MAT544 Fall 2009

Homework 3

Problem 1 Show that any $r \in [-1, 1]$ has an expansion $\sum_{m\geq 1} \frac{a_m}{5^m}$, $a_m \in \{-4, -2, 0, 2, 4\}$

Problem 2 Prove :

$$\left(1 - \frac{1}{k}\right)^k < \frac{1}{e} < \left(1 - \frac{1}{k}\right)^{k-1}, \quad k \geq 2$$

(see HW2) and then by induction

$$\left(\frac{n}{e}\right)^n < n! < e\left(\frac{n+1}{e}\right)^{n+1}, \quad n \geq 1$$

Problem 3 Compute the limits

$$\lim_{n\to\infty} \int_0^\infty x^{n^2} e^{-n^\alpha x} dx, \quad \alpha = 1, 2$$

You are not expected to justify the change of variables and integration by parts in improper integrals.

Problem 4 Define

$$var(f)(x) = \overline{\lim}_{y\to x} f(y) - \underline{\lim}_{y\to x} f(y)$$

for any function $f : \mathbb{R} \to \mathbb{R}$.

Prove that $A = \{x | var(f)(x) \geq \epsilon\}$ is closed.

Deduce from that that the set of discontinuity points of any function can not be $\mathbb{R}\backslash\mathbb{Q}$.

Problem 5 The radial limit of a function $f(x, y)$ at a point (x_0, y_0) is $\lim_{t\to 0} f(x_0 + at, y_0 + bt)$, where $a^2 + b^2 > 0$. Give an example of a function, whose limit at (x_0, y_0) does not exist, but all radial limits do and coincide.

Problem 6 A Liouville number $r \in \mathbb{R}$ by definition satisfies the condition that for any $n \geq 1$ there exists a rational number p/q which satisfy

$$0 < |r - p/q| < 1/q^n$$

. Prove that all Liouville numbers are irrational.

Give an example of a Liouville number.

Problem 7 Let x_n be a sequence that satisfies $\lim_{n\to\infty} x_n = a$. Define $\xi_n = \frac{1}{n}\sum_{i=1}^{n} x_n$. Prove that $\lim_{n\to\infty} \xi_n = a$.