**MAT544 Fall 2009**

## Homework 2

**Problem 1** Prove the inequalities

- $$(1+x_1)\cdots(1+x_n) \geq 1 + x_1 + \cdots + x_n$$

  the numbers $x_i > -1$ have equal sign. Also prove

  $$(1+x)^n \geq 1 + nx, \tag{1}$$

  $n \geq 2$

- $$n! < (\frac{n+1}{2})^n, n > 1$$

  (Hint use the identity $(\frac{n+2}{n+1})^{n+1} = (1 + \frac{1}{n+1})^{n+1}$)

- $$2!4!\cdots(2n)! > [(n+1)!]^n$$

- $$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$$

- $$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n} \quad (n \geq 2)$$

**Problem 2**

Compute the limits

- $$\lim_{n\to\infty} \frac{1^2}{n^3} + \frac{3^2}{n^3} + \cdots + \frac{(2n-1)^2}{n^3}$$

- $$\lim_{n\to\infty} \sqrt[2]{2}\sqrt[4]{2}\cdots\sqrt[2^n]{2}$$

**Problem 3**

Prove the identities

- $$\lim_{n\to\infty} \frac{a^n}{n!} = 0$$
- $$\lim_{n\to\infty} \sqrt[n]{a}$$
- $$\lim_{n\to\infty} \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} = 0$$

**Problem 4**

- Show that the sequence
  $$x_n = (1 + \frac{1}{n})^n, \quad n \geq 1$$
  is strictly increasing and is bounded above
- Show that the sequence
  $$y_n = (1 + \frac{1}{n})^{n+1}, \quad n \geq 1$$
  is strictly decreasing and is bounded below.
- Deduce from this that the sequences have a common limit
  $$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = e$$

(Hint define fractions $\frac{x_{n+1}}{x_n}, \frac{y_n}{y_{n-}}$ and use the inequality from 1)

**Problem 5**

Knowing that $\lim_{n\to\infty}(1 + \frac{1}{n})^n = e$ prove that

$\lim_{n\to\infty} 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} = e$

**Problem 6**

Prove that

- $$\underline{\lim}_{n\to\infty} x_n + \underline{\lim}_{n\to\infty} y_n \le \underline{\lim}_{n\to\infty}(x_n + y_n) \le \underline{\lim}_{n\to\infty} x_n + \overline{\lim}_{n\to\infty} y_n$$

- $$\underline{\lim}_{n\to\infty} x_n + \overline{\lim}_{n\to\infty} y_n \le \overline{\lim}_{n\to\infty}(x_n + y_n) \le \overline{\lim}_{n\to\infty} x_n + \overline{\lim}_{n\to\infty} y_n$$

Give examples with strict inequalities

**Problem 7** Prove that if $x_n > 0$ $n \ge 1$ and

$$\overline{\lim}_{n\to\infty} x_n \cdot \overline{\lim}_{n\to\infty} \frac{1}{x_n} = 1$$

Then the sequence is converging.

**Problem 8**

For sequences $x_n, n \ge 1$ find $\inf x_n$, $\sup x_n$ $\overline{\lim}_{n\to\infty} x_n$ $\underline{\lim}_{n\to\infty} x_n$.

- $$x_n = 1 - 1/n$$

- $$x_n = \frac{(-1)^n}{n} + \frac{1 + (-1)^n}{2}$$

- $$x_n = n^{(-1)^n}$$

- $$x_n = \frac{n}{n+1} \sin^2 \frac{\pi n}{4}$$

**Problem 9**

- Let $\lim_{n\to\infty} x_n = 0$ and $y_n, n \ge 1$ is an arbitrary sequence. Is it true that $\lim_{n\to\infty} x_n y_n = 0$?
- Let $\lim_{n\to\infty} x_n y_n = 0$ is it true that $\lim_{n\to\infty} x_n = 0$ or $\lim_{n\to\infty} y_n = 0$?