3.5.38 $y'' + 2y' + 2y = \sin 3x, \quad y(0)=2, \quad y'(0)=0$

(complimentary)
Homogeneous solution:

$$r^2+2r+2=0 \Rightarrow r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$y_c = e^{-x}(c_1 \cos x + c_2 \sin x)$$

Particular solution:

Guess $y_p = A\cos 3x + B\sin 3x$

$y_p' = -3A\sin 3x + 3B\cos 3x$

$y_p'' = -9A\cos 3x - 9B\sin 3x$

Then $y_p'' + 2y_p' + 2y_p = \sin 3x$ implies

$$\begin{cases} -9A + 2(3B) + 2A = 0 & \leftarrow \text{coefficients of } \cos 3x \\ -9B + 2(-3A) + 2B = 1 & \leftarrow \text{coeff. of } \sin 3x \end{cases}$$

$$\Rightarrow \begin{Bmatrix} -7A + 6B = 0 \\ -6A + {}^-7B = 1 \end{Bmatrix} \Rightarrow \begin{matrix} A = -6/85 \\ B = -7/85 \end{matrix}$$

So $y_p = \frac{-6}{85}\cos 3x - \frac{7}{85}\sin 3x$

General solution:

$$y = \frac{-6}{85}\cos 3x - \frac{7}{85}\sin 3x + e^{-x}[c_1\cos x + c_2 \sin x]$$

$$y' = \frac{18}{85}\sin 3x - \frac{21}{85}\cos 3x - e^{-x}[c_1\cos x + c_2\sin x] + e^{-x}[-c_1\sin x + c_2\cos x]$$

$$\begin{rcases} y(0)=2 \\ y'(0)=0 \end{rcases} \Rightarrow \begin{Bmatrix} 2 = \frac{-6}{85} + c_1 \\ 0 = \frac{-21}{85} - c_1 + c_2 \end{Bmatrix} \Rightarrow \begin{cases} c_1 = \frac{176}{85} \\ c_2 = \frac{197}{85} \end{cases}$$

$$\boxed{y = \frac{1}{85}\left[-6\cos 3x - 7\sin 3x + e^{-x}(176\cos x + 197\sin x)\right]}$$

3.5.55 $y'' + 4y = \sin^2 x$

$r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$\Rightarrow \left\{ \begin{array}{l} y_1 = \cos 2x \\ y_2 = \sin 2x \end{array} \right\}$ are homogeneous ~~complimentary~~ solutions

$$\left\{ \begin{array}{l} u_1' \cos 2x + u_2' \sin 2x = 0 \\ -2u_1' \sin 2x + 2u_2' \cos 2x = \sin^2 x \end{array} \right.$$

$$\Rightarrow u_1' = -\frac{1}{2} \sin^2 x \sin 2x$$

$$= -\frac{1}{2} \sin^2 x \, (2 \sin x \cos x)$$

$$= - \sin^3 x \cos x$$

$$\Rightarrow \underline{u_1 = -\frac{1}{4} \sin^4 x}$$

$$u_2' = \frac{1}{2} \sin^2 x \cos 2x$$

$$= \frac{1}{4} (1 - \cos 2x) \cos 2x$$

$$= \frac{1}{4} \left[\cos 2x - \cos^2 2x \right]$$

$$= \frac{1}{4} \left[\cos 2x - \frac{1}{2}(1 + \cos 4x) \right]$$

$$\Rightarrow \underline{u_2 = \frac{1}{8} \sin 2x - \frac{1}{8} x - \frac{1}{32} \sin 4x}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$\boxed{y_p = -\frac{1}{4} \sin^4 x \cos 2x + \frac{1}{8} \left(\sin 2x - x - \frac{1}{4} \sin 4x \right) \sin 2x}$$

$$= -\frac{1}{8} \cos 2x + \frac{1}{8} - \frac{1}{8} x \sin 2x \quad \text{(using trig identities)}$$

But $-\frac{1}{8} \cos 2x$ is a homogeneous solution, so

$y_p = \frac{1}{8} - \frac{1}{8} x \sin 2x$ is another particular solution.
(a different correct answer)