Selected proofs

230,231,236,242

230. Prove that the shortest of all chords, passing through a point A taken in the interior of a given circle, is the one which is perpendicular to the diameter drawn through A.

It is evidently equivalent to prove that among isosceles triangles with given congruent sides, in which a segment of length d can be drawn from the vertex to the base, the triangle with altitude d has the smallest base.
The altitude of such a triangle cannot be greater than d, by the theorem on slants.
Given such a triangle with altitude a less than d: break it into 2 congruent right triangles sharing leg a and having congruent hypotenuses (the congruent sides of the original isosceles triangle). Do the same for the triangle with altitude exactly d. In 2 right triangles with congruent hypotenuses, the one with smaller first leg has larger second leg. Therefore the base of the test triangle is larger than that of the triangle with altitude d.

231. Prove that the closest and the farthest points of a given circle from a given point lie on the secant passing through this point and the center.

Let d be the distance from the given point to the center of the given circle. Let r be the radius of the circle. Any new point on the circle forms the third vertex of a triangle with sides d, r, and a new distance x. According to the triangle inequality, $x \leq d + r$, with $x = d + r$ if and only if the three points are collinear (in this case, the third vertex lies on the secant line on the opposite side of the center of the circle). Thus the farthest point lies on this secant.
For the closest point, there are two cases. If the given point is inside the circle, so $d < r$, the triangle inequality $r \leq x + d$ implies $x \geq r - d$, with $x = r - d$ if and only if the three points are collinear, thus the closest point lies on this secant as well. If the given point is outside, $r < d$, and the triangle inequality $d \leq x + r$ implies $x \geq d - r$, with $x = d - r$ if and only if the three points are collinear, so that the closest point in this case also lies on the secant.

236. Given a chord in a disk, draw another chord which is bisected by the first one and makes a given angle with it. (Find out for which angles this is possible).

A diagram with many parallel chords suggests that this is possible for the following chords: Those whose perpendicular bisector meets both arcs cut by the given chord (as opposed to just one). Given such a perpendicular bisector b, construct the chords perpendicular to it through the two endpoints of the given chord. These endpoints and the 2 new vertices so

obtained form a trapezoid for which b is a line of symmetry. We want an interior segment, parallel to the base and the top, which is bisected by the diagonal.
Construct such a parallel segment through the point where the line b meets the diagonal, and extend it to a chord of the circle. It is bisected by the diagonal (the original chord).

242. Two lines passing through a point M are tangent to a circle at the points A and B. Through a point C taken on the smaller of the arcs AB, a third tangent is drawn up to its intersection points D and E with MA and MB respectively. Prove that (1) the perimeter of triangle DME and (2) the angle DOE (where O is the center of the circle) do not depend on the position of the point C.

The perpendicular bisector of CB passes through O, and hence is a line of symmetry for the circle. The reflection in this line exchanges C and B as well as the tangent lines at C and B, and therefore this reflection fixes the intersection E of these two lines. This establishes a congruence between CE and BE. The same reasoning shows $CD \cong AD$. Therefore the triangles CEB and CDA are isosceles, and the perimeter of the triangle MDE is equal to $AM + BM$.
As a result of the isosceles property, their are triangle congruences $DOA \cong DOC$ and $EOB \cong EOC$. Thus angle $\angle AOB$ is comprised of 2 pairs of congruent angles, with the sum of one from each equal to $\angle DOE$. Therefore $\angle DOE \cong \frac{1}{2}\angle AOB$.

Good luck and let me know if you spot any errors. jmath@math.sunysb.edu