solution to problem 5 of pract. midt.

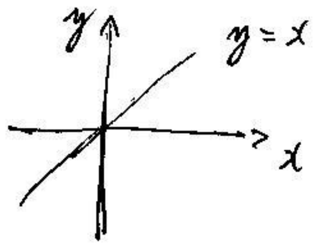
I think there is a typo, and the differential equation is

$$\frac{dy}{dx} = f(x, y)$$

1) If $(x, y) \neq (0,0)$, then $f(x,y) = \dfrac{xy}{x^2+y^2}$

is the ratio of two polynomials, and so it is continuous where the denominator x^2+y^2 is non-zero, i.e. for all $(x, y) \neq (0,0)$.

It remains to check if f is continuous or not at $(0,0)$.



(and $(x,y) \neq (0,0)$)

If (x, y) is on the line $y = x$ then

$$f(x,y) = \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

Hence for f restricted to $y = x$,

$$\lim_{x \to 0} f(x,x) = \frac{1}{2} \neq f(0,0) = 0$$

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Hence $f(x,y)$ is not continuous at $(0,0)$.

~~[illegible]~~. Hence $f(x,y)$ is continuous for $(x,y) \neq (0,0)$ and is discontinuous at $(0,0)$.

$\boxed{\frac{\partial f}{\partial y}}$ Now let's consider $\frac{\partial f}{\partial y}$. If $(x,y) \neq (0,0)$ then

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{xy}{x^2+y^2}\right)$$

$$= \frac{x(x^2+y^2) - 2xy^2}{(x^2+y^2)^2} \qquad \text{(quotient rule)}$$

$$= \frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0}\left(\frac{f(0,h) - f(0,0)}{h}\right) \qquad \text{(by definition)}$$

$$= \lim_{h \to 0} \frac{0-0}{h} = 0$$

Hence $$\frac{\partial f}{\partial y} = \begin{cases} \dfrac{x(x^2-y^2)}{(x^2+y^2)^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

If $(x,y) \neq (0,0)$, then f is continuous at (x,y).
It remains to consider the point $(0,0)$.

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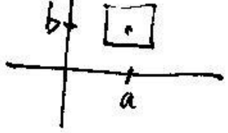
consider the restriction of $f(x, y)$ to the x-axis ($y = 0$); if $y = 0$ and $(x, y) \neq (0,0)$, then

$$\frac{\partial f}{\partial y} = \frac{x(x^2 - 0)}{x^4} = \frac{1}{x}$$

and $\lim_{x \to 0} \frac{\partial f}{\partial y}(x, 0) \neq 0 = \frac{\partial f}{\partial y}(0,0)$.

Hence $\frac{\partial f}{\partial y}$ is discontinuous at $(0,0)$.

2) If $(a, b) \neq (0,0)$, then both f and $\frac{\partial f}{\partial y}$ are continuous on a small rectangle around the point (a, b),



so that theorem 1 guarantees the existence of a unique solution of

$$\begin{cases} \frac{dy}{dx} = f(x, y) \\ y(a) = b \end{cases}.$$

If $(a, b) = (0,0)$, the hypotheses of theorem 1 are not satisfied, so theorem 1 doesn't tell us anything for that case.

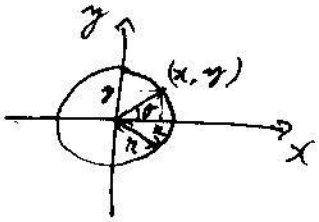
Solution to problem 5

If $(x,y) \neq (0,0)$, then $f(x,y) = \dfrac{xy}{x^2+y^2}$

Using polar coordinates (r,θ), where

$x = r\cos\theta$

$y = r\sin\theta$

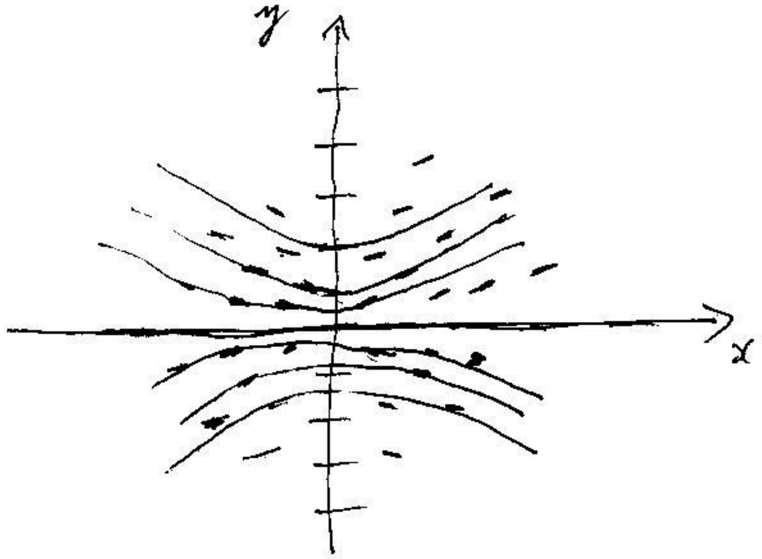


We get that

$$f(x,y) = \frac{r\cos\theta \, r\sin\theta}{r^2}$$

$$= \sin\theta\cos\theta$$

$$\boxed{f(x,y) = \tfrac{1}{2}\sin(2\theta) \quad , \text{ if } (x,y) \neq (0,0)}$$



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4) It appears from the slope field that there is a solution with $y(0)=0$, namely $y(x)=0$, and it seems that this solution is unique (from the plot).

5) $$\frac{dy}{dx} = \frac{xy}{x^2+y^2} = \frac{\frac{y}{x}}{1+\frac{y^2}{x^2}}$$

Let $v = \frac{y}{x}$

$\Rightarrow y = vx$

$\Rightarrow y' = v'x + v$

$\Rightarrow v'x + v = \frac{v}{1+v^2}$

$\Rightarrow v'x = -\frac{v^3}{1+v^2}$

$\Rightarrow \frac{(1+v^2)v'}{v^3} = -\frac{1}{x}$ (separable).

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From previous page

$$\frac{(1+v^2)v'}{v^3} = -\frac{1}{x}$$

Integrating, one gets:

$$-\frac{1}{2}v^{-2} + \ln|v| = -\ln|x| + C$$

$$\Rightarrow -\frac{1}{2}\frac{x^2}{y^2} + \ln|y| - \ln|x| = -\ln|x| + C \qquad \left(v = \frac{y}{x}\right)$$

$$\Rightarrow -\frac{1}{2}\frac{x^2}{y^2} + \ln|y| = C$$

skipping a few steps (because I need to sleep):

$$\boxed{x^2 = 2y^2(\ln|y| - C)}$$

We should add to this family of solutions, the solution

$\boxed{y(x) = 0}$ (solution of $\frac{dy}{dx} = f(x,y)$ that is...).

QED

solution to problem 6 of pract. midt.

(6) $$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy$$ (if $F(x,y)$ is a differentiable function).

1) $F(x,y) = x^2 + xy + y^3$

$$\boxed{dF = (2x+y)\,dx + (x+3y^2)\,dy}$$

The solutions of $dF = 0$ are given implicitly by $F(x,y) = C$, i.e.

$$\boxed{x^2 + xy + y^3 = C}$$ (where C is a constant).

2) $F(x,y) = x^2\cos(y)$

$$\boxed{dF = 2x\cos(y)\,dx - x^2\sin(y)\,dy}$$

Solutions of $dF = 0$ are given by:

$$\boxed{x^2\cos(y) = C}$$ (where C is a constant).

QED