MAT544 Fall 2009

Homework 7

Problem 1 Let f be the mapping of $\mathbb{R}^2$ into $\mathbb{R}^2$ which sends the point (x, y) into the point (u, v) given by $u = x, v = xy$. Draw some curves $u = const, v = const$ in the (x, y)-plane and some curves $x = const, y = const$ in the (x, y)-plane. Is this mapping one-one? Does f map onto all of $\mathbb{R}^2$? Show that if $x \neq 0$, then f maps some neighborhood of (x, y) in a one-one fashion onto a neighborhood of (x, xy). Into what region in the (u, v)-plane does f map the rectangle $\{(x, y)|1 \leq x \leq 2, 0 \leq y \leq 2\}$? What points in the (x, y)-plane map under/into the rectangle $\{(u, v)|1 \leq u \leq 2, 0 \leq v \leq 2\}$?

Problem 2 Let f be defined on $\mathbb{R}$ to $\mathbb{R}$ by

$$f(x) = \begin{cases} x + 2x^2 \sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then $Df(0)$ is one-one but f has no inverse near $x = 0$.

Problem 3 Suppose that L_0 is a one-one linear function on $\mathbb{R}^p$ to $\mathbb{R}^p$. Show that there exists a positive number a such that if L is a linear function on $\mathbb{R}^p$ to $\mathbb{R}^p$ satisfying $|L(z) - L_0(z)| < a|z|$ for $z \in \mathbb{R}^p$, then L is one-one.

Problem 4 Let F be the function on $\mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{R}^2$ defined for $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ by the formula

$$F(x, y) = (\xi_1^3 + \xi_2\eta_1 + \eta_2, \xi_1\eta_2 + \xi_2^3 - \eta_1).$$

At what points (x, y) can one solve the equation $F(x, y) = 0$ for $x = \phi(y)$ in terms of y. Calculate the derivative of this solution function ϕ, when it exists. In particular, calculate the partial derivatives of the coordinate functions of ϕ with respect to (η_1, η_2).

Problem 5 Let f be defined and continuous on the set $B = \{x \in \mathbb{R}^p : ||x|| \leq 1\}$ with values in $\mathbb{R}$. Suppose that f is differentiable at every interior point of B and that $f(x) = 0$ for all $||x|| = 1$. Prove that there exists an interior point c of B and that $Df(c) = 0$ (This result may be regarded as a generalization of Rolle's Theorem.)

Problem 6 Take dimensions $p = q = 1$ in the inverse function theorem ("steepest" descent lemma), and interpret the theorem (as well as its proof) graphically.

Problem 7 For $(x, y) \neq (0, 0)$, define $f = (x, y)$ by

$$f_1(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, f_2(x, y) = \frac{xy}{x^2 + y^2}$$

Compute the rank of $Df_{(x,y)}$, and find the range of f.

Problem 8 Put $f(0, 0) = 0$, and

$$f_1(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

if $(x, y) \neq (0, 0)$.

Prove that

1. $f, D_1 f, D_2 f$ are continuous in $\mathbb{R}^2$;
2. $D_{12} f$ and $D_{21} f$ exist at every point of $\mathbb{R}^2$, and are continuous except at $(0, 0)$;
3. $D_{12} f(0, 0) = 1$, and $D_{21} f(0, 0) = -1$.

In the above formulas D_i, D_{ij} stand for partial and mixed derivatives.