Practice Mid 2 Solutions (Even Numbers)

Note Title 11/13/2012

2\. 1) For any λ, $y(t) \equiv 0$ is a solution to $y'' = \lambda y$ with $y(0) = y(1) = 0$

2) the roots of the auxiliary polynomial

$$t^4 + 2t^3 + 10t^2 + 18t + 9$$

$$= (t^4 + 2t^3 + t^2) + (9t^2 + 18t + 9)$$

$$= (t^2 + 9)(t+1)^2$$

are $t = \pm 3i$ and $t = -1$ with multiplicity 2.

$\Rightarrow$ the general solution of the differential equation is in the form

$$y = C_1 \sin 3x + C_2 \cos 3x + C_3 e^{-x} + C_4 x e^{-x}$$

4. 1)

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & & & \\ -1 & 1 & -3 & 1 & & 1 & & \\ 0 & 2 & 3 & 0 & & & 1 & \\ -5 & 1 & -15 & 0 & & & & 1 \end{array}\right) \xrightarrow[5\times① + ④]{① + ②} \left(\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{② \leftrightarrow ④} \left(\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 & 0 & 1 \\ 0 & 2 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow[-② + ④]{-2\times② + ③} \left(\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & -10 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -4 & 1 & 0 & -1 \end{array}\right)$$

$$\xrightarrow{-③ + ①} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 11 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 5 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & -10 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -4 & 1 & 0 & -1 \end{array}\right) \xrightarrow{③ \div 3} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 11 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 5 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{10}{3} & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 1 & -4 & 1 & 0 & -1 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 3 & 0 \\ -1 & 1 & -3 & 1 \\ 0 & 2 & 3 & 0 \\ -5 & 1 & -15 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 11 & 0 & -1 & 2 \\ 5 & 0 & 0 & 1 \\ -\frac{10}{3} & 0 & \frac{1}{3} & -\frac{2}{3} \\ -4 & 1 & 0 & -1 \end{pmatrix}$$

2) let $A = \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}$ $\quad A^{-1} = \frac{1}{7}\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$

$A\vec{x} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ $\quad \vec{x} = A^{-1}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

6. $$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 1 & 1 & -2 & 3 \\ -2 & 0 & 6 & -10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ a \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 5 \\ 1 & 1 & -2 & 3 & 6 \\ -2 & 0 & 6 & -10 & a \end{array}\right)$$

$$\xrightarrow[2\times (1) + (3)]{-(1) + (2)} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 5 \\ 0 & -1 & -1 & 2 & 1 \\ 0 & 4 & 4 & -8 & 10+a \end{array}\right)$$

$$\xrightarrow{4\times (2) + (3)} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 5 \\ 0 & -1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 14+a \end{array}\right)$$

$\Rightarrow$ the system is consistent when $14 + a = 0$,

i.e. $a = -14$.

When $a = -14$,

$$x_1 + 2x_2 - x_3 + x_4 = 5$$
$$-x_2 - x_3 + 2x_4 = 1$$

let $x_3 = s$, $x_4 = t$,

then
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5t + 3s + 5 \\ 2t - s - 1 \\ s \\ t \end{pmatrix}$$

$$= s\begin{pmatrix} 3 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 5 \\ 2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

Therefore $\begin{pmatrix} 3 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 5 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ form a "basis"

of the solution set. (The solution set isn't really a vector space!)

8. 1) False $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

2) Incorrectly stated.

The matrix is $n \times k$. Determinant is not defined if $n \neq k$.

3) False. $E_3 = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \lambda & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}$

4) True

5) False $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

6) True

7) False $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

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$$A = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[\substack{E_1 \\ E_2}]{\substack{2 \times ① + ② \\ -① + ③}} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow[E_3]{② \leftrightarrow ③} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 4 & 3 \end{pmatrix} \xrightarrow[E_4]{② \times -1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 4 & 3 \end{pmatrix}$$

$$\xrightarrow[\substack{E_5 \\ E_6}]{\substack{-2 \times ② + ① \\ -4 \times ② + ③}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \xrightarrow[E_7]{③ \times \frac{1}{3}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[E_8]{-③ + ①} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_8 E_7 E_6 E_5 E_4 E_3 E_2 E_1 A = I$$

$$\Rightarrow \quad A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} E_7^{-1} E_8^{-1}$$

$$= \begin{pmatrix} 1 & & \\ -2 & 1 & \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$