Solution to problem I from practice midterm II

1) $y' = (y+3)^2(y+1)(y-3)$

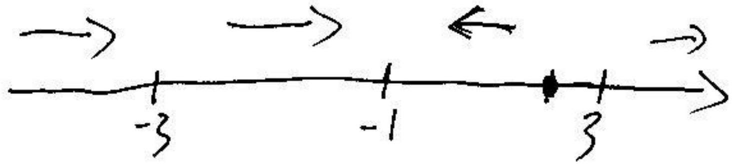
y	$-\infty$		-3		-1		3		$+\infty$
y'		$+$	0	$+$	0	$-$	0	$+$	

Equilibrium solutions:

$y = -3 \rightarrow$ stable from left, unstable from right ($\Rightarrow$ unstable)

$y = -1 \rightarrow$ stable

$y = 3 \rightarrow$ unstable



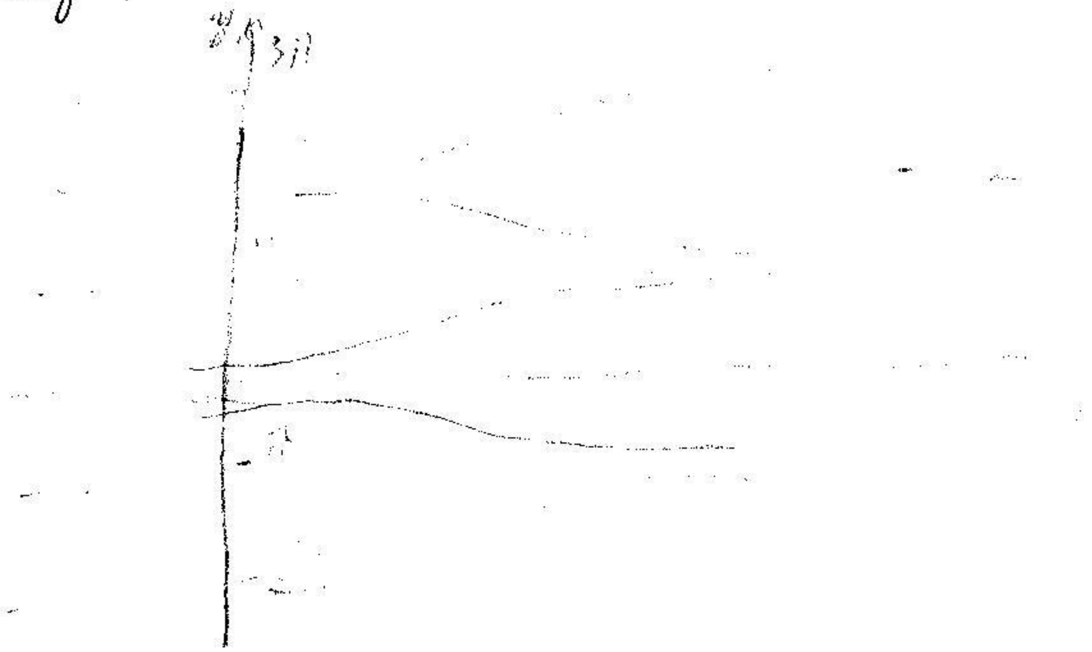
Solution to problem 2 from practice midterm

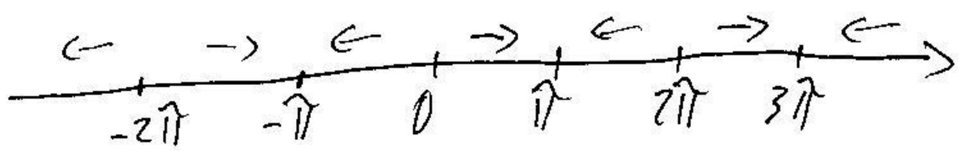
$y' = \sin y$

Equilibrium solutions correspond to the zeros of $\sin y$:

$y = n\pi \ , \quad n \in \mathbb{Z}$

y		-2π		$-\pi$		0		π		2π		3π	
$y' = \sin y$	$-$	0	$+$	0	$-$	0	$+$	0	$-$	0	$+$	0	$-$





$y = 2k\pi \ , \quad k \in \mathbb{Z} \longrightarrow$ unstable

$y = (2k+1)\pi \ , \quad k \in \mathbb{Z} \longrightarrow$ stable

Solution to problem 4 from practice midterm II

If the solution curve is concave up, we see from the picture above that Euler's method would give an underestimate.

- Improved Euler:

$y' = xy$

$y(1) = 2$ $\qquad \underline{\underline{h = 1}}$

$x_0 = 1$

$y_0 = 2$

$k_1 = f(1,2) = 2$

$u_1 = y_0 + hk_1 = 2 + 1\times 2 = 4$

$k_2 = f(x_1, u_1) = f(2,4) = 8$

$y_1 = y_0 + \frac{h}{2}(k_1 + k_2)$

$\quad = 2 + \frac{1}{2}(2+8)$

$y_1 = 7$

So improved Euler's method gives

$y(2) \simeq 7$

Solution to problem 6 from practice midterm II

1) $y'' - 2y' + y = -x e^x$ (*)

$y(0) = 1$, $y'(0) = 1$

$s^2 - 2s + 1 = 0 \iff (s-1)^2 = 0$

$s = 1$ (mult. 2)

The RHS of (*) corresponds to $s = 1$, which is a characteristic root, with multiplicity 2. So we take:

$y_p = x^2(A x e^x + B e^x)$

$= A x^3 e^x + B x^2 e^x$

We want (from *):

$\left(\frac{d}{dx} - 1 \times Id\right)^2 y_p = -x e^x$

<u>proposition</u>: $\left(\frac{d}{dx} - 1 \times Id\right) x^n e^x = n x^{n-1} e^x$

<u>proof</u>: by direct computation. ◻

Solution to 6 (continues)

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$$\left(\frac{d}{dx} - 1\times Id\right)^2 y_p$$

$$= \left(\frac{d}{dx} - 1\times Id\right)\circ\left(\frac{d}{dx} - 1\times Id\right)\left(Ax^3e^x + Bx^2e^x\right)$$

($\circ \to$ composition)

$$= \left(\frac{d}{dx} - 1\times Id\right)\left(3Ax^2e^x + 2Bx e^x\right)$$

$$= 6Ax e^x + 2Be^x \overset{\text{Want}}{=} -xe^x$$

So
$$\begin{cases} 6A = -1 & \Rightarrow \boxed{A = -\frac{1}{6}} \\ 2B = 0 & \Rightarrow \boxed{B = 0} \end{cases}$$

$$\Rightarrow \boxed{y_p = -\frac{1}{6}x^3e^x}$$

$$\Rightarrow \quad y = y_c + y_p$$

$$y = ae^x + bxe^x - \frac{1}{6}x^3e^x$$

$$y(0) = 1 \Rightarrow a = 1$$

$$\Rightarrow y = e^x + bxe^x - \frac{1}{6}x^3e^x$$

$$\cancel{y'(0) = 1 \Rightarrow} \Rightarrow y' = e^x + be^x + bxe^x - \frac{1}{2}x^2e^x - \frac{1}{6}x^3e^x$$

$$y'(0) = 1 \Rightarrow 1 = 1 + b \Rightarrow \boxed{b = 0}$$

$$\Rightarrow \boxed{y = e^x - \frac{1}{6}x^3e^x}$$

Solution to 6 (continued)

b) 2.

$$y'' + 4y = -\cos(2x) \qquad (**)$$

$$\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

char. eq. : $s^2 + 4 = 0$

$\Rightarrow s = \pm 2i$

$\Rightarrow y_c = A\cos(2x) + B\sin(2x)$

The RHS of $(**)$, $-\cos(2x)$, corresponds to the pair of roots $\pm 2i$, which is also a characteristic pair of roots with multiplicity 1. So we take:

$$y_p = x\left(c\cos(2x) + d\sin(2x)\right)$$

$$y_p = c\,x\cos(2x) + d\,x\sin(2x)$$

$$y_p' = -2c\,x\sin(2x) + 2d\,x\cos(2x) + c\cos(2x) + d\sin(2x)$$

$$y_p'' = -4c\,\cancel{x\cos(2x)} - 4d\,\cancel{x\sin(2x)} - 2c\sin(2x) + 2d\cos(2x)$$
$$- 2c\sin(2x) + 2d\cos(2x)$$

$$\Rightarrow y_p'' + 4y_p = -4c\sin(2x) + 4d\cos(2x) \overset{\text{Want}}{=} -\cos(2x)$$

Solution to 6 (continued)

so we get the following system:

$$\begin{cases} 4d = -1 \\ -4c = 0 \end{cases}$$

$$\Rightarrow \boxed{d = -\frac{1}{4}}$$
$$\boxed{c = 0}$$

$$\Rightarrow \boxed{y_p = -\frac{1}{4} x \sin(2x)}$$

$$\Rightarrow y = y_c + y_p$$
$$\boxed{y = A\cos(2x) + B\sin(2x) - \frac{1}{4} x \sin(2x)}$$

$$y(0) = 0 \Rightarrow A = 0$$
$$\Rightarrow y = B\sin(2x) - \frac{1}{4} x \sin(2x)$$
$$\Rightarrow y' = 2B\cos(2x) - \frac{1}{4}\sin(2x) - \frac{1}{2} x \cos(2x)$$
$$y'(0) = 0 \Rightarrow 0 = 2B \Rightarrow B = 0$$

$$\Rightarrow \boxed{y = -\frac{1}{4} x \sin(2x)}$$

QEF