MAT313 Fall 2017
Practice Midterm II

The actual midterm will consist of 5-6 problems.

It will be based on subsections 127-234 from sections 6-11. This is a preliminary version. I might add few more problems to make sure that all important concepts and methods are covered.

Problem 1.

Let $\mathbf{P}$ be a set of lines through 0 in $\mathbb{R}^2$. The group $\mathrm{SL}(2,\mathbb{R})$ acts on X by linear transformations. Let H be the stabilizer of a line defined by the equation $y = 0$.

(1) Describe the set of matrices H.

(2) Describe the orbits of H in X. How many orbits are there?

(3) Identify X with the set of cosets of $\mathrm{SL}(2,\mathbb{R})$.

Solution.

(1) Denote the set of lines by $\mathbf{P} = \{L_{m,n}\}$, where $L_{m,n}$ is equal to $\{(x,y)|mx+ny = 0\}$. Note that $L_{km,kn} = L_{m,n}$ if $k \neq 0$. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $ad - bc = 1$ be an element of $\mathrm{SL}(2,\mathbb{R})$. Then $gL_{m,n} = \{(x,y)|m(ax+by) + n(cx+dy) = 0\} = L_{ma+nc,mb+nd}$. By definition the line $\{(x,y)|y = 0\}$ is equal to $L_{0,1}$. Then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) L_{0,1} = L_{c,d}$. The line $L_{c,d}$ coincides with $L_{0,1}$ if $c = 0$. Then the stabilizer $Stab(L_{0,1})$ coincides with $H = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\} \subset \mathrm{SL}(2,\mathbb{R})$.

(2) We already know one trivial H-orbit $\mathcal{O} = \{L_{0,1}\}$. The complement $\mathbf{P}\backslash\{L_{0,1}\}$ is equal to $\{L_{m.n}|m \neq 0\} = \{L_{1.n/m}|m \neq 0\} = \{L_{1.t}\}$. The line $L_{1,0}$ is an element of $\mathbf{P}\backslash\{L_{0,1}\}$. Its H orbit is equal to $\{gL_{1,0}|g \in H\} = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) L_{1,0}\} = \{L_{a,b}|a \neq 0\} = \{L_{1,b/a}|a \neq 0\} = \mathbf{P}\backslash\{L_{0,1}\}$.

We conclude that H consists of two H-orbits $\{L_{0,1}\}$ and $\mathbf{P}\backslash\{L_{0,1}\}$.

(3) The action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbf{P}$ is transitive. This is because $gL_{0,1} = L_{a,b}$ with $(a,b) \neq (0,0)$. For any pair (a,b), defined up to a multiplicative constant there is g such that $gL_{0,1} = L_{a,b}$. This is because for any such a pair (a,b) the equation $ad-bc = 1$ always has a solution.

□

Problem 2.

(1) Describe all elements of order eight in $\mathbb{Q}/\mathbb{Z}$.
(2) Find all elements of infinite order in $\mathbb{Q}/\mathbb{Z}$.
(3) Identify $\mathbb{Q}/\mathbb{Z}$ with a subgroup of $\mathbb{C}^*$

Solution.

(1) a is an element of order 8 in $\mathbb{Q}/\mathbb{Z}$ iff a representative $\tilde{a} \in \mathbb{Q}$ such that $\tilde{a} \in a = \tilde{a} + \mathbb{Z}$ satisfies $8\tilde{a} \in \mathbb{Z}$. Then $\tilde{a} = s/8 + k, 0 \leq s < 8, k \in \mathbb{Z}$. Element $s_1/8 + k_1$ and $s_2/8 + k_3$ define the same element in $\mathbb{Q}/\mathbb{Z}$ id $s_1 = s_2$. We conclude that there are precisely eight elements of order 8 in $\mathbb{Q}/\mathbb{Z}$.
(2) Any rational number $a = p/q$ satisfies $qa \in \mathbb{Z}$. Thus $\mathbb{Q}/\mathbb{Z}$ contains no elements of infinite order.
(3) Define a homomorphism $\psi : \mathbb{Q} \to \mathbb{C}^*$ by the formula $\psi(a) = exp(2\pi i a)$. The image coincides with group of roots of unity. The kernel is the set of integers. By the first isomorphism theorem $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the group of unity, i.e. the set of solution of the equations $z^k = 1, k \in \mathbb{Z}$ in the complex numbers.

□

Problem 3. Let G be the group of quaternions, i.e., $G = \{1, -1, i, j, k, (-1)i, (-1)j, (-1)k\}$. The elements $1, -1$ are central and satisfy $-1^2 = 1$. In addition $i^2 = j^2 = k^2 = -1$, $(-1)ji = ij = k, (-1)ki = ik = (-1)j, (-1)jk = kj = (-1)i$.

Find orders of all elements in $G/Z(G)$, Is $G/Z(G)$ isomorphic to $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$. Why?

Solution. Define a homomorphism $G \to \mathbb{Z}_2 + \mathbb{Z}_2$ by the formula $\psi(1) = \psi(-1) = (0,0), \psi(i) = \psi(-1i) = (1,0), \psi(j) = \psi(-1j) = (0,1), \psi(k) = \psi(-1k) = (1,1)$. The kernel of this homomorphism is $\{1, -1\}$. The homomorphism is surjective. So $G/Z(G)$ is isomorphic to $\mathbb{Z}_2 + \mathbb{Z}_2$. The latter is not isomorphic to $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$, because the groups have different orders. □

Problem 4. Give the definition of a factor group.

Solution. Let G be a group with a normal subgroup H. Define the group structure on the set of cosets $\{gH | g \in G\}$ by the formula $g_1H \times g_2H = g_1g_2H$. The set of cosets with this group structure is the factor(quotient) group G/H. $\square$

Problem 5. Fix a group G. Prove that $|x| = |gxg^{-1}| \forall x, g \in G$. Deduce that $|xy| = |yx| \forall x, y \in G$.

Solution. Use the identity $gx^ng^{-1} = gxg^{-1}gxg^{-1} \cdots gxg^{-1} = (gxg^{-1})^n$ to see that $1 = x^n \Leftrightarrow 1 = g1g^{-1} = gx^ng^{-1} = (gxg^{-1})^n$. The order $|x|$ is the minimal n with this property. Identity $|x| = |gxg^{-1}|$ follows from this. The second assertion follows from $xy = x(yx)x^{-1}$.

$\square$

Problem 6. Compute the order of $GL_2(\mathbb{Z}_p)$ where p is a prime number.

Solution. A matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Mat_2(\mathbb{Z}_p)$ is an element of $GL_2(\mathbb{Z}_p)$ if an only if the columns $v_1 = \left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)$ and $v_2 = \left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)$ are not proportional.

Let us count the number of pairs (v_1, v_2), where v_1, v_2 are proportional. We have the following mutually exclusive possibilities:

(1) $v_1 = v_2 = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) =: 0$. The set of such has one element.
(2) $v_1 = 0, v_2 \neq 0$. The set of such has $p^2 - 1$ elements
(3) $v_1 \neq 0, v_2 = 0$. Again, the set of such has $p^2 - 1$ elements
(4) $v_1 \neq 0, v_2 = cv_1, c \neq 0$. The set of such pairs has $(p-1)(p^2-1)$ elements. $(p-1)$ stands for distinct values of c, (p^2-1) the number of nonzero v_1.

The set of proportional v_1, v_2 contain $1+2(p^2-1)+(p-1)(p^2-1)$ elements. The number of elements in $Mat_2(\mathbb{Z}_p)$ is p^4. We conclude $|GL_2(\mathbb{Z}_p)| = p^4-(1+2(p^2-1)+(p-1)(p^2-1)) = (p-1)^2p(p+1)$

$\square$

Problem 7.

(1) Prove that dihedral group D_{12} (the group of symmetries of the regular 12-gon) is not isomorphic to symmetric group S_4.
(2) Prove that dihedral group $\mathbb{Z} \neq \mathcal{Q}$.

Solution. (1) Note first of all that the orders of both groups are equal to 24. The group D_{12} contain an element of order 12-rotation of angle $2\pi/12$. The maximal order of an element of S_4 is 4. This is a cycle of length 4. Isomorphism preserves orders of elements. From this we conclude that existence of an isomorphism is impossible.
(2) Let us assume that there is an isomorphism $\psi : \mathbb{Z} \to \mathcal{Q}$ and deduce a contradiction. We know that if m is a generator of $\mathbb{Z}$, then $m = \pm 1$. In particular equation $2x = 1$ has no solutions in $\mathbb{Z}$. $\psi(1) = a$, then $\psi^{-1}(a/2)$ is a solution of $2x = 1$ in $\mathbb{Z}$, which is a contradiction.

□

A group G action on the set X is faithful if the corresponding homomorphism $\rho : G \to S_X$ has a trivial kernel.

Problem 8. Prove that the group of rigid symmetries of a cube doesn't act faithfully on the set of opposite faces of the cube. Find the kernel.

Solution. In class we proved that rigid symmetries of a cube is isomorphic to S_4.Let us identify vertices of the cube with points $\{(\pm 1, \pm 1, \pm 1) \in \mathbb{R}^3\}$. It should be clear then that rotations on angle π about axises x, y, z fix any pair of opposite faces. The number of elements in the set $\{\pi_1, \pi_2, \pi_3\}$of pairs of opposite faces is 3. Thus the action defines a homomorphism $\mu : S_4 \to S_3$. Rotation on angle $\pi/2$ about one of the axis x, y, z fixes π_i and interchanges π_j and π_k,$j, k \neq i$. This way we see that permutations $(1, 2)$, $(1, 3)$ and $(2, 3)$ are in the image of μ. We know $(1, 2)$, $(1, 3)$ and $(2, 3)$ generate S_3. We conclude that μ is onto and $|\mathrm{Ker}\mu| = \frac{|S_4|}{|S_3|} = 4$. Tree rotations on angle π mentioned above and the trivial symmetry is the list of all the elements in the kernel.

□

Problem 9. Give an example of a noncommutative group G and a normal subgroup $N \subset G$ such that G/N is abelian of order ≥ 3.

Solution. $G = \{\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)\} \subset \mathrm{SL}(2, \mathbb{Z}_p)$ contains a normal subgroup $N = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\}$. N is a kernel of homomorphism $\psi : G \to \mathbb{Z}_p^\times (\mathbb{Z}_p^\times := \mathbb{Z}_p \backslash \{0\})$. By definition $\psi(\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)) = a$. Since $\mathbb{Z}_p^\times$ is a commutative group, ψ defines an isomorphism $\psi : G/N \to \mathbb{Z}_p^\times$ and G/N is commutative. $\square$

Problem 10. Let G be a finite group, let H be a subgroup and $N \trianglelefteq G$. Prove that if $\gcd(|H|, |G : N|) = 1$, then $H \subset N$.

Solution. Let $\pi : G \to G/N$ be the canonical homomorphism. We restrict it on $H \subset G$ and observe that by Lagrange theorem $\frac{|G:N|}{|\pi(H)|} \in \mathbb{Z}$ and $\frac{|H|}{|\pi(H)|} \in \mathbb{Z}$. We conclude $|\pi(H)| \mid \gcd(|H|, |G : N|)$ and $|\pi(H)| = 1$. Thus $H \subset \operatorname{Ker} \pi = N$ $\square$

Problem 11. Prove that if H has a finite index n in G then there is $N \trianglelefteq G$ such that $|G : N| \leq n!$

Solution. Consider the left action of G on $X = G/H$ is in Cayley theorem. The action defines a homomorphism $\rho : G \to S_X \cong S_n$. By construction $|G : \operatorname{Ker} \rho| = |\rho(G)| \leq |S_n| = n!$. We choose N to be $\operatorname{Ker} \rho$. $\square$

Problem 12. Let $A = \mathbb{Z}_{60} \times \mathbb{Z}_{45} \times \mathbb{Z}_{12} \times \mathbb{Z}_{36}$. Find the number of elements of order 2 and the number of subgroups of index 2 in A.

Solution. First we use the theorem about the direct product decomposition of cyclic groups $\mathbb{Z}_{p_1^{\alpha_1} \dots p_n^{\alpha_n}} \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \dots \times \mathbb{Z}_{p_n^{\alpha_n}}$. From this we conclude that

$$\mathbb{Z}_{2^2 \times 3 \times 5} \times \mathbb{Z}_{5 \times 3^2} \times \mathbb{Z}_{2^2 \times 3} \times \mathbb{Z}_{2^2 \times 3^2} \cong \mathbb{Z}_4^3 \times (\mathbb{Z}_3^2 \times \mathbb{Z}_9^2) \times \mathbb{Z}_5^2$$

All elements of order two belong to $\mathbb{Z}_2^3 \subset \mathbb{Z}_4^3$. We conclude that the number of elements of order two is $|\mathbb{Z}_2^3 \backslash \{1\}| = 2^3 - 1 = 7$.

Let $N \subset A$ be a subgroup of index 2. It coincides with the kernel of some homomorphism $\phi : A \to \mathbb{Z}_2$. As the order of $(\mathbb{Z}_3^2 \times \mathbb{Z}_9^2) \times \mathbb{Z}_5^2$ is not divisible by 2 this subgroup contains in $\operatorname{Ker}\phi$. Fix generators $e_i \in 1 \times \cdots \underset{i}{\mathbb{Z}_4} \cdots \times 1 \subset \mathbb{Z}_4^3$. The homomorphism ϕ is completely determined by its values $a_i = \phi(e_i), a_i \in \mathbb{Z}_2$ on generators. The number of sequences $\{(a_1, a_2, a_3)\}$ (excluding $(0,0,0)$) is $2^3 - 1 = 7$. $\square$