ is one-to-one.

#4 $x = (\xi_1, \xi_2)$ $\quad y = (\eta_1, \eta_2)$

$$F(x, y) = (\xi_1^3 + \xi_2\eta_1 + \eta_2,\ \xi_1\eta_2 + \xi_2^3 - \eta_1)$$

$$DF(x, y) = \begin{pmatrix} 3\xi_1^2 & \eta_1 & \xi_2 & 1 \\ \eta_2 & 3\xi_2^2 & -1 & \xi_1 \end{pmatrix}$$

$x = \phi(y)$

By the implicit function theorem, a sufficient condition of existence $x = \phi(y)$ is that

$$\begin{pmatrix} 3\xi_1^2 & \eta_1 \\ \eta_2 & 3\xi_2 \end{pmatrix} \text{ is invertible}$$

$$\Leftrightarrow 9\xi_1^2\xi_2^2 - \eta_1\eta_2 \neq 0$$

$$F(x, y) = F(\phi(y), y) = 0 \quad \text{Then}$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial x}\frac{\partial x}{\partial y} = 0$$

Hence, $\dfrac{\partial x}{\partial y} = D\phi = -\left(\dfrac{\partial F}{\partial x}\right)^{-1}\dfrac{\partial F}{\partial y}$

$$= -\begin{pmatrix} 3\xi_1^2 & \eta_1 \\ \eta_2 & 3\xi_2^2 \end{pmatrix}^{-1}\begin{pmatrix} \xi_2 & 1 \\ -1 & \xi_1 \end{pmatrix} = \frac{1}{9\xi_1^2\xi_2^2 - \eta_1\eta_2}\begin{pmatrix} 3\xi_2^3 + \eta_1 & 3\xi_2^2 - \xi_1\eta_1 \\ -3\xi_1^2 - \xi_2\eta_1 & 3\xi_1^3 - \eta_2 \end{pmatrix}$$

#5. $B = \{x \in \mathbb{R}^p : \|x\| < 1\}$

$f : B \to \mathbb{R}$, f differentible on $\mathrm{int}(B)$.

and $f(x) = 0$, $\forall x \in \partial B$, namely, $\|x\| = 1$.

! $\exists c \in \mathrm{int}\,B$ s.t $Df(c) = 0$.

B is compact, then f has a maximum <u>and</u> minimum in B. Both max and min are 0 then $f \equiv 0$ on B. (trivial case)

Thus we may assume that $f(c)$ = max (or min) in $\mathrm{int}(B)$ and $f(c) \neq 0$.

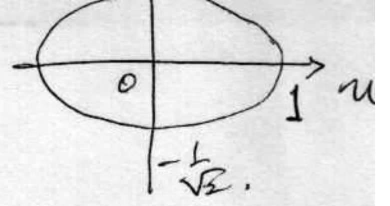
$f(c)$ is max on B then $f(c)$ is also max in $\pi_i(B)$, where π_i is the projection to i-th coordinate. Moreover, there exists partial derivatives on every coordinate.

Then $\frac{\partial f}{\partial x_i}(c) = 0 \quad \forall\ 1 \le i \le p$.

$$Df = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_p}\right)$$

Hence $Df(c) = 0$.

#6 (I skipped the proof)



#7. $f_1(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ $\quad f_2(x,y) = \dfrac{xy}{x^2+y^2}.$ $\quad u^2+2v^2=1$

$$\frac{\partial f_1}{\partial x} = \frac{2x(x^2+y^2)-2x(x^2-y^2)}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2}$$

$$\frac{\partial f_1}{\partial y} = \frac{-2y(x^2+y^2)-2y(x^2-y^2)}{(x^2+y^2)^2} = \frac{-4x^2y}{(x^2+y^2)^2}$$

$$\frac{\partial f_2}{\partial x} = \frac{y(x^2+y^2)-2x^2y}{(x^2+y^2)^2} = \frac{-y(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial f_2}{\partial y} = \frac{x(x^2+y^2)-2xy^2}{(x^2+y^2)^2} = \frac{x(x^2-y^2)}{(x^2+y^2)^2}.$$

$$Df = \frac{1}{(x^2+y^2)^2}\begin{pmatrix} 4xy^2 & -4x^2y \\ -y(x^2-y^2) & x(x^2-y^2) \end{pmatrix}$$

$$\det Df = \frac{1}{(x^2+y^2)^2}\left[4x^2y^2(x^2-y^2) - 4x^2y^2(x^2-y^2)\right]$$

$$= 0 \quad \forall (x,y) \neq (0,0).$$

#8 $f(x,y) = \dfrac{xy(x^2-y^2)}{x^2+y^2} = \dfrac{x^3y - xy^3}{x^2+y^2}$

$$= \frac{r^2\cos\theta\sin\theta\, r^2(\cos^2\theta - \sin^2\theta)}{r^2} = \frac{r^2}{2}\sin 2\theta\cos 2\theta$$

$\lim_{(r,\theta)\to 0} f = 0$

$\Rightarrow f$ is conti.

$$D_1 f = \frac{(3x^2y - y^3)(x^2+y^2) - 2x(x^3y - xy^3)}{(x^2+y^2)^2}$$

$$= \frac{3x^4y - y^3x^2 + 3x^2y^3 - y^5 - 2x^4y + 2x^2y^3}{(x^2+y^2)^2}$$

$$= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2}$$

$$= \frac{r\sin\theta(r^4\cos^4\theta + 4r^4\cos^2\theta\sin^2\theta - r^4\sin^4\theta)}{r^4}$$

$$= r\sin\theta(\cos^4\theta + 4\cos^2\theta\sin^2\theta - \sin^4\theta)$$

$\lim_{(r,\theta)\to 0} Df = 0$. Conti.

$$\lim_{h\to 0}\frac{1}{h}(f(h,0) - f(0,0)) = 0$$

$$D_2 f = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2+y^2)^2}$$

$$= \frac{r\cos\theta\,(r^4\cos^4\theta - 4r^4\cos^2\theta\sin^2\theta - r^4\sin^4\theta)}{r^4}$$

$$= r\cos\theta\,(\cos^4\theta - 4\cos^2\theta\sin^2\theta - \sin^4\theta)$$

$$\lim_{(r,\theta)\to 0} D_2 f = 0 \qquad \text{conti.}$$

$$\lim_{k\to 0}\frac{1}{k}\left(f(0,k) - f(0,0)\right) = 0.$$

$$D_{12} f = \frac{\cancel{4y^4}\,(x^4+12x^2y^2-5y^4)(x^2+y^2)^2}{(x^2+y^2)^4} - \frac{4(x^4y+4x^2y^3-y^5)\,y\,(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{x^4+12x^2y^2-5y^4}{(x^2+y^2)^2} - \frac{4y^2(x^4+4x^2y^2-y^4)}{(x^2+y^2)^3}$$

$$= \frac{r^4(\cos^4\theta + 12\sin^2\theta\cos^2\theta - 5\sin^4\theta)}{r^4}$$

$$- \frac{4r^2\sin^2\theta\,(r^4\cos^4\theta + 4r^4\cos^2\theta\sin^2\theta - r^4\sin^4\theta}{r^6}$$

$= \cos^4\theta + 12\cos^2\theta\sin^2\theta - 5\sin^4\theta$

$- 4\sin^2\theta\cos^4\theta - 16\cos^2\theta\sin^4\theta + 4\sin^6\theta$

$\lim_{(r,\theta)\to 0} D_{12}f$ doesn't exist.

$$D_{21}f = \frac{(5x^4 - 12x^2y^2 - y^4)(x^2+y^2)^2}{(x^2+y^2)^4} - \frac{4(x^5 - 4x^3y^2 - xy^4)\,2x(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{5x^4 - 12x^2y^2 - y^4}{(x^2+y^2)^2} - \frac{4x^2(x^4 - 4x^2y^2 - y^4)}{(x^2+y^2)^3}$$

$$= \frac{5r^4\cos^4\theta - 12r^4\cos^2\theta\sin^2\theta - r^4\sin^4\theta}{r^4} - \frac{4r^2\cos^2\theta\,(r^4\cos^4\theta - 4r^4\cos^2\theta\sin^2\theta - r^4\sin^4\theta)}{r^6}$$

$= 5\cos^4\theta - 12\cos^2\theta\sin^2\theta - \sin^4\theta$

$- 4\cos^6\theta + 16\cos^4\theta\sin^2\theta + 4\cos^2\theta\sin^4\theta$.

$\lim_{(r,\theta)\to 0} D_{21}f$ doesn't exist.

$$\lim_{(0,k)\to(0,0)} \frac{1}{k}\left(D_1 f(0,k) - D_1 f(0,0)\right) =$$

$$= \lim_{(0,k)\to(0,0)} \frac{1}{k} \cdot \frac{-k^5}{k^4} = -1 .$$

$$\lim_{(h,0)\to(0,0)} \frac{1}{h} \left(D_2 f(h,0) - D_2 f(0,0)\right)$$

$$= \lim_{(h,0)\to(0,0)} \frac{1}{h} \frac{h^5}{h^4} = 1$$