Homework #2 Solutions (MAT 360)

Exercises 76, 77, 80, 81, 90, 93 in the textbook (Kiselev).

Exercise 76. In a given triangle in which one altitude is a bisector, the triangle is isosceles.

Let AX be the altitude of triangle ABC emanating from A. By definition AX is perpendicular to BC, so $\angle AXB$ and $\angle AXC$ are congruent because they are both right.
If AX is also a bisector, meaning the bisector of $\angle BAC$, then by definition $\angle BAX$ is congruent to $\angle CAX$. The congruences

$$\angle BAX \cong \angle CAX$$
$$AX \cong AX$$
$$\angle AXB \cong \angle AXC$$

imply that the triangles AXB and AXC are congruent (by the Angle Side Angle theorem, page 30). Then AB is congruent to AC; ABC is isosceles.

Exercise 77. In a given triangle in which one altitude is a median, the triangle is isosceles.

Let AX be the altitude of a triangle ABC emanating from A. By definition AX is perpendicular to BC, so $\angle AXB$ and $\angle AXC$ are congruent because they are both right.
If AX is also a median, then by definition XB and XC are congruent. The congruences

$$XB \cong XC$$
$$\angle AXB \cong \angle AXC$$
$$AX \cong AX$$

imply that the triangles AXB and AXC are congruent (by the Side Angle Side theorem, page 30). Then AB is congruent to AC; ABC is isosceles.

Exercise 80. If two sides and the median drawn to the first side of one triangle are congruent to those of another triangle, then the triangles are congruent.

Let ABC and $A'B'C'$ be triangles with median XC drawn to AB and $X'C'$ drawn to $A'B'$. Suppose that $AB \cong A'B'$, $BC \cong B'C'$, and $XC \cong X'C'$. Since XC and $X'C'$ are medians,

$$XA \cong XB$$
$$X'A' \cong X'B'$$

Of course, since $AB \cong A'B'$, all four of the above segments are congruent to half of the segment AB or $A'B'$ (that is, each of these four can be added to themselves to produce a segment congruent to $AB \cong A'B'$.)
Then the congruences

$$XB \cong X'B'$$
$$XC \cong X'C'$$
$$BC \cong B'C'$$

imply that triangles XBC and $X'B'C'$ are congruent. As a result, $\angle BXC$ and $\angle B'X'C'$ are congruent. Their supplements are also congruent:

$$\angle AXC \cong \angle A'X'C'$$

Then the congruences

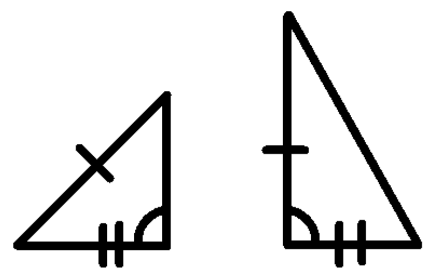
$$XA \cong X'A'$$
$$XC \cong X'C'$$
$$\angle AXC \cong \angle A'X'C'$$

imply that the triangles AXC and $A'X'C'$ are congruent (by the Side Angle Side theorem). In particular, segments AC and $A'C'$ are congruent. Finally, the congruences

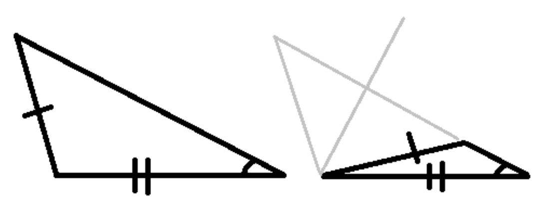
$$AB \cong A'B'$$
$$BC \cong B'C'$$
$$AC \cong A'C'$$

imply that the triangles ABC and $A'B'C'$ are congruent (by the Side Side Side theorem).

Exercise 81. Give an example of two non-congruent triangles such that two sides and one angle of the first triangle are congruent to two sides and one angle of the other triangle.
Here is an example



You might say this doesn't count, since the order has changed from Side Side Angle to Side Angle Side.
Here is one where the order is the same (of course, the order can't be Side Angle Side for both, because of the Side Angle Side theorem, but it is possible to have two non-congruent triangles with Side Side Angle congruences):



Exercise 90. A median of a triangle is smaller than its semiperimeter.
Let a, b, c be the sides of a triangle, and x, y, z the corresponding medians. Consider x first: it divides the triangle into two triangles. The triangle inequality (page 38) applied to these triangles implies that

$$x < (a/2) + c$$
$$x < (a/2) + b,$$
$$2x < a + b + c$$

That is, $x < s$, where $s = (a + b + c)/2$ is the semiperimeter of the original triangle. The same reasoning applies to y and z.

Exercise 93. The sum of the diagonals of a quadrilateral is smaller than its perimeter but larger than its semiperimeter.
Let $ABCD$ be a quadrilateral with diagonals AC and BD. By the triangle inequality,

$$\begin{aligned} AC &< AB + BC \\ AC &< AD + DC \\ BD &< BC + CD \\ BD &< AB + AD \end{aligned}$$

By segment addition,

$$2(AC + BD) < 2(AB + BC + CD + AD)$$

Then the sum of the diagonals $AC + BD$ is less than the perimeter $AB + BC + CD + AD$. For the second statement assume that the diagonals AC and BD meet at a point X inside the quadrilateral, so that the diagonals are broken into 4 segments XA, XB, XC, XD. Then according to the triangle inequality,

$$\begin{aligned} XA + XB &> AB \\ XB + XC &> BC \\ XC + XD &> CD \\ XD + XA &> AD \end{aligned}$$

Then by addition,

$$\begin{aligned} &2(XA + XB + XC + XD) > AB + BC + CD + AD \\ &XA + XB + XC + XD > s \end{aligned}$$

where s is the semiperimeter of the quadrilateral. The sum $XA + XB + XC + XD = (XA + XC) + (XB + XD)$ is the sum of the diagonals, which is therefore greater than s.