MAT313 Fall 2013
Practice Midterm I

Problem 1.

Let $\mathbf{P}$ be a set of lines through 0 in $\mathbb{R}^2$. The group $\mathrm{SL}(2,\mathbb{R})$ acts on X by linear transformations. Let H be the stabilizer of a line defined by the equation $y = 0$.

(1) Describe the set of matrices H.

(2) Describe the orbits of H in X. How many orbits are there?

(3) Identify X with the set of cosets of $\mathrm{SL}(2,\mathbb{R})$.

Solution.

(1) Denote the set of lines by $\mathbf{P} = \{L_{m,n}\}$, where $L_{m,n}$ is equal to $\{(x,y)|mx+ny = 0\}$. Note that $L_{km,kn} = L_{m,n}$ if $k \neq 0$. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $ad - bc = 1$ be an element of $\mathrm{SL}(2,\mathbb{R})$. Then $gL_{m,n} = \{(x,y)|m(ax + by) + n(cx + dy) = 0\} = L_{ma+nc,mb+nd}$. By definition the line $\{(x,y)|y = 0\}$ is equal to $L_{0,1}$. Then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) L_{0,1} = L_{c,d}$. The line $L_{c,d}$ coincides with $L_{0,1}$ if $c = 0$. Then the stabilizer $Stab(L_{0,1})$ coincides with $H = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\} \subset \mathrm{SL}(2,\mathbb{R})$.

(2) We already know one trivial H-orbit $\mathcal{O} = \{L_{0,1}\}$. The complement $\mathbf{P}\backslash\{L_{0,1}\}$ is equal to $\{L_{m.n}|m \neq 0\} = \{L_{1.n/m}|m \neq 0\} = \{L_{1.t}\}$. The line $L_{1,0}$ is an element of $\mathbf{P}\backslash\{L_{0,1}\}$. Its H orbit is equal to $\{gL_{1,0}|g \in H\} = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) L_{1,0}\} = \{L_{a,b}|a \neq 0\} = \{L_{1,b/a}|a \neq 0\} = \mathbf{P}\backslash\{L_{0,1}\}$.

We conclude that H consists of two H-orbits $\{L_{0,1}\}$ and $\mathbf{P}\backslash\{L_{0,1}\}$.

(3) The action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbf{P}$ is transitive. This is because $gL_{0,1} = L_{a,b}$ with $(a,b) \neq (0,0)$. For any pair (a,b), defined up to a multiplicative constant there is g such that $gL_{0,1} = L_{a,b}$. This is because for any such a pair (a,b) the equation $ad-bc = 1$ always has a solution.

$\square$

Problem 2.

(1) Describe all elements of order eight in $\mathbb{Q}/\mathbb{Z}$.

(2) Find all elements of infinite order in $\mathbb{Q}/\mathbb{Z}$.

(3) Identify $\mathbb{Q}/\mathbb{Z}$ with a subgroup of $\mathbb{C}^*$

Solution.

(1) a is an element of order 8 in $\mathbb{Q}/\mathbb{Z}$ iff a representative $\tilde{a} \in \mathbb{Q}$ such that $\tilde{a} \in a = \tilde{a} + \mathbb{Z}$ satisfies $8\tilde{a} \in \mathbb{Z}$. Then $\tilde{a} = s/8 + k, 0 \leq s < 8, k \in \mathbb{Z}$. Element $s_1/8 + k_1$ and $s_2/8 + k_3$ define the same element in $\mathbb{Q}/\mathbb{Z}$ id $s_1 = s_2$. We conclude that there are precisely eight elements of order 8 in $\mathbb{Q}/\mathbb{Z}$.

(2) Any rational number $a = p/q$ satisfies $qa \in \mathbb{Z}$. Thus $\mathbb{Q}/\mathbb{Z}$ contains no elements of infinite order.

(3) Define a homomorphism $\psi : \mathbb{Q} \to \mathbb{C}^*$ by the formula $\psi(a) = exp(2\pi i a)$. The image coincides with group of roots of unity. The kernel is the set of integers. By the first isomorphism theorem $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the group of unity, i.e. the set of solution of the equations $z^k = 1, k \in \mathbb{Z}$ in the complex numbers.

$\square$

Problem 3. Give an example of a non commutative group with a normal subgroup of index $p-1$, where p is prime

Solution. $H = \{\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)\} \subset \mathrm{SL}(2, \mathbb{Z}_p)$. There is a homomorphism $\psi : H \to \mathbb{Z}_p^*$ defined by the formula $\psi(\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)) = a$. Its kernel is equal to $K = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\}$. Since $|\mathbb{Z}_p^*| = p - 1$, the index of K is $p - 1$. $\square$

Problem 4. Give an example of a non commutative group that contains a subgroup of prime order.

Solution. The group K from Problem 3. $\square$

Problem 5. Let G be the group of quaternions, i.e., $G = \{1, -1, i, j, k, (-1)i, (-1)j, (-1)k\}$. The elements $1, -1$ are central and satisfy $-1^2 = 1$. In addition $i^2 = j^2 = k^2 = -1$, $(-1)ji = ij = k, (-1)ki = ik = (-1)j, (-1)jk = kj = (-1)i$.

Find orders of all elements in $G/Z(G)$, Is $G/Z(G)$ isomorphic to $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$. Why?

Solution. Define a homomorphism $G \to \mathbb{Z}_2 + \mathbb{Z}_2$ by the formula $\psi(1) = \psi(-1) = (0,0), \psi(i) = \psi(-1i) = (1,0), \psi(j) = \psi(-1j) = (0,1), \psi(k) = \psi(-1k) = (1,1)$. The kernel of this homomorphism is $\{1,-1\}$. The homomorphism is surjective. So $G/Z(G)$ is isomorphic to $\mathbb{Z}_2 + \mathbb{Z}_2$. The latter is not isomorphic to $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$, because the groups have different orders. □

Problem 6. Give the definition of a factor group.

Solution. Let G be a group with a normal subgroup H. Define the group structure on the set of cosets $gH|g \in G$ by the formula $g_1H \times g_2H = g_1g_2H$. The set of cosets with this group structure is the factor(quotient) group G/H. □

Problem 7. Describe all the subgroups in $\mathbb{Z}_{18}$ and their generators.

Solution. *Fact:* Subgroups of a cyclic group G are $1:1$ with divisors of $|G|$. In our case $|G| = 18 = 2 \cdot 3^2$. This means that we have the following subgroups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6, \mathbb{Z}_9$.

Fact: If $a \in G$ is an element of order n, then a^k has order $\frac{n}{(n,k)}$.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$(18,k)$	1	2	3	2	1	6	1	2	9	2	1	6	1	2	3	2	1
order $= \frac{18}{(18,k)}$	18	9	6	9	18	3	18	9	2	9	18	3	18	9	6	9	18

The generator of $\mathbb{Z}_2$ is 9 . $\mathbb{Z}_3$ is generated by either element of the set $\{6,12\}$. $\mathbb{Z}_6$ is generated by either element of the set $\{3,15\}$. Likewise $\mathbb{Z}_9$ is generated by either of $\{2,4,8,10,14,16\}$ □

Problem 8. Give your prove that $\mathbb{Z}_{10} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_5$ but $\mathbb{Z}_8 \ncong \mathbb{Z}_2 \oplus \mathbb{Z}_4$.

Solution. We have a map $\psi : \mathbb{Z}_2 \oplus \mathbb{Z}_5 \to \mathbb{Z}_{10}$, defined by the formula $\psi(x \oplus y) = 5x + 2y$. The map is correctly defined because $5, 2 \in \mathbb{Z}_{10}$ are elements of order $2, 5$ respectively. $2, 5$ are relatively prime $\Rightarrow -2 \times 2 + 1 \times 5 = 1 \Rightarrow x = 2(-2x) + 5x \Rightarrow$ the map ψ is onto. $|\mathbb{Z}_2 \oplus \mathbb{Z}_5| = |\mathbb{Z}_{10}| = 10 \Rightarrow \psi$ is a bijection.

If $2x \cong 0 \mathrm{mod} 8 \Rightarrow x \cong 0 \mathrm{mod} 4 \Rightarrow$ the only nontrivial element of order 2 is $4 \in \mathbb{Z}_8$. On the other hand $\{(1,0),(0,2),(1,2)\}$ are nontrivial element of order 2 in $\mathbb{Z}_2 \oplus \mathbb{Z}_4$. An isomorphism defines a bijection between sets of elements of the same order. $\Rightarrow \mathbb{Z}_8 \not\cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$ $\square$