#1. 1. $\|x\|_p$, $p \geq 1$ and $\|x\|_\infty$ are equivalent on $\mathbb{R}^n$.

$$\sum_{i=1}^{n} |x_i|^p \leq n \cdot \|x\|_\infty^p$$

Then $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \leq n^{\frac{1}{p}} \cdot \|x\|_\infty$.

$\mathbb{R}^n$ is finite dimensional space.

$\|x\|_\infty^p = |x_i|^p$ for some i (depending on x)

Then $\|x\|_\infty^p \leq \sum_{i=1}^{n} |x_i|^p$ for every $x \in \mathbb{R}^n$.

Hence, $\|x\|_\infty \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$.

$\|x\|_p$ and $\|x\|_\infty$ equivalent in $\mathbb{R}^n$.

2. l_p, $p \geq 1$ is a Banach space.

– $\|x\|_p = 0$ then $x = 0$ by def. of $\|\cdot\|_p$

– $\|cx\|_p = \left(\sum |cx_i|^p\right)^{\frac{1}{p}} = \left(|c|^p \sum |x_i|^p\right)^{\frac{1}{p}}$

$$= |c| \left(\sum |x_i|^p\right)^{\frac{1}{p}} = |c| \, \|x\|_p$$

#1.

- $\|x+y\|_p \le \|x\|_p + \|y\|_p$ (Minkowski inequality)

Then $\|\cdot\|_p$ is a norm.

Consider a Cauchy seq. in l^p, namely,
for all $n, m \ge N$ $\|x_n - x_m\|_p < \varepsilon$ for any $\varepsilon > 0$
$\|x\|_p$ and $\|x\|_\infty$ are norm equivalent, then it's sufficient to show that l^∞ is complete.

Let $x_n = (x_{n1}, x_{n2}, \cdots, x_{np})$
$x_m = (x_{m1}, x_{m2}, \cdots, x_{mp})$

$|x_{mi} - x_{ni}| \le \|x_m - x_n\|_\infty$ for all i

and $\|x_m - x_n\|_\infty \le \sum_{i\ge 1} |x_{mi} - x_{ni}|$

$(\mathbb{R}, |\cdot|)$ is complete, then l^∞ is complete.
on $\mathbb{R}^n$.

Hence l^p on $\mathbb{R}^n$ is complete.

#2. $\overline{\lim_{i\to\infty}} E_i = \bigcap_{i\geq 1} \bigcup_{n\geq i} E_n$, $\underline{\lim_{i\to\infty}} E_i = \bigcup_{i\geq 1} \bigcap_{n\geq i} E_n$

1. $x \in \bigcap_{i\geq 1} \bigcup_{n\geq i} E_n$ then $x \in E_n$ infinitely many times by definition.

If $x \in E_n$ for finitely many E_n, then

$x \notin E_M$ for $\widehat{\text{all}}$ sufficiently large $M \geq N$

Then $x \notin \bigcap_{i\geq 1} \bigcup_{n\geq i} E_n . \supseteq \bigcap_{i\geq M} \bigcup_{n\geq i} E_n$

2. $x \in \bigcap_{n\geq i} E_n$ means that $x \in E_n$ for all $n \geq i$

Thus $x \in \bigcup_{i\geq 1} \bigcap_{n\geq i} E_n$ implies that

x is contained in E_n for all $n \geq i_0$

where i_0 is a number ≥ 1.

If not, that is, $x \notin E_n$ for infinitely many $n \geq i_0$ then $x \notin \bigcap_{n\geq i_0} E_n$

Then $x \notin \bigcup_{i\geq 1} \bigcap_{n\geq i} E_n$.

3. $\frac{1}{p}+\frac{1}{q}=1$, $\ell_p \times \ell_q \to \mathbb{R}$.

$(x_1,\dots,x_n,\dots)\times(y_1,\dots,y_n,\dots) \longrightarrow \sum_{n\geq 1} x_n y_n$

is well-defined and continuous.

Let $x=(x_1,\dots x_n,\dots)$, $x'=(x_1',x_2',\dots,x_n',\dots)$

$y=(y_1,\dots y_n,\dots)$, $y'=(y_1',y_2',\dots,y_n',\dots)$

$x=x'$ and $y=y'$ means

$x_i=x_i'$ and $y_i=y_i'$ $\forall i\geq 1$

Then $x_iy_i=x_i'y_i'$ $\forall i\geq 1 \Rightarrow \sum x_iy_i=\sum_{i\geq 1} x_i'y_i'$

$\sum_{n\geq 1} x_ny_n. \leq \|x\|_p\cdot\|y\|_q$, $\frac{1}{p}+\frac{1}{q}=1$

by Hölder's inequality. Then $\sum x_ny_n<\infty$

Then it's well-defined.

Define $\|(x,y)\| = \sup\{\|x\|_p, \|y\|_q\}$.

then. $\|(x,y)-(x',y')\| < \sqrt{\varepsilon}$

implies that $\sum (x_n-x_n')\cdot(y_n-y_n')$

$\leq \|x-x'\|_p\|y-y'\|_q < \varepsilon. \Rightarrow \therefore$ pairing is continuous

#2.3. $\underline{\lim}_{i\to\infty} E_i \subset \overline{\lim}_{i\to\infty} E_i$

$x \in \underline{\lim}_{i\to\infty} E_i$ means that for some i_0

$x \in \bigcap_{n \geq i_0} E_n$, then x is contained in

infinitely many E_ns.

$\Rightarrow \underline{\lim} E_i \subset \overline{\lim} E_i$

#3. $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ s.t. $\mu(A) < \infty$

$\hookrightarrow$ locally measurable. $E \in \tilde{\mathcal{M}}$.

$\mathcal{M} \subset \tilde{\mathcal{M}}$, If $\mathcal{M} = \tilde{\mathcal{M}}$, then μ is called saturated

a) μ is σ-finited $\Rightarrow$ μ is saturated

For $E \in \tilde{\mathcal{M}}$ and $A \in \mathcal{M}$ such that $\mu(A) < \infty$

$E = E \cap X = E \cap (\bigcup_i E_i)$ for $\mu(E_i) < \infty$

$= \bigcup_i (E \cap E_i)$ $\llcorner$ and $E_i \in \mathcal{M}$.

$E \cap E_i \in \mathcal{M}$ then the countable union is in $\mathcal{M}$

#3. $E = \bigcup_i (E \cap E_i) \in \mathcal{M}$.

Then $\tilde{\mathcal{M}} = \mathcal{M}$.

b) ! $\tilde{\mathcal{M}}$ is σ-algebra. ($\mathcal{M}$ is already σ-algebra)

· $\phi \in \tilde{\mathcal{M}} \supset \mathcal{M}$

· For $E \in \tilde{\mathcal{M}}$, $E \cap A \in \mathcal{M}$ for any $A \in \mathcal{M}$

s.t $\mu(A) < \infty$.

$\Rightarrow (E \cap A)^c \cap A \in \mathcal{M}$

then $E^c \cap A \in \mathcal{M}$ Hence $E^c \in \tilde{\mathcal{M}}$.

· For $E_i \in \tilde{\mathcal{M}}$, for any $A \in \mathcal{M}$ $\mu(A) < \infty$

$\bigcup_i (E_i \cap A) = \bigcup_i E_i \cap A \in \mathcal{M}$.

$E_i \cap A \in \mathcal{M} \Rightarrow \bigcup_i (E_i \cap A) \in \mathcal{M}$

then $\bigcup_i E_i \in \tilde{\mathcal{M}}$

Hence $\tilde{\mathcal{M}}$ is σ-algebra.

#3 c) Define $\tilde{\mu}$ on $\tilde{\mathcal{M}}$ $\tilde{\mu}(E) = \mu(E)$, $E \in \mathcal{M}$.
and $\tilde{\mu}(E) = \infty$ otherwise. Then $\tilde{\mu}$ is a measure

- $\tilde{\mu}(\phi) = \mu(\phi) = 0$, $\phi \in \mathcal{M}$.
- Consider any countable disjoint set $\bigcup_j S_j = S$
  - When $S_j \in \tilde{\mathcal{M}} \setminus \mathcal{M}$ for some j
    $\tilde{\mu}(S) \geq \tilde{\mu}(S_j) = \infty$.
  - $S_j \in \mathcal{M}$ for all j, then. $\tilde{\mu}(S) = \tilde{\mu}(\bigcup_j S_j)$
    $= \mu(\bigcup_j S_j) = \mu(S) = \sum \mu(S_j) = \sum \tilde{\mu}(S_j)$
    (Countable additivity)

$\Rightarrow$ $\tilde{\mu}$ is a measure.

d) For $E \in \tilde{\mathcal{M}}$ where $\tilde{\mu}(E) = 0$
Any subset $E' \subset E$, $\tilde{\mu}(E') = 0$. However,
by the definition of $\tilde{\mu}$, $\tilde{\mu}(E') = \mu(E') = 0$
and $E' \in \mathcal{M}$. If not, $\tilde{\mu}(E') \neq \mu(E')$ then
$\tilde{\mu}(E') = \infty$ ↯.

#3. Since μ is complete,

$\tilde{\mu}(E') = \mu(E') = 0$ implies that $E' \in \mathcal{M} \subseteq \tilde{\mathcal{M}}$

Hence $\tilde{\mu}$ is complete.

e) μ is semifinite. (For $E \in \tilde{\mathcal{M}}$, $\mu(E) = \infty$,

$\exists F \subset E$ s.t. $0 < \mu(F) < \infty$). Define

$\underline{\mu}(E) = \sup\{\mu(A) \mid A \in \mathcal{M}$ and $A \subset E\}$.

! $\underline{\mu}$ is saturated measure on $\tilde{\mathcal{M}}$ extending μ.

· $\underline{\mu}(\phi) = 0$ (clear!).

· Take a countable disjoint sets $S_j \in \tilde{\mathcal{M}}$. Let $S = \bigcup_j S_j$

$$\underline{\mu}\left(\bigcup_j S_j\right) = \sup\{\mu(A) \mid A \subset \bigcup_j S_j\}$$
$$\geq \mu(A) = \mu\left(\bigcup_j (A \cap S_j)\right)$$

when $\mu(A) < \infty$ $= \sum_j \mu(A \cap S_j)$

Then $\underline{\mu}\left(\bigcup_j S_j\right) \geq \sum_j \mu(A \cap S_j)$

implies $\underline{\mu}\left(\bigcup_j S_j\right) \geq \sup \sum_j \mu(A \cap S_j)$

$$\mu(A\cap S_j) + \frac{\varepsilon}{2^{j+1}} \geq \sup \mu(A\cap S_j)$$

Hence $\underline{\mu}(\bigcup_j S_j) \geq \cancel{\sup} \sum_j \sup \mu(A\cap S_j) + \varepsilon.$

$$\Rightarrow \underline{\mu}(\bigcup_j S_j) \geq \sum_j \underline{\mu}(S_j)$$

By similar argument.

$$\underline{\mu}(\bigcup_j S_j) = \sup\{\mu(A) \mid A \subset \mathcal{B}, \bigcup_j S_j\}$$

$$\leq \mu(A) + \varepsilon \quad \text{for some } A \subset S_j \ \forall \varepsilon > 0.$$

$$= \mu(\bigcup_j (A\cap S_j)) + \varepsilon.$$

when $\mu(A) < \infty$ $\quad = \sum_j \mu(A\cap S_j) + \varepsilon.$

$$\Rightarrow \underline{\mu}(\bigcup_j S_j) \leq \sum_j \sup(A\cap S_j) + \varepsilon.$$

$$\Rightarrow \underline{\mu}(\bigcup_j S_j) \leq \sum_j \underline{\mu}(A\cap S_j).$$

Hence, $\underline{\mu}$ satisfies countable additivity.

($\mu(A) = \infty$ case., skip the proof).

$\underline{\mu}$ is an extension of μ (clear!). #3 e)

Show that for every $E \in \tilde{\tilde{\mathcal{M}}}$, that is,

$E \cap B \in \tilde{\mathcal{M}}$ for every $B \in \tilde{\mathcal{M}}$ s.t. $\underline{\mu}(B) < \infty$

$\Rightarrow E \in \tilde{\mathcal{M}}$.

Take $B \in \mathcal{M} \subset \tilde{\mathcal{M}}$ with $\mu(B) = \underline{\mu}(B) < \infty$

then $\quad$ select the same $B \in \mathcal{M}$.

$(E \cap \ ) \cap B \in \mathcal{M}$

$\Rightarrow E \cap B \in \mathcal{M}$

$\Rightarrow E \in \tilde{\mathcal{M}}$. Hence $\tilde{\tilde{\mathcal{M}}} = \tilde{\mathcal{M}}$

$\underline{\mu}$ is saturated on $\tilde{\mathcal{M}}$.

f) $X_1 \cap X_2 = \emptyset$, $X = X_1 \cup X_2$

uncountable $\quad$ $\mathcal{M}$: σ-algebra countable and co-countable of X

μ_0: counting measure on $\mathcal{P}(X_1)$

μ on $\mathcal{M}$. $\mu \overset{\text{def}}{=} \mu_0(E \cap X_1)$

! μ is a measure on $\mathcal{M}$, $\tilde{\mathcal{M}} = \mathcal{P}(X)$.

$\tilde{\mu} \neq \underline{\mu}$. #3 f)

- $\mu(\phi) = \mu_0(\phi) = 0$
- Let S_j is a countable disjoint set in $\mathcal{M}$.

$\mu(\bigcup_j S_j) = \mu_0(\bigcup_j S_j \cap X_1)$

$= \mu_0(\bigcup_j (S_j \cap X_1))$

$= \sum_j \mu_0(S_j \cap X_1)$

$= \sum_j \mu(S_j)$. countable additivity.

$\Rightarrow \mu$ is a measure on $\mathcal{M}$.

μ_0 is the counting measure, then for $P \in \mathcal{M}$

$\mu_0(P) < \infty$ implies that P is a finite set.

Then for any set $Y \subseteq X$. L in $\mathcal{P}(X_1)$

$P \cap Y$ is ϕ or finite set in $\mathcal{P}(X_1)$

$\Rightarrow P \cap Y \in \mathcal{P}(X_1)$ Hence $\tilde{\mathcal{M}} = \mathcal{P}(X)$

$\tilde{\mu}(\{x_1\} \cup X_2) = \infty$ where $x_1 \in X_1$ and

$\underline{\mu}(\{x_1\} \cup X_2) = \mu(\{x_1\}) = \mu_0(\{x_1\}) = 1$.

Hence $\tilde{\mu} \neq \underline{\mu}$

#4. $\mathcal{A} \subset \mathcal{P}(X)$: algebra

μ_0 : premeasure on $\mathcal{A}$, μ^* : induced outer measure.

1. For $E \subset X$ and $\varepsilon > 0$ $\exists A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) < \mu^*(E) + \varepsilon$

$\mu^*(E) = \inf \{ \sum_{j \geq 1} \mu(A_j) \mid A_j \in \mathcal{A} \text{ for all } j \geq 1 \}$

Let $A = \bigcup_{j \geq 1} A_j$, then $\mu_0(A) \leq \sum_{j \geq 1} \mu_0(A_j)$

$\mu^*(E) > \sum_{j \geq 1} \mu_0(A_j) - \varepsilon \geq \mu^*(A) - \varepsilon$.

Then $\mu^*(A) < \mu^*(E) + \varepsilon$

$\left(\mu^*(A) \leq \sum_{j \geq 1} \mu_0(A_j) \text{ because } A \subset \bigcup_j A_j \right)$

#4. 2. $\mu^*(E) < \infty$

! μ^*-measurable $\Leftrightarrow \exists B \in A_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.

$\Rightarrow)$ $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$ for all $A \in \mathcal{P}(X)$

Take a countable union $\bigcup_{j\geq 1} A_j \supseteq E$

~~then~~ for each $A_j \supseteq E$. such that $\mu^*(A_j) \leq \mu^*(E) + \frac{\varepsilon}{2^j}$

Then $E \subseteq \bigcap_{j\geq 1} \bigcup_{n\geq j} A_j \overset{\text{def}}{=\!=} B$

existence was proved on #4. 1

$\mu^*(\bigcap_{j\geq 1} \bigcup_{n\geq j} A_j) \geq \mu^*(E)$ and.

$$\mu^*(\bigcap_{j\geq 1} \bigcup_{n\geq j} A_j) = \mu^*(\bigcap_{j\geq 0} \bigcup_{n\geq j} (A_{j+1} \setminus A_j))$$

$$\leq \mu^*(\bigcup_{n\geq j} A_{j+1} \setminus A_j)$$

$$\leq \sum_{j\geq 0} \mu^*(A_{j+1} \setminus A_j) \quad , (A_0 = \phi.)$$

$$\leq \mu^*(A_1) + \sum_{j\geq 1} \frac{\varepsilon}{2^j}$$

$$< \mu^*(E) + 2\varepsilon. \quad \text{for all } \varepsilon > 0.$$

Hence, $\mu^*(B) = \mu^*(E)$.

$$\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$$
$$= \mu^*(E) + \mu^*(B \setminus E)$$

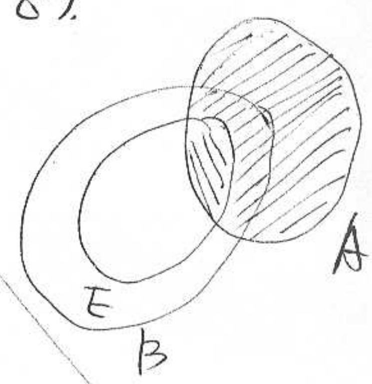
$$\Rightarrow \mu^*(B \setminus E) = 0.$$

$\Leftarrow$) In general, $\forall A \in \mathcal{P}(X)$, $\mu^*(B \setminus E) = 0$.

$$\mu^*(A) \le \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

countable subadditivity.

$$\mu^*(A \setminus E) \le \mu^*(A \setminus B) + \mu^*(A \cap (B \setminus E))$$
$$\le \mu^*(A \setminus B) + \mu^*(B \setminus E)$$
$$= \mu^*(A \setminus B)$$



$(A \setminus E = A \setminus B \cup (A \cap (B \setminus E))$

disjoint union.

$\boxed{B \text{ is measure of } \mu^*}$

$\mu^*(A)$

$\parallel$

$$\mu^*(A) \le \mu^*(A \cap E) + \mu^*(A \cap E^c) \le \mu^*(A \cap B) + \mu^*(A \cap B^c)$$

Hence $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E) \quad \forall E \in \mathcal{P}(X)$

E is μ^* measurable.

#5 μ_0 : premeasure which is finite.

μ^* : outer measure induced by μ_0 on X

μ_* : inner measure defined as

$$\mu_*(E) = \mu_0(X) - \mu^*(E^c)$$

! E is measurable $\Leftrightarrow$ $\mu^* = \mu_*$.

$\Rightarrow$) $\mu^*(X) = \mu^*(E) + \mu^*(X \setminus E)$ Then

$$\mu^*(E) = \mu^*(X) - \mu^*(X \setminus E) = \mu_*(E)$$

for any measurable set $E \subset X$.

$\cancel{\Leftrightarrow}$ ($\mu^*(X) = \mu_0(X) < \infty$ by the hypothesis).

$\Leftarrow$) $\mu^*(E) = \mu_*(E) < \infty$

Let $B = \limsup_{j \to \infty} A_j$ when $E \subseteq A_j$, $\forall j \geq 1$

(by #4.1) and $\mu^*(E) \geq \mu^*(A_j) - \frac{\varepsilon}{2^j}$

B is measurable.

#5. It is suffice to show that $\mu^*(B \setminus E) = 0$ (then by #4.2 E is measurable).

B is measurable then

$\mu_*(B) = \mu^*(B)$. However, $E \subset B$ then.

$\mu_*(B) \leq \mu_*(E) = \mu^*(E) \leq \mu^*(B)$.

So $\mu_*(B) = \mu_*(E)$. and $\boxed{\mu^*(E) = \mu^*(B).}$

$$\mu^*(E) = \mu^*(B \cap E) + \mu^*(B \setminus E)$$
$$= \mu^*(E) + \mu^*(B \setminus E).$$

Then $\mu^*(B \setminus E) = 0$.

#6. $\mathcal{A}$: finite unions of $(a, b] \cap \mathbb{Q}$

$-\infty \leq a < b \leq \infty$

a) $\mathcal{A}$ is an algebra on $\mathbb{Q}$

$E \in \mathcal{A}$. $\quad E = \bigcup_{i=1}^{n} (a_i\ b_i] \cap \mathbb{Q}$

We may assume that $(a_i\ b_i] \cap (a_j\ b_j] = \phi$, $i \neq j$

and $\quad a_1 < b_1 \leq a_2 < b_2 \leq a_3 < b_3 \leq \cdots \leq a_n < b_n$.

$-\infty \leq a_1 \qquad\qquad\qquad\qquad b_n \leq +\infty$

Then $\mathbb{Q} \setminus E = \left[(-\infty, a_1) \cup \left(\bigcup_{i=1}^{n-1} (b_i\ a_{i+1}]\right) \cup (b_n, +\infty)\right] \cap \mathbb{Q}$

where $-\infty < a_1$ and $b_n < +\infty$.

If $-\infty = a_1$ or $b_n = +\infty$ then $(-\infty, a_1), (b_n, +\infty) = \phi$.

Then $\mathbb{Q} \setminus E \in \mathcal{A}$.

For $E_1, \cdots, E_m \in \mathcal{A}$, let $E_i = \tilde{E}_i \cap \mathbb{Q}$.

$\bigcup_{i=1}^{m} E_i = \bigcup_{i=1}^{k} (a_i\ b_i] \cap \mathbb{Q}$ $\qquad$ [finite union of intervals of form $(a\ b]$

$= \bigcup_{i=1}^{m} (\tilde{E}_i \cap \mathbb{Q}) = \left(\bigcup_{i=1}^{m} \tilde{E}_i\right) \cap \mathbb{Q} \in \mathcal{A}$.

b) σ-algebra generated by $\mathcal{A}$ is $\mathcal{P}(\mathbb{Q})$.

$\mathbb{Q}$ is countable then any subset of $\mathbb{Q}$ is a countable union of rational #s.

#6 b) σ-algebra is closed under countable union.

(Then it suffice to show that every rational # in $\mathbb{Q}$ is contained in the countable union of the element of A.)

For every $n \in \mathbb{Z}$, $(n-1, n+1] \cap \mathbb{Q} \in A$.

Then σ-algebra generated by A contains the countable union $\bigcup_{n \in \mathbb{Z}} (n-1, n+1] \cap \mathbb{Q} = \mathbb{Q}$.

Moreover, σ-algebra is closed under countable intersections. Then for every $q \in \mathbb{Q}$.

$\bigcap_{n \in \mathbb{N}} (q, q+\frac{1}{n}] \cap \mathbb{Q} = \{q\}$ is contained that algebra. Then $P(\mathbb{Q})$ is the σ-algebra generated by A.

c) μ_0 on A, $\mu_0(\emptyset) = 0$, $\mu_0(A) = \infty$ for $A \neq \emptyset$. Then μ_0 is premeasure on A. and <u>more than one measure</u> on $P(\mathbb{Q})$, $\mu_i|_A = \mu_0$, $\mu_1 \neq \mu_2$. μ_1, μ_2. $i=1,2$

μ_0 is a premeasure on $\mathcal{A}$ (clear!)

Define μ_1 is the counting measure and

μ_2 is the trivial extension of μ_0 to $\mathcal{P}(\mathbb{Q})$

$(a\ b] \cap \mathbb{Q}$ is always infinite set

where $a<b$. Then $\mu_1(\phi)=0$ but $\mu_1(A)=\infty$

for any $\overset{\neq}{\phi}\, A \in \mathcal{A}$.

μ_2 is a measure on $\mathcal{P}(\mathcal{A})$ (skip the proof).

but $\mu_1(\{q\}) = 1$ and $\mu_2(\{q\}) = \infty$.

#7. m : Lebesgue measure , $E \subset [0,1]$

$\bar{E}$: closure of E .

$m(\bar{E}) = 0 \iff \forall \varepsilon > 0$ $\exists$ finite disjoint collection of intervals $I_j \subseteq \mathbb{R}$ with $E \subset \bigcup I_j$ and $\sum m(I_j) < \varepsilon$.

$\Rightarrow)$ $\bar{E}$ is compact. Then every open covering has a finite subcover.

$m(\bar{E}) = 0$ then for some $\bigcup_{j=1}^{\infty} I_j \supset \bar{E}$, $\sum_{j=1}^{\infty} m(I_j) < \varepsilon$ for all $\varepsilon > 0$. where each I_j is a connected interval.

We can assume that each I_j is open because.

$m(I_j) = l$ then I_j can be in the extended open interval of which measure $m(\tilde{I}_j) = l + \frac{\varepsilon}{2^{j+2}}$

$\bar{E} \subset \left(\bigcup_{j=1}^{\infty} \tilde{I}_j \right) \cap [0,1]$, By compactness of $\bar{E}$ we can choose finite (open) subcovering. Then.

$\bar{E} \subset \left(\bigcup_{j=1}^{N} \tilde{I}_j \right) \cap [0,1]$

$m\left(\bigcup_{j=1}^{N} \tilde{I}_j \cap [0,1] \right) \le m\left(\bigcup_{j=1}^{N} \tilde{I}_j \right) \le m\left(\bigcup_{j=1}^{\infty} \tilde{I}_j \right) < 2\varepsilon$

(~~For~~ Some I_j is of the form $[0,\delta)$ or $(\delta,1]$ we can extend these intervals a little bit with extending measure $\frac{\varepsilon}{2^{j+2}}$, $(-\frac{\varepsilon}{2^{j+2}},\delta)$ or $(\delta, 1+\frac{\varepsilon}{2^{j+2}})$)

$\bigcup_{j=1}^{N} \tilde{I}_j$ has at most N components and each component is open. in $\mathbb{R}$.

$\Leftarrow$) clear! (skip the proof).

#8 a.) $T: l_2 \to l_2$ bounded map with no kernel. $\|T(x)\| \leq K\|x\|$,

example) $(\dots, x_{-n}, \dots, x_{-1}, x_0, x_1, \dots, x_n, \dots) \in l_2$

Let $T(x_n) = x_{n+1}$ (bilateral shift).

Then T is an isometry $(k=1)$ with no kernel

b.) Bounded $T: l_2 \to l_2$ with dense image but not surjective.

$T - \lambda I$ has the dense image.

(See HW 8 #1)

Let $y = (\ldots, \frac{1}{n}, \frac{1}{n-1}, \ldots, \frac{1}{2}, 1, 0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots)$

$$\begin{array}{c} 0 \\ \| \\ y_0. \end{array}$$

$y \in \ell^2$

Solve $(T - \lambda I)(x) = y$.

$$\begin{cases} y_0 = 0 \\ y_i = \frac{1}{i}, \quad i \neq 0. \ \forall i \in \mathbb{Z} \end{cases}$$

$(T - \lambda I)(x_0) = y_0$

$x_1 - \lambda x_0 = 0$

$x_1 = \lambda x_0$

$(T - \lambda I)(x_1) = y_1$

$x_2 - \lambda x_1 = 1$

$x_2 = \lambda x_1 + 1 = \lambda^2 x_0 + 1$

$\vdots$

$x_n = \lambda x_{n-1} + \frac{1}{n}$

degree n ~~plog~~ polynomial of λ.

If $|\lambda| > 1$ then $x \notin \ell^2$.

Hence $y \notin \operatorname{Im}(T - \lambda I)$, $|\lambda| > 1$.