MAT 310 - Solutions to practice mid term 1

Problem 1 Let $F = \mathbb{R}^2, U = \{(x,0) \,|\, x \in \mathbb{R}\}, W = \{(0,y) \,|\, y \in \mathbb{R}\}$. Then $(1,1) = (1,0) + (0,1)$ is not in $U \cup W$ but it should have been if $U \cup W$ was a subspace.

If $U \cup W$ is a subspace of F and $U \not\subset W, W \not\subset U$, choose $u \in U \setminus W$ and $w \in W \setminus U$. Then $u + w \in U \cup W$, since it is a subspace. If $u + w \in U$ then $w = (u+w) - u \in U$, a contradiction. On the other hand, if $u + w \in W$ then $u = (u+w) - w \in W$, again a contradiction.

Problem 2 (i) Suppose $a + b(t-1) + c(t-1)^2 + d(t-1)^3 = 0$. Then

$$(a - b + c - d) + (b - 2c - 3d)t + (c + 3d)t^2 + dt^3 = 0.$$

Since $(1, t, t^2, t^3)$ is a basis of $\mathbb{P}_3$ we conclude that $d = 0$. Since $c + 3d = 0$ this forces $c = 0$. Now $b - 2c - 3d = 0$ implies $b = 0$ and $a - b + c - d = 0$ implies $a = 0$. This proves that $(1, t-1, (t-1)^2, (t-1)^3)$ is linearly independent. Since $\mathcal{P}_3$ has dimension 4 and $U = \text{span}(1, t-1, (t-1)^2, (t-1)^3)$ is a subspace of dimension 4, we conclude that $U = \mathcal{P}_3$.

(ii) Yes. For example, take $S = \{(1,0),(0,1)\}$ and $T = \{(1,1),(1,-1)\}$. The vectors in S span $\mathbb{R}^2$ and so does the vectors in T but $S \neq T$.

Problem 3 It is given that $\psi\phi : V \to V$ is an isomorphism, i.e., it is injective (and surjective as well). If $\phi(v) = 0$ then $\psi\phi(v) = 0$ whence $v = 0$. Therefore, ϕ is injective. On the other hand, given $v \in V$, let $v' \in v$ be the unique element such that $\psi\phi(v') = v$. This is possible since $\psi\phi$ is surjective. Then $\psi(\phi(v')) = v$ and $\phi(v') \in W$, whence ψ is surjective.

Problem 4 Let $\rho : V \to V$ be such that $\rho\rho = \rho$. Let $v \in \text{range}(\rho)$ and write $v = \rho(v')$ for some $v' \in V$. Then

$$\rho(v) = \rho\rho(v') = \rho(v') = v.$$

Thus, ρ is the identity on range(ρ).

Problem 5 It is enough to prove linear independence since then the span of the given vectors would be of dimension 3 and consequently has to be $\mathbb{R}^3$. Suppose

$$a(1,1,0) + b(2,0,-1) + c(-3,1,1) = (a + 2b - 3c, a + c, -b + c) = (0,0,0).$$

This implies that $b = c, a = -c$ and $a + 2b - 3c = 0$. The last equation can be written as $-c + 2c - 3c = 0$ whence $c = 0$ and $a = b = 0$.

Problem 6 Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\phi(x,y) = (0,x)$. Since $\phi\phi(x,y) = \phi(0,x) = (0,0)$ it defines a nilpotent endomorphism of order 2. Similarly, $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\psi(x,y) = (y,0)$ is also a nilpotent endomorphism of order 2. Now $\psi\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\psi\phi(x, y) = \psi(0, x) = (x, 0)$. It is clear that $(\psi\phi)^2(x, y) = \psi\phi(x, 0) = (x, 0) = \psi\phi(x, y)$. Therefore, $\psi\phi$ is an idempotent.

Problem 7 Since $x \in \text{span}\{M, y\}$ and $x \notin M$ we can write

$$x = a_1 v_1 + \cdots + a_k v_k + by$$

where v_i's are a basis of M and $b \neq 0$. Then

$$y = (-a_1/b)v_1 + \cdots + (-a_k/b)v_k + (1/b)x$$

and $y \in \text{span}\{M, x\}$. Clearly $M \subset \text{span}\{M, x\}$. Therefore, $\text{span}\{M, y\} \subset \text{span}\{M, x\}$. On the other hand, $x \in \text{span}\{M, y\}$ whence $\text{span}\{M, x\} \subset \text{span}\{M, y\}$. This proves that $\text{span}\{M, y\} = \text{span}\{M, x\}$.

Problem 8 Since $M \subset M + (L \cap N)$ this implies that $L \cap M \subset L \cap (M + (L \cap N))$. On the other hand

$$L \cap N = L \cap (L \cap N) \subset L \cap (M + (L \cap N)).$$

This means that $L \cap M$ and $L \cap N$ are both subspaces of $L \cap (M + (L \cap N))$ and therefore contains the sum as well, viz.,

$$(L \cap M) + (L \cap N) \subset L \cap (M + (L \cap N)).$$

On the other hand if $v \in L \cap (M + (L \cap N))$ then $v \in L$ and $v \in M + (L \cap N)$. Write $v = m + l$ where $m \in M$ and $l \in L \cap N$. Then $m = v - l \in L$ whence $m \in L \cap M$. Therefore, $v = m + l \in (L \cap M) + (L \cap N)$.

Problem 9 (i) If $(1, \alpha) = \lambda(1, \beta)$ then $\lambda = 1$ and $\alpha = \beta$. Therefore, $(1, \alpha)$ and $(1, \beta)$ are linearly independent if and only if $\alpha \neq \beta$.

(ii) No. If there were then $\mathbb{C}^2$ would contain the span of these three vectors which is a 3 dimensional subspace while $\mathbb{C}^2$ is only 2 dimensional.

(iii) No matter what $x \in \mathbb{C}$ is, the vectors $(1, 1, 1)$ and $(1, x, x^2)$ span a subspace of $\mathbb{C}^3$ of dimension at most 2. When $x = 1$ the span is $\{(z, z, z) \mid z \in \mathbb{C}\}$. When $x \neq 1$ the span is a 2 dimensional subspace. In either case, it does not span $\mathbb{C}^3$.

(iv) If these vectors are linearly independent then we'll be done since we're in $\mathbb{C}^3$. For any choice of $x \in \mathbb{C}$ we can write $(x, 1, 1 + x) = (x, 0, 1) + (0, 1, x)$ whence they are not linearly independent and therefore not a basis.

Problem 10 (i) The first and the third transformations are linear. The second is not since $T(2x, 2y) = 4T(x, y)$.

(ii) The first and the third are linear transformations. For example, in the first case

$$T(a_0 + a_1 x + \cdots + a_k x^k) = a_0 + a_1 x^2 + \cdots + a_k x^{2k} = T(a_0) + a_1 T(x) + a_2 T(x^2) + \cdots + a_k T(x^k)$$

which precisely means that T is linear. Similarly, in the third case

$$T(a_0 + a_1 x + \cdots + a_k x^k) = X^2(a_0 + a_1 x + \cdots + a_k x^k) = T(a_0) + a_1 T(x) + a_2 T(x^2) + \cdots + a_k T(x^k)$$

which implies linearity of T. In the second case, however, $T(2p(x)) = 4(p(x))^2 \neq 2T(p(x))$ whence T is not linear.

Problem 11 (i) Let $p(x) = a_0 + a_1x + \cdots + a_6x^6 \in \mathcal{P}_6$.

$$T(p(x)) := \int_{-3}^{x+9} p(t)dt = \sum_{i=0}^{6} a_i \int_{-3}^{x+9} t^i dt = \sum_{i=0}^{6} \frac{a_i}{i+1}((x+9)^{i+1} - (-3)^{i+1}).$$

If $T(p(x)) = 0$ then a_6, the coefficient of x^7, is zero. Therefore,

$$T(p(x)) = \sum_{i=0}^{5} \frac{a_i}{i+1}((x+9)^{i+1} - (-3)^{i+1}) = 0.$$

Again, the coefficient of x^6 is a_5 and it has to be zero. Doing this recursively leads one to $T(p(x) = a_0((x+9) - (-3)) = a_0(x+12) = 0$ whence $a_0 = 0$. Therefore, if $p(x) \in \text{null}(T)$ then $p(x) = 0$. So $\text{null}(T) = \{0\}$.

(ii) Let $p(x) = a_0 + a_1x + \cdots + a_5x^5 \in \mathcal{P}_5$ such that

$$0 = D(p(x)) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4.$$

Then $a_1 = a_2 = a_3 = a_3 = a_4 = a_5 = 0$. Therefore, $\text{null}(D) = \mathbb{R}$, the space of constant polynomials.

(iii) If $T(x, y) = 0$ then $2x + 3y = 0$ and $7x = 5y$. Combining both these we get $-2x/3 = 7x/5$ which means $x = 0$ and $y = 7x/5 = 0$. Therefore, $\text{null}(T) = \{0\}$.

(iv) We know that $(1, x, x^2, x^3, x^4, x^5)$ is a basis for $\mathcal{P}_5$. It follows from the definition of T that $T(x^i) = x^{4i} \neq 0$, i.e., T is injective on the basis elements and therefore injective on $\mathcal{P}_5$. Consequently, $\text{null}(T) = \{0\}$,

(v) If $T(x, y) = (x, 0) = (0, 0)$ then $x = 0$. Therefore, $\text{null}(T) = \{(0, y) \,|\, y \in \mathbb{R}\}$.

(vi) If $T(x, y) = x + 2y = 0$ then $y = -x/2$. Therefore, $\text{null}(T) = \{(2x, -x) \,|\, x \in \mathbb{R}\}$.

Problem 12 (i) We compute ST ans TS and then compare them. On the one hand

$$ST(p(x)) = S(x^2p(x)) = x^4p(x^2)$$

while on the other hand

$$TS(p(x)) = T(p(x^2)) = x^2p(x^2).$$

Therefore S and T don't commute.

(ii) As before, on the one hand

$$ST(a + bx + cx^2 + dx^3) = S(a + cx^2) = a + c(x+2)^2 = (a + 4c) + 2cx + cx^2$$

while on the other hand

$$\begin{aligned} TS(a + bx + cx^2 + dx^3) &= T(a + 2b + 4c + 8d + (b + 2c + 12d)x + (c + 6d)x^2 + dx^3) \\ &= a + 2b + 4c + 8d + (c + 6d)x^2. \end{aligned}$$

Therefore, S and T don't commute.

Problem 13 (i) No. Any invertible linear transformation must be surjective, viz., the image must have full dimension. In this case, the image of $T : \mathbb{R}^2 \to \mathbb{R}^2$ is $\{(x, x) \,|\, x \in \mathbb{R}\}$ is 1 dimensional.

(ii) Yes. The inverse of T is T itself. For example, $TT(x, y) = T(y, x) = (x, y)$ whence $TT = \text{Id}$.

(iii) No. Any invertible linear transformation must be injective, viz., it must have no null space. As we saw in 11 (ii), D on $\mathcal{P}_5$ has a 1 dimensional space as its null space and hence not invertible.

Note Title 10/1/2012

Problem 4.

(1) Let $\{v_1, \cdots, v_\ell\}$ be a basis of L, where $\ell = \dim L$. For any $\varphi(v) \in \varphi(L)$, $v \in L$, write $v = a_1 v_1 + \cdots + a_\ell v_\ell$. Then

$\varphi(v) = \varphi(a_1 v_1 + \cdots + a_\ell v_\ell) = a_1 \varphi(v_1) + \cdots + a_\ell \varphi(v_\ell)$ (φ is linear)

That is, $\varphi(v_1), \cdots, \varphi(v_\ell)$ span $\varphi(L)$. Therefore, $\dim \varphi(L) \le \ell = \dim L$.

(2) When φ is one to one, if $a_1 \varphi(v_1) + \cdots + a_\ell \varphi(v_\ell) = 0$, which is equivalent to $\varphi(a_1 v_1 + \cdots + a_\ell v_\ell) = 0$, then $a_1 v_1 + \cdots + a_\ell v_\ell = 0$. ($Null(\varphi) = \{0\}$)

Since $\{v_1, \cdots, v_\ell\}$ is a basis of L, we must have $a_1 = \cdots = a_\ell = 0$.

Therefore $\varphi(v_1), \cdots, \varphi(v_\ell)$ are linearly independent.

Combining (1), we have $\{\varphi(v_1), \cdots, \varphi(v_\ell)\}$ is a basis of $\varphi(L)$,

so $\dim \varphi(L) = \ell = \dim L$.

Problem 15.

$\{1, x, x^2, x^3\}$ is a basis of P_3. Denote it by β.

Denote γ the standard basis $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$.

Then $[T]_\gamma^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

$\text{rank}([T]_\gamma^\beta) = 2 \Rightarrow \dim N(T) = 2$

Actually $\{x^3 - x^2, x^2 - x\}$ is a basis of $N(T)$, denote it by α.

Define $\mathcal{A}: N(T) \longrightarrow \mathbb{R}^2$ by $\mathcal{A}(f) = (f'(0), f'(1))$,

then $[\mathcal{A}]_\gamma^\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, γ as defined above.

$\det\left([\mathcal{A}]_\gamma^\alpha\right) \neq 0 \Rightarrow \mathcal{A}$ isomorphism.

α can be extended to $\{x^3 - x^2, x^2 - x, x, 1\}$, a basis of P_3.

Problem 16.

$T(1,3) = (-7, 26) = -54(1,3) + 47(1,4)$

$T(1,4) = (-10, 33) = -76(1,3) + 66(1,4)$

$\Rightarrow [T]_\beta = \begin{pmatrix} -54 & -76 \\ 47 & 66 \end{pmatrix}$

$T(3,2) = (0, 29) = -203(3,2) + 87(7,5)$

$T(7,5) = (-1, 70) = -495(3,2) + 212(7,5)$

$\Rightarrow [T]_{\beta'} = \begin{pmatrix} -203 & -495 \\ 87 & 212 \end{pmatrix}$

$Q = [I]_{\beta'}^{\beta} = \begin{pmatrix} -16 & -23 \\ 7 & 10 \end{pmatrix}$

$[T]_{\beta'} Q = \begin{pmatrix} -217 & -281 \\ 92 & 119 \end{pmatrix} = Q\,[T]_\beta$