

ON PFAFFIAN

For any skew-symmetric $2n \times 2n$ matrix $A = \{a_{ij}\}$ define its *Pfaffian* $\text{Pf}(A)$ by

$$(1) \quad \text{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \varepsilon(\sigma) a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2n-1)\sigma(2n)}.$$

The Pfaffian has the following basic properties:

$$(2) \quad \text{Pf}(B^t AB) = \text{Pf}(A) \det B,$$

$$(3) \quad \text{Pf}(A)^2 = \det A.$$

They follow from the following result.

Lemma 1. *Let u_1, \dots, u_{2n} be a basis in \mathbb{R}^{2n} and*

$$\omega(A) = \frac{1}{2} \sum_{i,j=1}^{2n} a_{ij} u_i \wedge u_j.$$

Then

$$\wedge^n \omega(A) = n! \text{Pf}(A) u_1 \wedge \cdots \wedge u_{2n}.$$

Proof. Direct computation using the properties of wedge product and the definition of Pfaffian. \square

Indeed, to prove (2) observe that

$$\omega(B^t AB) = \frac{1}{2} \sum_{i,j=1}^{2n} \sum_{k,l=1}^{2n} b_{ki} a_{ij} b_{lj} u_i \wedge u_j = \frac{1}{2} \sum_{k,l=1}^{2n} a_{kl} v_k \wedge v_l,$$

where

$$v_i = \sum_{j=1}^{2n} b_{ij} u_j, \quad i = 1, \dots, 2n.$$

Thus

$$\begin{aligned} n! \text{Pf}(B^t AB) u_1 \wedge \cdots \wedge u_{2n} &= \wedge^n \omega(B^t AB) \\ &= n! \text{Pf}(A) v_1 \wedge \cdots \wedge v_{2n} \\ &= n! \text{Pf}(A) \det B u_1 \wedge \cdots \wedge u_{2n}. \end{aligned}$$

To prove (3), let O be an orthogonal matrix with determinant 1 such that $O^{-1}AO$ has a block diagonal form with 2×2 blocks $\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}$, $i = 1, \dots, n$. The Pfaffian of this block-diagonal matrix is $\lambda_1 \cdots \lambda_n$ and we get by (2)

$$\text{Pf}(A) = \lambda_1 \cdots \lambda_n,$$

so that $\text{Pf}(A)^2 = \lambda_1^2 \cdots \lambda_n^2 = \det A$.

Remark 1. Note that in the derivation of (2)–(3) we did not take any square roots.

Remark 2. Consider a polynomial ring $\mathbb{Z}[x_{ij}]$ in the variables x_{ij} , where $1 \leq i < j \leq 2n$, and denote by X a skew-symmetric $2n \times 2n$ matrix with elements x_{ij} for $i < j$. Let $\det X \in \mathbb{Z}[x_{ij}]$ be its determinant. Then there is a polynomial $\text{Pf}(X) \in \mathbb{Z}[x_{ij}]$ such that

$$\det X = \text{Pf}(X)^2.$$

It is given by the same formula (1) written ‘without repetitions’:

$$\text{Pf}(X) = \sum_{\sigma \in \Pi_{2n}} \varepsilon(\sigma) x_{\sigma(1)\sigma(2)} \cdots x_{\sigma(2n-1)\sigma(2n)},$$

where

$$\Pi_{2n} =$$

$$\{\sigma \in S_{2n} : \sigma(1) < \sigma(2), \dots, \sigma(2n-1) < \sigma(2n); \sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)\}.$$

The homomorphism $\mathbb{Z} \rightarrow \mathbb{R}$ shows that properties of the Pfaffian hold for skew-symmetric matrices over an arbitrary commutative ring R with 1.