solution to problem 5 of prat.matt.

I think there is a typo, and the differential equation is

\[ \frac{dy}{dx} = f(x, y) \]

1) If \((x, y) \neq (0, 0)\), then \(f(x, y) = \frac{x^2}{x^2 + y^2}\)

is the ratio of two polynomials, and so it is continuous where the denominator \(x^2 + y^2\) is non-zero, i.e., for all \((x, y) \neq (0, 0)\).

It remains to check if \(f\) is continuous or not at \((0, 0)\).

\[ \frac{y}{x} \quad (\text{and} \ (x, y) \neq (0, 0)) \]

If \((x, y)\) is on the line \(y = x\), then

\[ f(x, y) = \frac{x^2}{x^2 + x^2} = \frac{1}{2} \]

Hence for \(f\) restricted to \(y = x\),

\[ \lim_{x \to 0} f(x, x) = \frac{1}{2} \neq f(0, 0) = 0 \]
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Hence \( f(x, y) \) is not continuous at \((0, 0)\).

Hence \( f(x, y) \) is continuous \( \forall \) for \((x, y) \neq (0, 0)\)

and is discontinuous at \((0, 0)\).

Now let's consider \( \frac{\partial f}{\partial y} \). If \((x, y) \neq (0, 0)\) then

\[
\frac{\partial f}{\partial y} = \frac{1}{2y} \left( \frac{x^2y^2}{x^2 + y^2} \right)
= \frac{x(x^2y^2 - 2xy^4)}{(x^2 + y^2)^2} \quad \text{(quotient rule)}
= \frac{x(x^2 - y^4)}{(x^2 + y^2)^2}
\]

\[
\frac{\partial f}{\partial y}(0, 0) = \lim_{h \to 0} \left( \frac{f(h, h) - f(0, 0)}{h} \right) \quad \text{(by definition)}
= \lim_{h \to 0} \frac{0 - 0}{h} = 0
\]

Hence \( \frac{\partial f}{\partial y} = \begin{cases} 
\frac{x(x^2 - y^4)}{(x^2 + y^2)^2}, & \text{if } (x, y) \neq (0, 0) \\
0, & \text{if } (x, y) = (0, 0) 
\end{cases} \)

If \((x, y) \neq (0, 0)\), then \( f \) is continuous at \((x, y)\).

It remains to consider the point \((0, 0)\).
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Consider the restriction of \( f(x,y) \) to the x-axis \((y = 0)\); if \( y = 0 \) and \((x,y) \neq (0,0)\), then

\[
\frac{\partial f}{\partial y} = \frac{x^2 - 0}{x^y} = \frac{1}{x}
\]

and

\[
\lim_{x \to 0} \frac{\partial f}{\partial y} (x,0) \neq 0 = \frac{\partial f}{\partial y} (0,0).
\]

Hence \( \frac{\partial f}{\partial y} \) is discontinuous at \((0,0)\).

2) If \((a,b) \neq (0,0)\), then both \( f \) and \( \frac{\partial f}{\partial y} \) are continuous on a small rectangle around the point \((a,b)\), so that theorem 1 guarantees the existence of a unique solution of

\[
\begin{cases}
\frac{dy}{dx} = f(x,y) \\
y(a) = b
\end{cases}
\]

If \((a,b) = (0,0)\), the hypotheses of theorem 1 are not satisfied, so theorem 1 doesn't tell us anything for that case.
Solution to problem 5

If \((x, y) \neq (0, 0)\), then \(f(x, y) = \frac{xy}{x^2 + y^2}\).

Using polar coordinates \((r, \theta)\), where
\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta
\end{align*}
\]
we get that
\[
f(x, y) = \frac{r \cos \theta \cdot r \sin \theta}{r^2} = \cos \theta \sin \theta
\]
\[
f(x, y) = \frac{1}{2} \sin(2\theta), \text{ if } (x, y) \neq (0, 0)
\]
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4) It appears from the slope field that there is a solution with \( y(x) = 0 \), namely \( y(x) = 0 \), and it seems that this solution is unique (from the plot).

5) \[
\frac{dy}{dx} = \frac{xy}{x^2 + y^2}
\]

Let \( z = \frac{y}{x} \)

\[
=> y = zx
\]

\[
=> y' = z'x + z
\]

\[
=> z'x + z = \frac{z^2}{1 + z^2}
\]

\[
=> z'x = \frac{z^2}{1 + z^2}
\]

\[
=> (1 + z^2)z' = -\frac{1}{x} \quad \text{(separable)}
\]
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From previous page

\[
\frac{(1 + x^2) \, y'}{y^3} = -\frac{1}{2}
\]

Integrating, one gets:

\[-\frac{1}{2} \, x^{-2} + \ln |y| = -\ln |x| + C\]

\[-\frac{1}{2} \, \frac{x}{y^2} + \ln |y| = -\ln |x| + C \quad (x \neq 0)\]

\[-\frac{1}{2} \, \frac{x}{y^2} + \ln |y| = C\]

skipping a few steps (because I need to sleep):

\[x^2 = 2 \, y^3 (\ln |y| - C)\]

We should add to this family of solutions, the solution

\[y(x) = 0\] (solution of \(\frac{dy}{dx} = f(x, y)\) that is...).

\[\text{QED}\]
solution to problem 6 of prob. whth.

\[ dF = \frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial y} \, dy \]  (if \( F \) is a differntiable function).

1) \( F(x, y) = x^2 + xy + y^2 \)
\[ dF = (2x + y) \, dx + (x + 3y^2) \, dy \]

The solutions of \( dF = 0 \) are given implicitly by
\[ F(x, y) = C, \quad \text{id.} \]
\[ x^2 + xy + y^2 = C \]  (where \( C \) is a constant).

2) \( F(x, y) = x \cos(y) \)
\[ dF = 2x \cos(y) \, dx - x \sin(y) \, dy \]

Solutions of \( dF = 0 \) are given by:
\[ x \cos(y) = C \]  (where \( C \) is a constant).