Selected proofs

252, 269, 272, 401, 407.

252. A secant to two congruent circles is parallel to the line of centers, meeting the first in A and B and the second in A' and B'. Then $OO' = AA' = BB'$ (where O and O' are the centers).

The translation T by OO' evidently sends the first circle onto the second. This translation preserves the secant, because it is parallel to OO'. Therefore it maps the intersections of the secant with the first circle onto the intersections with the second circle. We have $T(O) = O'$, $T(A) = A'$, $T(B) = B'$. The defining property of a translation in this case says that AA' and OO' form opposite sides of a parallelogram and that BB' and OO' form opposite sides of a parallelogram. Therefore $OO' = AA' = BB'$.

269. Two secant lines passing through the point of tangency of two circles each meet the circles at 2 other points. The chords formed from these 4 points are parallel.

The larger circle is the image of the smaller by a homothety centered at the point of tangency (with ratio that of the diameters of the circles). This homothety preserves the secant lines, since they pass through the center of the homothety. Therefore it sends the intersections with the first circle onto the intersections with the second circle. According to theorem 177 in the text, homotheties map each segment to a parallel segment, so the chords formed here are parallel.

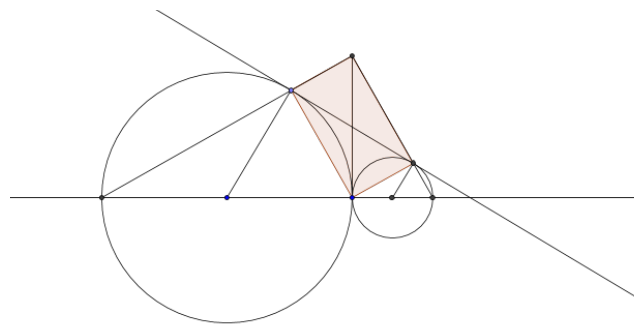
272. Through a point A of a circle, the tangent and a chord AB are drawn. The diameter perpendicular to the radius OB meets the tangent and the chord (or its extension) at the points C and D respectively. $AC = CD$.

Let $a = \angle CDA$ and $b = \angle CAD$.
Let F be the foot of the perpendicular dropped from O to AB. Triangles OFD and BOD are similar, since they are right and share an angle. Moreover BOA is isosceles. Consequently

$$a = \angle FDO = 2d - \angle FOD = 2d - \angle OBD = 2d - \angle OAD = b$$

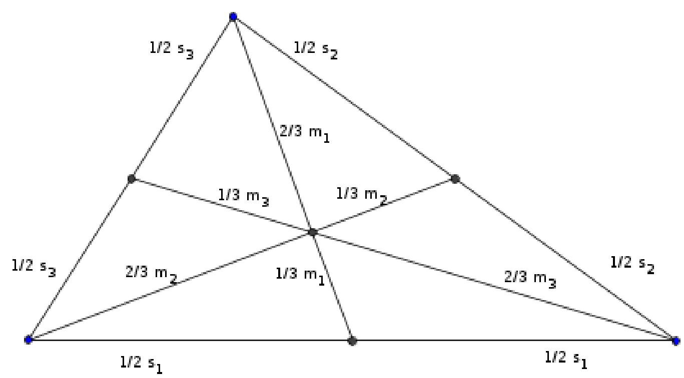
Therefore CDA is isosceles and $AC = CD$.

401. If two disks are tangent externally, then the segment of an external common tangent between the tangency points is the geometric mean between the diameters of the disks.
(Just a hint)



Try to prove it using the figure above.

407. For any triangle, the ratio of the sum of the squares of the medians to the sum of squares of the sides is 3 to 4.



We know that the medians are concurrent, and the point of concurrency divides them in the ratio 1 to 2. This is shown in the figure.
We can use theorem 190 in the text (essentially what you may know as the "Law of Cosines") to prove Appolonius' theorem:

The sides a and b of a triangle and the median m dividing the third side into $2c$ satisfy the equation

$$a^2 + b^2 = 2(m^2 + c^2)$$

Apply this to the triangle with sides s_1, $\frac{2}{3}m_2$, and $\frac{2}{3}m_3$:

$$\frac{4}{9}\left(m_2^2 + m_3^2\right) = 2\left(\frac{1}{9}m_1^2 + \frac{1}{4}s_1^2\right)$$

$$8m_2^2 + 8m_3^2 = 4m_1^2 + 9s_1^2$$

We can do the same for the other 2 triangles of this form, and take the sum:

$$\sum_i 8m_i^2 + 8m_i^2 = \sum_i m_i^2 + 9s_i^2$$

$$\sum_i 12m_i^2 = \sum_i 9s_i^2$$

$$4\sum_i m_i^2 = 3\sum_i s_i^2$$