solutions to 3.5 problems (from Homework)

6) $2y'' + 4y' + 7y = x^2$

char. equation: $2s^2 + 4s + 7 = 0$

$$s^2 + 2s + \frac{7}{2} = 0$$

$$s = \frac{-2 \pm \sqrt{4 - 14}}{2}$$

$$= -1 \pm i\frac{\sqrt{10}}{2}$$

x^2 corresponds to ~~the characteristic~~ 0, but 0 is not a characteristic root (since the characteristic roots are $-1 \pm i\frac{\sqrt{10}}{2}$).

So we try $y_p = Ax^2 + Bx + C$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$2y_p'' + 4y_p' + 7y_p = \cancel{4A} + \cancel{8Ax} + \cancel{4B} + \cancel{7Ax^2} + \cancel{7Bx} + \cancel{7C}$$

$$= 7Ax^2 + (8A + 7B)x + 4A + 7C + 4B$$

$$\overset{\text{want}}{=} x^2$$

$\Rightarrow$ we get the system:

$$\begin{cases} 7A = 1 \\ 8A + 7B = 0 \\ 4A + 7C + 4B = 0 \end{cases}$$

$$\Rightarrow \boxed{A = \frac{1}{7}}$$

$$\Rightarrow \boxed{B = -\frac{8}{49}}$$

$$7C = -\frac{4}{7} + \frac{32}{49}$$

$$= \frac{-28 + 32}{49}$$

$$= \frac{4}{49}$$

$$\Rightarrow \boxed{C = \frac{4}{343}}$$

$$\Rightarrow \boxed{y_p = \frac{1}{7}x^2 - \frac{8}{49}x + \frac{4}{343}}$$

3.5 problems (continued)

14) $y^{(4)} - 2y'' + y = xe^{x}$ $(*)$

char. eq.: $s^4 - 2s^2 + 1 = 0$

$\Rightarrow (s^2-1)^2 = 0$

$\Rightarrow [(s-1)(s+1)]^2 = 0$

$\Rightarrow s = 1$ (mult. 2)

$s = -1$ (mult. 2)

xe^x corresponds to $s=1$, which is a characteristic root of multiplicity 2. So we try:

$$y_p = x^2(Axe^x + Be^x)$$

$$= Ax^3e^x + Bx^2e^x$$

The ODE can be rewritten as: ($\triangle$ SEE THE NOTE BELOW)

$$\left(\frac{d}{dx} - 1\cdot \mathrm{Id}\right)^2 \circ \left(\frac{d}{dx} + 1\cdot \mathrm{Id}\right)^2 y = 0$$

(here $\circ$ ~~multiplication~~ is composition of operators; and $L^2 = L \circ L$)

$$\Leftrightarrow \left(\frac{d}{dx} + 1\cdot \mathrm{Id}\right)^2 \circ \left(\frac{d}{dx} - 1\cdot \mathrm{Id}\right)^2 y = 0$$

Here Id is the identity operator ($\mathrm{Id}\, y = y$).

Note: This method is more complicated conceptually, but

14) (continued) but involves less computations. You can use your own way to solve this problem if you want (by computing directly for instance). IF IT CONFUSES YOU, DON'T READ IT!

<u>prop.</u>: $\left(\frac{d}{dx} - 1\times Id\right)(fg) = f'g + f\left(\frac{d}{dx} - 1\times Id\right)g$

<u>proof</u>: $LHS = \frac{d}{dx}(fg) - fg$

$$= f'g + fg' - fg$$

$$= f'g + f\left(\frac{d}{dx} - 1\times Id\right)g$$

$$= RHS$$

QED

<u>corollary</u>: $\boxed{\left(\frac{d}{dx} - 1\times Id\right)(x^n e^x) = nx^{n-1}e^x}$

<u>proof</u>: $LHS = nx^{n-1}e^x + x^n\left(\frac{d}{dx} - 1\times Id\right)e^x$

But $\left(\frac{d}{dx} - 1\times Id\right)e^x = 0$, proving the corollary. QED.

14) (continued)

We want $\left(\frac{d}{dx} + 1 \times Id\right)^2 \circ \left(\frac{d}{dx} - 1 \times Id\right)^2 y_p = x e^x$

$$LHS = \left(\frac{d}{dx} + 1 \times Id\right)^2 \circ \left(\frac{d}{dx} - 1 \times Id\right) \circ \left(\frac{d}{dx} - 1 \times Id\right)\left(A x^3 e^x + B x^2 e^x\right)$$

$$= \left(\frac{d}{dx} + 1 \times Id\right)^2 \circ \left(\frac{d}{dx} - 1 \times Id\right)\left(3A x^2 e^x + 2B x e^x\right)$$

$$= \left(\frac{d}{dx} + 1 \times Id\right)^2 \left(6A x e^x + 2B e^x\right) \qquad \text{(I)}$$

Note $\left(\frac{d}{dx} + 1 \times Id\right)\left(x^n e^x\right) = n x^{n-1} e^x + x^n \left(\frac{d}{dx} + 1 \times Id\right) e^x$

(similar proof as the proposition).

But $\left(\frac{d}{dx} + 1 \times Id\right) e^x = 2 e^x$

$$\Rightarrow \left(\frac{d}{dx} + 1 \times Id\right)^2 \circ \left(\frac{d}{dx} - 1 \times Id\right)^2 y_p$$

$$= \left(\frac{d}{dx} + 1 \times Id\right)\left(6A e^x + 12 A x e^x + 4B e^x\right)$$

$$= \left(\frac{d}{dx} + 1 \times Id\right)\left(12 A x e^x + (6A + 4B) e^x\right)$$

$$= 12 A e^x + 24 A x e^x + (12A + 8B) e^x$$

$$= 24 A x e^x + (24A + 8B) e^x$$

3.5 problems (continued)

We want

$$24Axe^x + (24A+8B)e^x = xe^x$$

$$\Rightarrow 24A = 1$$

$$\Rightarrow \boxed{A = \frac{1}{24}}$$

Also $24A + 8B = 0$

$$\Rightarrow 1 + 8B = 0$$

$$\Rightarrow \boxed{B = -\frac{1}{8}}$$

$$\Rightarrow \boxed{y_p = \frac{1}{24}x^3e^x - \frac{1}{8}x^2e^x}$$

QED