

Homework #3 Solutions (MAT 360)

Exercises 86, 96, 101, 102 in the textbook (Kiselev), and the question about convex broken lines.

Exercise 86. Can an exterior angle of an isosceles triangle be smaller than the supplementary interior angle? Consider the cases (a) where the angle is at the base, and (b) where the angle is at the vertex.

In the case (a), the answer is no. For, let a and b be the base angles of an isosceles triangle, so a is congruent to b . If the exterior angle of a were smaller than the base angle, then this exterior angle is obviously also smaller than b . But this is a contradiction to theorem 42 (page 34), which states that an exterior angle in *any* triangle is actually larger than each of the 2 interior angles to which it is not supplementary.

In the case (b), the answer is yes. For an example, consider any obtuse angle (an angle greater than half of a straight angle). Mark two congruent segments on its sides with a compass at any given setting, then finish the triangle with a segment connecting these two marks. This triangle is isosceles and the exterior angle at the vertex is less than half a straight angle, and hence less than the obtuse angle to which it is supplementary.

Exercise 96. Each leg of a right triangle is smaller than the hypotenuse.

According to corollary 43 (page 35), in a right triangle, the two angles besides the right angle are actually acute (less than half a straight angle). Therefore the angles opposite the legs are smaller than the (right) angle opposite the hypotenuse. According to theorem 45 (page 36), this implies that the legs are smaller than the hypotenuse.

Exercise 101. A triangle is isosceles if two of its altitudes are congruent.

Let AX and BY be altitudes of a triangle ABC , so that foot X lies on line BC and foot Y lies on AC . Assume that AX and BY are congruent. According to the theorem 55 (page 43), the special congruence test for right triangles, triangles AXB and BYA are congruent (they are right at X and at Y , the legs AX and BY are congruent, and the hypotenuses AB and BA are congruent). Then the angles ABY and BAX are congruent. That is, angles ABC and BAC are congruent. Then triangle ABC and BAC are congruent by the Angle Side Angle congruence theorem, so AC and BC are congruent; the original triangle is isosceles.

Exercise 102. A median in a triangle is equidistant from the two vertices not lying on it.

According to theorem 51 (page 41), the smallest distance from a point to a line is achieved by the perpendicular segment dropped from the point to the line.

Let AX be a median of triangle ABC , so BX is congruent to CX . Let B' and C' be the feet of the perpendiculars to the median dropped from the vertices B and C . If the median AX is also an altitude, B' and C' will coincide with X . Then we would have $BB' \cong BX \cong CX \cong CC'$, so that the distance from B to the median equals the distance from C .

So assume that median AX is not also an altitude. In this case, the feet B' and C' lie on opposite sides of X , namely in the order such that $\angle B'XB$ and $\angle C'XC$ are acute. Therefore these angles are actually vertical, and hence congruent. According to theorem 55 (page 43), the special congruence test for right triangles, triangles $B'XB$ and $C'XC$ are congruent. Then BB' is congruent CC' , and the distance from B to the median is the same as the distance from C to the median.

Additional problem. Suppose that $ABCD...F$ and $A'B'C'D'...F'$ are convex broken lines sharing the same endpoints $A = A'$ and $F = F'$ and lying on the same side of line $AF = A'F'$. If broken line $A'B'...$ lies in the interior of the region formed by broken line $AB...$, then the sum of the line segments comprising $AB...$ is larger than that of $A'B'...$. Also, what happens if the broken lines are not convex?

First consider the case where the broken lines both have only 2 segments, so the broken lines are ABC and $AB'C$. By the assumption, B' lies inside the triangle ABC . Assume that the ray AB' intersects segment BC at X (the conclusion of the so-called “crossbar theorem”). Then according to the triangle inequality,

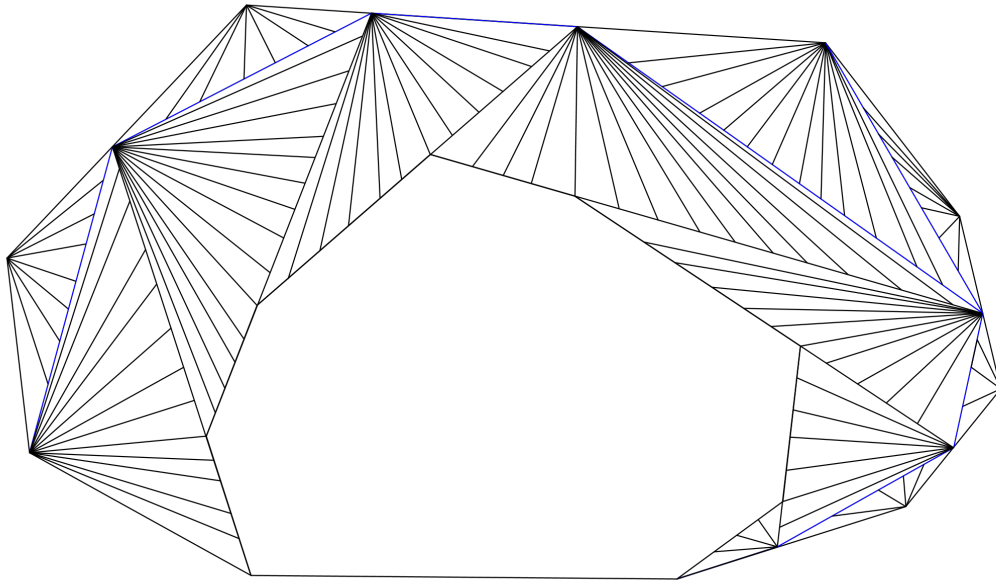
$$\begin{aligned} AB' + B'X &< AB + BX \\ B'C &< B'X + XC \end{aligned}$$

Then by segment addition,

$$\begin{aligned} AB' + B'X + B'C &< AB + BX + B'X + XC \\ AB' + B'C &< AB + BX + XC \\ AB' + B'C &< AB + BC \end{aligned}$$

This is the desired conclusion.

One of the joys of geometry is the albeit rare possibility of “proofs without words”. See the figure on the next page:



Each “exploding triangle” is a suggestion to write the triangle inequality, the two “sweeping” sides adding to something larger than the “spray” side. In the picture, the region between the inner broken line and the outer broken line is decomposed into exploding triangles such that adjacent triangles only meet along an edge to which they together contribute one spray side and one sweeping side. As a result, the sum of the triangle inequalities says that the sum of the unshared sweeping sides is larger than the sum of the unshared spray sides; in this case, the inner broken line is shorter than the outer one.

According to this reasoning, you just need to prove that if both broken lines are convex, you can always find such a decomposition into exploding triangles.

The following refinement suggests how to do this.

