

MAT 535: HOMEWORK 1

DUE THURSDAY ~~Feb 4~~ Feb 7

1. (a) Find an orthonormal eigenbasis for the operator $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ in the standard basis of \mathbb{R}^2 .
 (b) Prove that the cyclic shift operator T in \mathbb{R}^n , defined in the standard basis by $Te_k = e_{k-1}$, $k = 1, \dots, n$ and $e_0 = e_n$, is normal and find its orthonormal eigenbasis.
2. Let V be a finite-dimensional \mathbb{R} -vector space with Euclidean inner product (\cdot, \cdot) . Prove that then, for any symmetric operator A , the bilinear form $B_A(u, v) = (Au, v)$ is symmetric and that conversely, every symmetric bilinear form can be written in this form for some symmetric operator A .
3. Let A be the following tri-diagonal $n \times n$ matrix:

$$A = \begin{pmatrix} b & c & & & \\ a & b & c & & \\ & a & \ddots & \ddots & \\ & & \ddots & \ddots & c \\ & & & a & b \end{pmatrix}.$$

Prove that A determines a normal operator in \mathbb{R}^n if and only if $a^2 = c^2$. When $a = c$ diagonalize A for $n = 2, 3$.

4. Let V be a finite-dimensional \mathbb{C} -vector space with Hermitian inner product $\langle \cdot, \cdot \rangle$. Prove that $(\cdot, \cdot) = \operatorname{Re} \langle \cdot, \cdot \rangle$ is a Euclidean inner product on $V_{\mathbb{R}}$ — the space V considered as an \mathbb{R} -vector space, and that the bilinear form $\omega(u, v) = \operatorname{Im} \langle u, v \rangle$ on $V_{\mathbb{R}}$ is alternating.
5. Let F be a field and let \mathcal{A} be an associative F -algebra with 1. Let $S \subseteq \mathcal{A}$ be such that $st = ts$ for every $s, t \in S$. Let \mathcal{B} be the smallest F -subalgebra of \mathcal{A} which contains the subset S and 1. Prove that \mathcal{B} is commutative. (*Hint*: Prove first that if $st = ts$ then $s^n t^m = t^m s^n$ for all $m, n \in \mathbb{N}$).
6. Let V be a finite-dimensional \mathbb{C} -vector space with Hermitian inner product and let A, B be commuting self-adjoint operators on V . Prove that A and B have a common orthonormal eigenbasis.
7. Let V be a finite-dimensional \mathbb{C} -vector space with Hermitian inner product, and let A be an invertible, normal operator on V . Prove that there exists a unique factorization $A = UP = PU$, where U is unitary and P is positive, that is, $(Pv, v) > 0$ for all non-zero $v \in V$. (*Hint*: Relate P and A^*A and prove that $U = AP^{-1}$ is unitary. Note that commutativity of U and P is equivalent to A being a normal operator).