EXTRA MATERIAL ON TENSOR, SYMMETRIC AND EXTERIOR ALGEBRAS

1. Introduction

These are supplementary notes to the material in the homework assignements and in §11.5 of Dummit and Foote textbook. Here R is a commutative ring with 1, M, N, etc. are R-modules, $\otimes = \otimes_R$ is the tensor product over R, and V is vector space over a field F.

2. Tensor algebra

2.1. **Tensor algebra of a module.** The *tensor algebra* $T(M)$ of an R-module M is an R-module

$$T(M) = \bigoplus_{k=0}^{\infty} T^k(M),$$

where $T^0(M) = R$, $T^1(M) = M$ and $T^k(M) = M^{\otimes^k}$. Let $\imath_0 : R \to T(M)$ and $\imath_k : M^{\otimes^k} \to T(M)$ be the natural inclusion maps. Then $T(M)$ has an R-algebra structure with the unit $\mathbf{1} = \imath_0(1)$ and with the multiplication defined by

$$(m_1 \otimes \cdots \otimes m_k) \cdot (m_{k+1} \otimes \cdots \otimes m_{k+l}) \stackrel{\text{def}}{=} m_1 \otimes \cdots \otimes m_{k+l} \in T^{k+l}(M),$$

and extended to all $T^k(M) \times T^l(M)$ using distributive laws. The tensor algebra $T(M)$ is a *graded algebra*, $T^k(M) \cdot T^l(M) \subseteq T^{k+l}(M)$. When M is a free R-module of rank n, the tensor algebra $T(M)$ corresponds to the algebra of polynomials with coefficients in R in n non-commuting variables. Namely, every choice of free generators $x_1, \dots, x_n$ of M gives an isomorphism $T(M) \cong R\langle x_1, \dots, x_n\rangle$ — a free R-algebra generated by $x_1, \dots, x_n$.

The tensor algebra $T(M)$ is a *bialgebra* (actually a *Hopf algebra*, see HW 3) with the *coproduct*

$$\Delta : T(M) \to T(M) \otimes T(M)$$

and the *counit*

$$\varepsilon : T(M) \to R,$$

the R-algebra homomorphisms, defined by

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}, \quad \Delta(m) = m \otimes \mathbf{1} + \mathbf{1} \otimes m, \quad m \in M,$$

and $\varepsilon(m) = 0$ for all $m \in M$, $\varepsilon(\mathbf{1}) = 1$. Here multiplication on $T(M) \otimes T(M)$ is defined by $(a \otimes b) \bullet (c \otimes d) = (a \otimes c) \otimes (c \otimes d)$, where $\otimes$ in parentheses is the multiplication on $T(M)$. On $T^k(M)$ the coproduct is given by

$$\Delta(m_1\otimes\cdots\otimes m_k)=\Delta(m_1)\bullet\cdots\bullet\Delta(m_k)$$
$$=\sum_{i=0}^{k}\sum_{\sigma\in\mathrm{Sh}(i,k-i)}\left(m_{\sigma(1)}\otimes\cdots\otimes m_{\sigma(i)}\right)\otimes\left(m_{\sigma(i+1)}\otimes\cdots\otimes m_{\sigma(k)}\right),$$

where $\mathrm{Sh}(i,k-i)$ consists of $(i,k-i)$ *shuffles* — permutations $\sigma\in S_k$ satisfying $\sigma(1)<\cdots<\sigma(i)$ and $\sigma(i+1)<\cdots<\sigma(k)$, and for $i=0$ and $i=k$ the corresponding terms are, respectively, $\mathbf{1}\otimes(m_1\otimes\cdots\otimes m_k)$ and $(m_1\otimes\cdots\otimes m_k)\otimes\mathbf{1}$.

Hilbert-Poincaré series of $T(M)$, in case when M is a finitely generated R-module, is the following formal power series

$$H(t)\stackrel{\text{def}}{=}\sum_{k=0}^{\infty}\mathrm{rank}_R T^k(M)t^k\in R[[t]].$$

If M is a free module of rank n,

$$H(t)=\sum_{k=0}^{\infty}n^kt^k=\frac{1}{1-nt}.$$

2.2. **Tensor algebra of a vector space.** Let V be a vector space over F and V^* be its dual space. There is a natural (canonical) isomorphism of graded vector spaces

$$T(V^*)\cong T(V)^*,$$

defined by

$$(v_1^*\otimes\cdots\otimes v_k^*)(u_1\otimes\cdots\otimes u_l)\stackrel{\text{def}}{=}\delta_{kl}v_1^*(u_1)\cdots v_k^*(u_k),$$

where $u_1\otimes\cdots\otimes u_l\in T^l(V)$ and $v_1^*\otimes\cdots\otimes v_k^*\in T^k(V^*)$. The elements of $T^k(V)$ are called *contravariant k-tensors*, and elements of $T^k(V^*)$ — *covariant k-tensors*.

In differential geometry and in physics one uses more general type of tensors. Namely, the vector space of tensors of bi-degree (r,s), $r,s\geq 0$ is defined as

$$T^{r,s}(V)\stackrel{\text{def}}{=}\underbrace{V\otimes\cdots\otimes V}_{r}\otimes\underbrace{V^*\otimes\cdots\otimes V^*}_{s},$$

so that the general tensor algebra

$$T^{\bullet,\bullet}(V)=\bigoplus_{k=0}^{\infty}\bigoplus_{r,s\geq 0}^{r+s=k}T^{r,s}(V)$$

is a graded algebra with a multiplication defined by

$$u\otimes v\stackrel{\text{def}}{=}u_1\otimes\cdots\otimes u_{r_1}\otimes v_1\otimes\cdots\otimes v_{r_2}\otimes u_1^*\otimes\cdots\otimes u_{s_1}^*\otimes v_1^*\otimes\cdots\otimes v_{s_2}^*,$$

where

$$u=u_1\otimes\cdots\otimes u_{r_1}\otimes u_1^*\otimes\cdots\otimes u_{s_1}^*\text{ and }v=v_1\otimes\cdots\otimes v_{r_2}\otimes v_1^*\otimes\cdots\otimes v_{s_2}^*.$$

As graded vectors spaces, $T^{\bullet,\bullet}(V^*) \cong T^{\bullet,\bullet}(V)^*$.

The *contaction operator* is a map $c_{ij} : T^{r,s}(V) \to T^{r-1,s-1}(V)$, where $1 \leq i \leq r, 1 \leq j \leq s$, defined as follows

$$c_{ij}(v) \stackrel{\text{def}}{=} v_j^*(v_i) v_1 \otimes \cdots \otimes \check{v}_i \otimes \cdots \otimes v_r \otimes v_1^* \otimes \cdots \otimes \check{v}_j^* \otimes \cdots \otimes v_s^*,$$

where $v = v_1 \otimes \cdots \otimes v_r \otimes v_1^* \otimes \cdots \otimes v_s^*$ and the check over an argument means that it should be omitted. In particular, using the isomorphism $\operatorname{End} V \cong V^* \otimes V$, we see that the map $c_{11} : T^{1,1}(V^*) \to F$ is the *trace map*: $c_{11}(A) = \operatorname{Tr} A$ for $A \in \operatorname{End} V$.

3. Symmetric algebra

3.1. **Symmetric algebra of a module.** Let $\mathcal{C}(M)$ be a two-sided ideal in $T(M)$, generated by the elements $m_1 \otimes m_2 - m_2 \otimes m_1$ for all $m_1, m_2 \in M$. It is a *graded ideal* of $T(M)$ so that the corresponding quotient algebra is graded. By definition, it is a *symmetric algebra* of a module M,

$$\operatorname{Sym}^\bullet(M) \stackrel{\text{def}}{=} T(M)/\mathcal{C}(M) = \bigoplus_{k=0}^{\infty} \operatorname{Sym}^k(M).$$

The symmetric algebra $\operatorname{Sym}^\bullet(M)$ is commutative and we denote its multiplication by $\odot$. The symmetric algebra is a bialgebra (actually a Hopf algebra), which follows from the fact that the ideal $I = \mathcal{C}(M)$ is also a two-sided *coideal* in the bialgebra $A = T(M)$, that is,

$$\Delta(I) \subseteq A \otimes I + I \otimes A.$$

This property of I easily follows from the fact for $c = m_1 \otimes m_2 - m_2 \otimes m_1$ we have

$$\Delta(c) = \mathbf{1} \otimes c + c \otimes \mathbf{1}.$$

Indeed, it follows from the formula for Δ that

$$\begin{aligned}
\Delta(m_1 \otimes m_2) &= \Delta(m_1) \cdot \Delta(m_2) \\
&= (\mathbf{1} \otimes m_1 + m_1 \otimes \mathbf{1}) \cdot (\mathbf{1} \otimes m_2 + m_2 \otimes \mathbf{1}) \\
&= \mathbf{1} \otimes m_1 \otimes m_2 + m_1 \otimes m_2 + m_2 \otimes m_1 + m_1 \otimes m_2 \otimes \mathbf{1},
\end{aligned}$$

which gives the above formula for $\Delta(c)$.

If $M = M' \oplus M''$, a direct sum of two free modules, there is a canonical isomorphism of commutative graded algebras

$$\operatorname{Sym}^\bullet(M) \cong \operatorname{Sym}^\bullet(M') \otimes \operatorname{Sym}^\bullet(M'') = \bigoplus_{k=0}^{\infty} \bigoplus_{r,s \geq 0}^{r+s=k} \operatorname{Sym}^r(M') \otimes \operatorname{Sym}^s(M'').$$

In particular, if M is a free module of rank n with free generators $x_1, \ldots, x_n$, then

$$\operatorname{Sym}^\bullet(M) \cong R[x_1, \ldots, x_n].$$

In this case it easy to see (e.g. by using 'stars and bars') that

$$\operatorname{rank}_R \operatorname{Sym}^k(M) = \binom{n+k-1}{k}$$

and by the binomial formula the Hilbert polynomial of $\operatorname{Sym}^\bullet(M)$ is

$$H(t) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} t^k = \frac{1}{(1-t)^n}.$$

3.2. **Symmetric algebra of a vector space.** Let V be a vector space over F and V^* be its dual space. There is a canonical isomorphism $\operatorname{Sym}^\bullet(V) \cong \operatorname{Pol}(V^*)$, the polynomial algebra ov V^*. Indeed, $v \in V$ can be considered as a linear function on V^* with values in F, and an element $v_1 \odot \cdots \odot v_k$ — as a homogeneous polynomial function on V^* of degree k. There is also a natural isomorphism of graded vector spaces

$$\operatorname{Sym}^\bullet(V^*) \cong \operatorname{Sym}^\bullet(V)^*,$$

given by the identification of $u_1^* \odot \cdots \odot u_k^* \in \operatorname{Sym}^k(V^*)$ with $\mu(u_1^* \odot \cdots \odot u_k^*) \in \operatorname{Sym}^k(V)^*$ defined by

$$\mu(u_1^* \odot \cdots \odot u_k^*)(v_1 \odot \cdots \odot v_l) = \delta_{kl} \sum_{\sigma \in S_n} u_1^*(v_{\sigma(1)}) \cdots u_k^*(v_{\sigma(k)}).$$

The above expression is called a *permanent* of the $k \times k$ matrix $u_i^*(v_j)$. Correspondingly, the inner product $(\ ,\ )$ in V determines an inner product in $\operatorname{Sym}^\bullet(V)$ by the formula

$$(u_1 \odot \cdots \odot u_k, v_1 \odot \cdots \odot v_l) = \delta_{kl} \sum_{\sigma \in S_n} \left(u_1, v_{\sigma(1)}\right) \cdots \left(u_k, v_{\sigma(k)}\right).$$

Denote by $\operatorname{Sym}^k(V, F)$ the vector space of symmetric k-multilinear maps from V^k to F and let

$$\operatorname{Sym}^\bullet(V, F) \stackrel{\text{def}}{=} \bigoplus_{k=0}^{\infty} \operatorname{Sym}^k(V, F).$$

The map μ defines the isomorphism $\operatorname{Sym}^\bullet(V^*) \cong \operatorname{Sym}^\bullet(V, F)$, and the multiplication $\odot$ induces a multiplication $\odot_s$ on $\operatorname{Sym}^\bullet(V, F)$ such that the following diagram is commutative

$$\begin{array}{ccc}
\operatorname{Sym}^k(V^*) \times \operatorname{Sym}^l(V^*) & \xrightarrow{\ \odot\ } & \operatorname{Sym}^{k+l}(V^*) \\
{\scriptstyle \mu\times\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\
\operatorname{Sym}^k(V, F) \times \operatorname{Sym}^l(V, F) & \xrightarrow{\ \odot_s\ } & \operatorname{Sym}^{k+l}(V, F)
\end{array}$$

Explicitly the map $\odot_s$ is given by the *shuffle product*:

$$\begin{aligned}
&(f \odot_s g)(v_1, \ldots, v_{k+l}) \\
&= \sum_{\sigma \in \operatorname{Sh}(k,l)} f\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) g\left(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}\right)
\end{aligned}$$

for $f \in \mathrm{Sym}^k(V, F), g \in \mathrm{Sym}^l(V, F)$.

3.3. **Weyl algebra.** For $u \in V$ define a 'multiplication by u operator' $\hat{u} : \mathrm{Sym}^\bullet(V) \to \mathrm{Sym}^\bullet(V)$ by

$$\hat{u}(u_1 \odot \cdots \odot u_k) \stackrel{\text{def}}{=} u \odot u_1 \odot \cdots \odot u_k,$$

so that $\hat{u} : \mathrm{Sym}^k(V) \to \mathrm{Sym}^{k+1}(V)$ and $\deg \hat{u} = +1$. For $v^* \in V^*$ define a 'directional derivative operator' $\partial_{v^*} : \mathrm{Sym}^\bullet(V) \to \mathrm{Sym}^\bullet(V)$ by

$$\partial_{v^*}(u_1 \odot \cdots \odot u_k) \stackrel{\text{def}}{=} \sum_{i=1}^k v^*(u_i) u_1 \odot \cdots \odot \check{u}_i \odot \cdots \odot u_k,$$

so that $\partial_{v^*} : \mathrm{Sym}^k(V) \to \mathrm{Sym}^{k-1}(V)$ and $\deg \partial_{v^*} = -1$.

For $A, B \in \mathrm{End}\,\mathrm{Sym}^\bullet(V)$ denote by $[A, B] = A \circ B - B \circ A \in \mathrm{End}\,\mathrm{Sym}^\bullet(V)$ the commutator of operators A and B.

Lemma 1 (Heisenberg commutation relations)**.** *The operators $\hat{u}$ and ∂_{v^*} satisfy the following commutation relations*

$$[\hat{u}_1, \hat{u}_2] = [\partial_{v_1^*}, \partial_{v_2^*}] = 0,$$
$$[\partial_{v^*}, \hat{u}] = v^*(u) I,$$

where I is the identity operator in $\mathrm{Sym}^\bullet(V)$.

Proof. Direct computation using definition of ∂_{u^*} and $\hat{v}$. □

Remark 1. Let $e_1, \dots, e_n$ be a basis of V and $e_1^*, \dots, e_n^*$ be the corresponding dual basis of V^*. Under the isomorphism $\mathrm{Sym}^\bullet(V) \cong F[x_1, \dots, x_n]$ (variables x_i correspond to e_i) the operators $\hat{e}_i$ become the multiplication by x_i operators and $\partial_{e_i^*}$ become the differentiation operators $\dfrac{\partial}{\partial x_i}$.

Remark 2. For an inner product $(\ ,\ )$ on V denote by $\varphi : V \xrightarrow{\sim} V^*$ the induced isomorphism between V and V^*. Then $\partial_{\varphi(v)} = \hat{v}^*$, the adjoint operator to $\hat{v}$ with respect to the inner product on $\mathrm{Sym}^\bullet(V)$ determined by $(\ ,\ )$.

On the vector space $W = V \oplus V^*$ define a non-degenerate alternating form $\omega : W \times W \to F$ by

$$\omega(w_1, w_2) \stackrel{\text{def}}{=} v_1^*(u_2) - v_2^*(u_1), \quad \text{where} \quad w_1 = u_1 + v_1^*,\ w_2 = u_2 + v_2^* \in W.$$

The *Weyl algebra* $\mathscr{W}$ is defined as a quotient algebra of $T(W)$ by the two-sided ideal J in $T(W)$, generated by $w_1 \otimes w_2 - w_2 \otimes w_1 - \omega(w_1, w_2)\mathbf{1}$ for all $w_1, w_2 \in W$,

$$\mathscr{W} \stackrel{\text{def}}{=} T(W)/J.$$

It follows from Lemma 1 that multiplication and differentiation operators give a *representation* of the Weyl algebra $\mathscr{W}$ in $\mathrm{Sym}^\bullet(V)$ — an algebra homomorphism $\rho : \mathscr{W} \to \mathrm{End}\,\mathrm{Sym}^\bullet(V)$, such that $\rho(w) = \hat{u} + \partial_{v^*}$ for $w = u + v^* \in W$. It is easy to see that ρ is injective and it follows from Remark 1 that $\rho(\mathscr{W})$ is isomorphic to the algebra of differential operators in variables $x_1, \dots, x_n$ with polynomial coefficients.

Remark 3. The ideal J is not a graded ideal of $T(W)$ so that the Weyl algebra $\mathscr{W}$ is not a graded algebra. However, it is a *filtered algebra* — there is a *filtration*

$$F_0\mathscr{W} \subset F_1\mathscr{W} \subset \cdots \subset F_k\mathscr{W} \subset F_{k+1}\mathscr{W} \subset \cdots$$

on $\mathscr{W}$ given by the subspaces $F_k\mathscr{W} = \pi(T^0(W) \oplus \cdots \oplus T^k(W))$, where $\pi : T(W) \to \mathscr{W}$ is a canonical projection, satisfying

$$F_k\mathscr{W} \cdot F_l\mathscr{W} \subseteq F_{k+l}\mathscr{W} \quad \text{and} \quad \mathscr{W} = \bigcup_{k=0}^{\infty} F_k\mathscr{W}.$$

Correspondingly, the *associated graded algebra* $\mathrm{gr}(A)$ of a filtered algebra A is defined by

$$\mathrm{gr}(A) = \bigoplus_{k=0}^{\infty} F_kA/F_{k-1}A, \quad F_{-1}A = 0,$$

and

$$\mathrm{gr}(\mathscr{W}) \cong \mathrm{Sym}^\bullet(V \oplus V^*).$$

The Weyl algebra is a *quantization* of the symmetric algebra.

Remark 4. In general, the Weyl algebra $\mathscr{W}(V)$ of the *symplectic* vector space (V, ω), where $\omega : V \times V \to F$ is non-degenerate alternating form on V[1] is defined by

$$\mathscr{W}(V) = T(V)/J,$$

where J is a two-sided ideal in $T(V)$, generated by $v_1 \otimes v_2 - v_2 \otimes v_1 - \omega(v_1, v_2)\mathbf{1}$ for all $v_1, v_2 \in V$. Let $L \subset V$ be a *Lagrangian subspace* of V, the subspace of the dimension $\frac{1}{2}\dim_F V$ such that $\omega|_L = 0$. Then the Weyl algebra $\mathscr{W}(V)$ admits a representation in $\mathrm{Sym}^\bullet(L)$. The Weyl algebra $\mathscr{W}(V)$ is a filtered algebra and

$$\mathrm{gr}(\mathscr{W}(V)) \cong \mathrm{Sym}^\bullet(V).$$

4. Exterior algebra

4.1. **Exterior algebra of a module.** Let $\mathcal{A}(M)$ be a two-sided ideal in $T(M)$ generated by $m_1 \otimes m_2 + m_2 \otimes m_1$ for all $m_1, m_2 \in M$. If $2 \neq 0$ in R, the identity

$$2(m_1 \otimes m_2 + m_2 \otimes m_1) = (m_1 + m_2) \otimes (m_1 + m_2) - m_1 \otimes m_1 - m_2 \otimes m_2$$

shows that $\mathcal{A}(M)$ is generated by $m \otimes m$ for all $m \in M$. It is a graded ideal in $T(M)$ and the *exterior algebra* of an R-module M is the corresponding quotient algebra

$$\Lambda^\bullet(M) \stackrel{\text{def}}{=} T(M)/\mathcal{A}(M) = \Lambda^0(M) \oplus \Lambda^1(M) \oplus \Lambda^2(M) \oplus \cdots,$$

[1]Note that V is necessarily even-dimensional.

where $\Lambda^0(M) = R$ and $\Lambda^1(M) = M$. The exterior is *graded commutative* algebra with a product $\wedge$, that is

$$\alpha \wedge \beta = (-1)^{\deg \alpha \cdot \deg \beta} \beta \wedge \alpha,$$

where $\deg \alpha$ and $\deg \beta$ are degrees of the homogeneous elements $\alpha, \beta \in \Lambda^\bullet(M)$, $\deg \Lambda^k(M) = k$.

If A and B are graded commutative R-algebras, their tensor product carries a graded commutative algebra structure defined on homogeneous elements by

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{\deg b \cdot \deg c}(ac \otimes bd).$$

We will denote this algebra by $A \hat{\otimes} B$.

If M is a free module of rank n, $\Lambda^k(M) = 0$ for $k > n$ and

$$\Lambda^\bullet(M) = \bigoplus_{k=0}^{n} \Lambda^k(M).$$

However this is not true if M is not a free module. Thus for $R = \mathbb{Z}[x, y]$ the module $M = (x, y)$ has rank 1 but is not free and $M \wedge M \neq 0$ (see example on p. 449 in D&F).

If $M = M' \oplus M''$, a direct sum of two free modules, there is a canonical isomorphism of graded commutative algebras

$$\Lambda^\bullet(M) \cong \Lambda^\bullet(M') \hat{\otimes} \Lambda^\bullet(M'') = \bigoplus_{k=0}^{\infty} \bigoplus_{r,s \geq 0}^{r+s=k} \Lambda^r(M') \otimes \Lambda^s(M'').$$

In particular, if M is a free module of rank n with free generators $\theta_1, \dots, \theta_n$, then

$$\Lambda^\bullet(M) \cong \mathrm{Gr}[x_1, \dots, x_n]$$

— the *Grassmann algebra* with the generators θ_i satisfying relations

$$\theta_i \theta_j + \theta_j \theta_i = 0, \quad i, j = 1, \dots, n.$$

4.2. **Exterior algebra of a vector space.** Let V be a vector space over a field F of dimension n. It is easy to see that $\dim_F \Lambda^k(V) = \binom{n}{k}$ and the Hilbert series of the exterior algebra is

$$H(t) = \sum_{k=0}^{n} \dim_F \Lambda^k(V) t^k = (1 + t)^n.$$

Denoting by $H_{\mathrm{Sym}}(t)$ and $H_\Lambda(t)$ respectively the Hilbert series for symmetric and exterior algebras of V, we get (see Sect. 3.1)

$$H_{\mathrm{Sym}}(t) H_\Lambda(-t) = 1.$$

This is an example of a *Koszul duality.* Namely, let $\mathcal{I} = (\mathcal{R})$ be a two-sided ideal in $T(V)$ generated by the subspace $\mathcal{R}$ of $T^2(V)$ and let

$$A \stackrel{\text{def}}{=} T(V)/\mathcal{I}$$

be the corresponding graded algebra, the so-called *Koszul quadratic algebra*. Let $\mathcal{R}'$ be the orthogonal subspace to $\mathcal{R}$ in $T^2(V^*)$,

$$\mathcal{R}' = \{q^* \in T^2(V^*) : q^*(r) = 0 \quad \text{for all} \quad r \in \mathcal{R}\}.$$

The *Koszul dual* of A is a quadratic algebra $A^!$ defined by

$$A^! \stackrel{\text{def}}{=} T(V^*)/\mathcal{I}',$$

where $\mathcal{I}' = (\mathcal{R}')$ is a two-sided ideal in $T(V^*)$ generated by $\mathcal{R}'$. The Koszul duality reads

$$H_A(t)H_{A^!}(-t) = 1.$$

In our case $\mathcal{R}$ is the subspace of $T^2(V)$ spanned by $u \otimes v - v \otimes u$ and $\mathcal{R}'$ is the subspace of $T^2(V^*)$ spanned by $v^* \otimes v^*$. Thus $A = \mathrm{Sym}^\bullet(V)$ and $A^! = \Lambda^\bullet(V^*)$. Indeed, every $l \in T^2(V^*) \simeq T^2(V)^*$ can be uniquely written as the sum of symmetric and antisymmetric functionals $l = l_+ + l_-$, where

$$l_+(u \otimes v) = l_+(v \otimes u) \quad \text{and} \quad l_-(u \otimes v) = -l_-(v \otimes u), \;\; u, v \in V.$$

Then $l|_{\mathcal{R}} = 0$ if and only if $l_- = 0$ and $\mathcal{R}' = \{l \in T^2(V^*) : l = l_+\}$. Since every symmetric bilinear form can be diagonalized there are $v_i^* \in V^*$ and $c_i \in F$ such that

$$l_+ = \sum_i c_i v_i^* \otimes v_i^*.$$

There is a natural isomorphism of graded vector spaces

$$\Lambda^\bullet(V^*) \cong \Lambda^\bullet(V)^*$$

given by the identification of $u_1^* \wedge \cdots \wedge u_k^* \in \Lambda^k(V^*)$ with $\mu(u_1^* \wedge \cdots \wedge u_k^*) \in \Lambda^k(V)^*$, defined by

$$\mu(u_1^* \wedge \cdots \wedge u_k^*)(v_1 \wedge \cdots \wedge v_l) = \delta_{kl} \sum_{\sigma \in S_n} (-1)^{\varepsilon(\sigma)} u_1^*(v_{\sigma(1)}) \cdots u_k^*(v_{\sigma(k)}),$$

a determinant of the $k \times k$ matrix $u_i^*(v_j)$. Correspondingly, the inner product $(\, , \,)$ in V determines an inner product in $\Lambda^\bullet(V)$ by the formula

$$(u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_l) = \delta_{kl} \det(u_i, v_j).$$

Denote by $\mathrm{Alt}^k(V, F)$ the vector space of symmetric k-multilinear maps from V^k to F and let

$$\mathrm{Alt}^\bullet(V, F) \stackrel{\text{def}}{=} \bigoplus_{k=0}^{\infty} \mathrm{Alt}^k(V, F).$$

The map μ defines the isomorphism $\Lambda^\bullet(V^*) \cong \mathrm{Alt}^\bullet(V, F)$, and the multiplication $\wedge$ induces a multiplication $\wedge_s$ on $\mathrm{Alt}^\bullet(V, F)$ such that the following diagram is commutative

$$\begin{array}{ccc}
\Lambda^k(V^*) \times \Lambda^l(V^*) & \xrightarrow{\quad \wedge \quad} & \Lambda^{k+l}(V^*) \\
{\scriptstyle \mu \times \mu} \big\downarrow & & \big\downarrow {\scriptstyle \mu} \\
\mathrm{Alt}(V, F) \times \mathrm{Alt}^l(V, F) & \xrightarrow{\wedge_s} & \mathrm{Alt}^{k+l}(V, F)
\end{array}$$

Explicitly the map $\wedge_s$ is given by the *shuffle product*:

$$(f \wedge_s g)(v_1, \dots, v_{k+l}) = \sum_{\sigma \in \mathrm{Sh}(k,l)} (-1)^{\varepsilon(\sigma)} f\left(v_{\sigma(1)}, \dots, v_{\sigma(k)}\right) g\left(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}\right)$$

for $f \in \mathrm{Alt}^k(V, F), g \in \mathrm{Alt}^l(V, F)$.

4.3. **Clifford algebra.** For $u \in V$ define a 'multiplication by u operator' $\hat{u} : \Lambda^\bullet(V) \to \Lambda^\bullet(V)$ by

$$\hat{u}(u_1 \wedge \cdots \wedge u_k) \stackrel{\mathrm{def}}{=} u \wedge u_1 \wedge \cdots \wedge u_k,$$

so that $\hat{u} : \Lambda^k(V) \to \Lambda^{k+1}(V)$ and $\deg \hat{u} = +1$. For $v^* \in V^*$ define a 'directional derivative operator' $\partial_{v^*} : \Lambda^\bullet(V) \to \Lambda^\bullet(V)$ by

$$\partial_{v^*}(u_1 \wedge \cdots \wedge u_k) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{k} (-1)^{i-1} v^*(u_i) u_1 \wedge \cdots \wedge \check{u}_i \wedge \cdots \wedge u_k,$$

so that $\partial_{v^*} : \Lambda^k(V) \to \Lambda^{k-1}(V)$ and $\deg \partial_{v^*} = -1$.

For $A, B \in \operatorname{End} \Lambda^\bullet(V)$ denote by $[A, B]_+ = A \circ B + B \circ A \in \operatorname{End} \Lambda^\bullet(V)$ the *anti-commutator* of operators A and B.

Lemma 2 (Fermi-Dirac anti-commutation relations)**.** *The operators $\hat{u}$ and ∂_{v^*} satisfy the following anti-commutation relations*

$$[\hat{u}_1, \hat{u}_2]_+ = [\partial_{v_1^*}, \partial_{v_2^*}]_+ = 0,$$
$$[\partial_{v^*}, \hat{u}]_+ = v^*(u) I,$$

where I is the identity operator in $\Lambda^\bullet(V)$.

Proof. Direct computation. Formulas $[\hat{u}_1, \hat{u}_2]_+ = 0$ and $[\partial_{v^*}, \hat{u}]_+ = v^*(u)I$ are proved exactly as analogous formulas in Lemma 1. To prove that $[\partial_{v_1^*}, \partial_{v_2^*}]_+ = 0$ observe that

$$\partial_{v_1^*}(\partial_{v_2^*}(u_1 \wedge \cdots \wedge u_k)) = \partial_{v_1^*}\left(\sum_{j=1}^{k} (-1)^{i-1} v_2^*(u_j) u_1 \wedge \cdots \wedge \check{u}_j \wedge \cdots \wedge u_k\right)$$
$$= \sum_{\substack{i,j=1 \\ i \neq j}}^{k} (-1)^{i-1+j-1+\theta(i-j)} v_1^*(u_i) v_2^*(u_j) u^{ij},$$

where $\theta(i-j) = 1$ for $i > j$, $\theta(i-j) = 0$ for $i < j$ and u^{ij} is $u_1 \wedge \cdots \wedge u_k$ with i-th and j-th factors omitted. Since $(-1)^{\theta(i-j)} = -(-1)^{\theta(j-i)}$, the formula folows. □

Remark 5. Let $e_1, \dots, e_n$ be a basis of V and $e_1^*, \dots, e_n^*$ be the corresponding dual basis of V^*. Under the isomorphism $\mathrm{Sym}^\bullet(V) \cong \mathrm{Gr}[\theta_1, \dots, \theta_n]$ (variables θ_i correspond to e_i) the operators $\hat{e}_i$ become the multiplication by θ_i

operators and $\partial_{e_i^*}$ become the 'differentiation operators' $\dfrac{\partial}{\partial\theta_i}$ in Grassmann variables.

Remark 6. For an inner product $(\ ,\ )$ on V denote by $\varphi : V \xrightarrow{\sim} V^*$ the induced isomorphism between V and V^*. Then $\partial_{\varphi(v)} = \hat{v}^*$, the adjoint operator to $\hat{v}$ with respect to the inner product on $\Lambda^\bullet(V)$ determined by $(\ ,\ )$.

On the vector space $W = V \oplus V^*$ define a symmetric non-degenerate bilinear form $c : W \times W \to F$ by

$$c(w_1, w_2) \stackrel{\text{def}}{=} v_1^*(u_2) + v_2^*(u_1), \quad \text{where} \quad w_1 = u_1 + v_1^*,\, w_2 = u_2 + v_2^* \in W.$$

The *Clifford algebra* $\mathscr{C}$ is defined as a quotient algebra of $T(W)$ by a two-sided ideal I in $T(W)$, generated by $w_1 \otimes w_2 + w_2 \otimes w_1 - c(w_1, w_2)\mathbf{1}$ for all $w_1, w_2 \in W$,

$$\mathscr{C} \stackrel{\text{def}}{=} T(W)/I.$$

It follows from Lemma 2 that multiplication and differentiation operators give a *representation* of the Clifford algebra $\mathscr{C}$ in $\Lambda^\bullet(V)$ — an algebra homomorphism $\rho : \mathscr{C} \to \operatorname{End}\Lambda^\bullet(V)$, such that $\rho(w) = \hat{u} + \partial_{v^*}$ for $w = u + v^* \in W$. It is easy to see that ρ is injective and it follows from Remark 5 that $\rho(\mathscr{C})$ is isomorphic to the algebra of differential operators in Grassmann variables $\theta_1, \dots, \theta_n$ with polynomial coefficients.

Remark 7. The ideal I is not a graded ideal of $T(W)$ so that the Clifford algebra $\mathscr{C}$ is not a graded algebra. However, it is a filtered algebra with the filtration

$$F_0\mathscr{C} \subset \cdots \subset F_k\mathscr{C} \subset F_{k+1}\mathscr{C} \subset \cdots \subset F_n\mathscr{C}$$

on $\mathscr{C}$ given by the subspaces $F_k\mathscr{C} = \pi(T^0(W) \oplus \cdots \oplus T^k(W))$, where $\pi : T(W) \to \mathscr{C}$ is a canonical projection, satisfying

$$F_k\mathscr{C} \cdot F_l\mathscr{C} \subseteq F_{k+l}\mathscr{C} \quad \text{and} \quad \mathscr{C} = \bigcup_{k=0}^{n} F_k\mathscr{C}.$$

Correspondingly, the associated graded algebra of $\mathscr{C}$ is $\Lambda^\bullet(V \oplus V^*)$,

$$\operatorname{gr}(\mathscr{C}) \cong \Lambda^\bullet(V \oplus V^*),$$

and the Clifford algebra is a *Fermi-Dirac quantization* of the exterior algebra.

Remark 8. In general, the Clifford algebra $\mathscr{C}(V)$ of the vector space V with a non-degenerate symmetric form $c : V \times V \to F$ is defined by

$$\mathscr{C}(V) = T(V)/I,$$

where I is a two-sided ideal in $T(V)$, generated by $v_1 \otimes v_2 + v_2 \otimes v_1 - c(v_1, v_2)\mathbf{1}$ for all $v_1, v_2 \in V$. The Clifford algebra $\mathscr{C}(V)$ is a filtered algebra and

$$\operatorname{gr}(\mathscr{C}(V)) \cong \Lambda^\bullet(V).$$

4.4. **Determinants.** For $A \in \operatorname{End} V$ define $\Lambda^n A \in \Lambda^n V$ by

$$\Lambda^n A(v_1 \wedge \cdots \wedge v_n) = Av_1 \wedge \cdots \wedge Av_n.$$

Since $\Lambda^n V$ is one-dimensional there is a canonical identification

$$\imath : \operatorname{End} \Lambda^n V \xrightarrow{\sim} F$$

(a matrix of an operator on a one-dimensional vector space does not depend on the choice of a basis). We define $\det A = \imath(\Lambda^n A)$, so that for every basis $e_1, \dots, e_n$ of V,

$$Ae_1 \wedge \cdots \wedge Ae_n = \det A(e_1 \wedge \cdots \wedge e_n),$$

and one gets the standard formula for the determinant of a matrix. From here it is immediate that

$$\det(AB) = \det A \det B$$

and all other properties of determinants like row expansion, Laplace theorem, etc., easily follow. In particular,

$$\begin{aligned} \Lambda^k A(e_{i_1} \wedge \cdots \wedge e_{i_k}) &= Ae_{i_1} \wedge \cdots \wedge Ae_{i_k} \\ &= \sum_{1 \leq j_1 < \cdots < j_k \leq n} \det A^{j_1 \dots j_k}_{i_1 \dots i_k} (e_{j_1} \wedge \cdots \wedge e_{j_k}), \end{aligned}$$

where the $k \times k$ matrix $A^{j_1 \cdots j_k}_{i_1 \dots i_k}$ is obtained by choosing the columns numbered by $i_1, \dots i_k$ and the rows $j_1, \dots, j_k$ from the matrix A.

4.5. **Hodge star product.** Let V be a vector space over $\mathbb{R}$ with Euclidean inner product $(\ ,\ )$. The *orientation* is determined by a choice of an orthonormal basis $e_1, \dots, e_n$. Another orthonormal basis $e'_1, \dots, e'_n$ is said to be positively oriented if it is related to $e_1, \dots, e_n$ by an orthogonal matrix with determinant 1. The basis $e_1, \dots, e_n$ determines an isomorphism $*_n : \Lambda^n V \xrightarrow{\sim} \mathbb{R}$ by

$$*_n(c\, e_1 \wedge \cdots \wedge e_n) = c,$$

which does not depend on the choice of positively oriented orthonormal basis. The Hodge star operator $*_k : \Lambda^k V \to \Lambda^{n-k} V$ is defined by the requirement that

$$(\alpha, \beta) = *_n(\alpha \wedge *_k \beta)$$

for all $\alpha, \beta \in \Lambda^k V$. Indeed, $*_n$ defines the isomorphism

$$\psi : \Lambda^{n-k} V \xrightarrow{\sim} (\Lambda^k V)^*$$

by $\psi(\gamma)(\alpha) = *_n(\alpha \wedge \gamma)$, $\alpha \in \Lambda^k V, \gamma \in \Lambda^{n-k} V$. Therefore

$$*_k \beta = (\psi^{-1} \circ \varphi)(\beta),$$

where the isomorphism $\varphi : \Lambda^k V \xrightarrow{\sim} (\Lambda^k V)^*$ is given by the Euclidean inner product (see Remark 2). The Hodge star operator satisfies

$$*_{n-k} \circ *_k = (-1)^{k(n-k)} I$$

on $\Lambda^k V$, and the same formula holds on $\Lambda^{n-k} V$.

In a similar fashion the Hodge star operator can be defined for vector spaces over $\mathbb{C}$ with Hermitian inner product.